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THE GROUP PROPERTY OF THE INVARIANT S OF VON NEUMANN ALGEBRAS

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Abstract.

We prove that if M is any countably decomposable factor, the invariant $S(M)$ defined in [1] is a closed subgroup of the group of positive real numbers. Moreover multiplication by any element of $S(M)$ leaves the spectrum of any state on M invariant.

THEOREM 1. (a) *Let M be a countably decomposable factor, then the non zero elements of the intersection $S(M)$ of the spectra of the modular operators Δ_φ associated with φ, when φ runs through all faithful normal states on M, is a closed subgroup of the multiplicative group of positive real numbers.*
(b) *For any faithful normal state φ on M the spectrum of Δ_φ is invariant under multiplication by $S(M)$.*

To prove the theorem we need a few lemmas.

Let $\mathscr{A}$ be an achieved generalized left Hilbert algebra, Δ the modular operator of $\mathscr{A}$.

LEMMA 2. *Let V be any compact interval of $]0,\infty[$ and χ the characteristic function of V. If $\xi \in \mathscr{A}$ such that $\chi(\Delta)\xi=\xi$ then for all integers $n \in \mathsf{Z}$ we have $\xi \in \mathscr{D}(\Delta^n)$ and $\Delta^n\xi \in \mathscr{A}$.*

PROOF. It is not hard to see that there exists a function $f \in L_1(\mathsf{R})$ such that

$$\lambda^n = \int_{-\infty}^{+\infty} \lambda^{it} f(t)\,dt \qquad \text{for all } \lambda \in V .$$

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It then follows that

$$\Delta^n\xi = \Delta^n\chi(\Delta)\xi = \int_{-\infty}^{+\infty} \Delta^{it}\chi(\Delta)\xi f(t)\,dt$$
$$= \int_{-\infty}^{+\infty} \Delta^{it}\xi f(t)\,dt\,.$$

Clearly $\Delta^n\xi \in \mathscr{D}(\Delta^{\frac{1}{2}})$ and $\|\pi(\Delta^n\xi)\| \leqq \|f\|_1\|\pi(\xi)\|$ so that $\Delta^n\xi \in \mathscr{A}$.

LEMMA 3. *Let V_1 and V_2 be two compact intervals of* $]0,\infty[$ *and*

$$V = \{pq \mid p \in V_1,\ q \in V_2\}\,.$$

Let χ_1, χ_2 and χ be the characteristic functions of respectively V_1, V_2 and V. Then for any $\xi_1 \in \mathscr{A}$, $\xi_2 \in \mathscr{A}$ such that $\chi_1(\Delta)\xi_1 = \xi_1$ and $\chi_2(\Delta)\xi_2 = \xi_2$ we have

$$\chi(\Delta)\xi_1\xi_2 = \xi_1\xi_2\,.$$

PROOF. By lemma 2 we know that $\Delta^n\xi_1 \in \mathscr{A}$ for all $n \in \mathbf{Z}$. With the notations of [2] and using [2, lemma 8.3] this implies that $\Delta^n\xi_1 \in \mathscr{A}'$ for all $n \in \mathbf{Z}$ and therefore $\Delta^n\xi_1 \in \mathscr{A}^\sharp$. This holds also for $\Delta^n\xi_2$ and by induction we get that $\xi_1\xi_2 \in \mathscr{D}(\Delta^n)$ and that

$$\Delta^n(\xi_1\xi_2) = (\Delta^n\xi_1)(\Delta^n\xi_2)\,.$$

Let $\Delta_1 = \Delta\chi_1(\Delta) + \alpha(1-\chi_1(\Delta))$ for some $\alpha \in V_1$; then $\mathrm{Sp}\,\Delta_1 \subset V_1$ and $\Delta^n\xi_1 = \Delta_1{}^n\xi_1$. For any simply closed smooth curve Γ enclosing V_1 we have

$$\Delta^n(\xi_1\xi_2) = \pi'(\Delta^n\xi_2)\Delta_1{}^n\xi_1$$
$$= (2\pi)^{-1}i \oint_\Gamma \pi'((\lambda\Delta)^n\xi_2)(\Delta_1-\lambda)^{-1}\xi_1\,d\lambda\,.$$

As in the proof of lemma 2 we can find a function $f \in L_1(\mathbf{R})$ such that

$$(\Delta_1-\lambda)^{-1}\xi_1 = \int_{-\infty}^{+\infty} \Delta^{it}\xi_1 f(t)\,dt$$

and by the same arguments $(\Delta_1-\lambda)^{-1}\xi_1 \in \mathscr{A}$ whenever $\lambda \notin V_1$. So for any polynomial p we have

$$p(\Delta)\xi_1\xi_2 = (2\pi)^{-1}i \oint_\Gamma ((\Delta_1-\lambda)^{-1}\xi_1)p(\lambda\Delta)\xi_2\,d\lambda\,.$$

Now let V_0 be any compact interval disjoint from V and $E_0 = \chi_0(\Delta)$ where χ_0 is the characteristic function of V_0. Then

$$\begin{aligned} \|p(\Delta)E_0\xi_1\xi_2\| &= \|E_0 p(\Delta)\xi_1\xi_2\| \\ &\leqq (2\pi)^{-1} \sup_{\Gamma} \|\pi(\Delta_1-\lambda)^{-1}\xi_1\| \|p(\lambda\Delta)\xi_2\| \, |\Gamma| \,. \end{aligned}$$

Choose ε sufficiently small such that the two open sets

$$\begin{aligned} W_0 &= \{z \mid z \in \mathsf{C},\ \text{distance}(z, V_0) < \varepsilon\}, \\ W &= \{z \mid z \in \mathsf{C},\ \text{distance}(z, V) < \varepsilon\} \end{aligned}$$

have disjoint closures.

Then it is possible to choose Γ such that the set

$$\{pq \mid p \in V_2,\ q \text{ is inside } \Gamma\}$$

is contained in W. Let f be the analytic function on $W_0 \cup W$ which is 1 on W_0 and 0 on W. By Runge's theorem it is possible to find a sequence of polynomials p_k tending uniformly to f on $W_0 \cup W$. Then

$$p_k(\Delta)E_0\xi_1\xi_2 \ \text{tends to}\ E_0\xi_1\xi_2$$

and

$$p_k(\lambda\Delta\chi_2(\Delta))\xi_2 \ \text{tends to}\ 0$$

uniformly in $\lambda \in \Gamma$. Moreover $\|\pi((\Delta_1-\lambda)^{-1}\xi_1)\|$ is uniformly bounded on Γ. Therefore $E_0\xi_1\xi_2 = 0$ and since this holds for all compact closed intervals disjoint from V,

$$\chi(\Delta)\xi_1\xi_2 = \xi_1\xi_2 \,.$$

This completes the proof.

Let φ be a faithful normal state on the von Neumann algebra M. Let (M, H, ξ_0) be the G.N.S.-construction of φ on M. As in [2] let $S = J\Delta^{\frac{1}{2}}$ be the corresponding involution. Remind that $JMJ = M'$, and that

$$\sigma_t(x) = \Delta^{it}x\Delta^{-it} \quad \text{for } x \in M$$

defines a one parameter group of automorphisms of M. In [2, lemma 15.8] it is proved that the subalgebra

$$\{x \in M \mid \sigma_t(x) = x \ \text{for all}\ t \in \mathsf{R}\}$$

equals the set

$$\{x \in M \mid \varphi(xy) = \varphi(yx) \ \text{for all}\ y \in M\} \,.$$

As in [2] we call this subalgebra M_φ.

Let e be a non zero projection of M_φ, we shall first determine the modular operator of the state φ_e defined on the reduced von Neumann algebra M_e by

$$\varphi_e(x) = \varphi(x)/\varphi(e) .$$

The closed subspace

$$H_e = \text{Image}\, e \cap \text{Image}\, JeJ = eJeJH$$

is invariant by any element of the algebra M_e. So we can consider the algebra M_1 induced by M_e in H_e and the canonical homomorphism π of M_e onto M_1. The element $e\xi_0$ of H is in H_e because

$$JeJ\xi_0 = Je\xi_0 = Je\Delta^{\frac{1}{2}}\xi_0 = J\Delta^{\frac{1}{2}}e\xi_0 = e\xi_0$$

hence $e\xi_0 \in eJeJH$. Let $\xi_1 = e\xi_0/\|e\xi_0\|$; then it is easy to check that (π, H_e, ξ_1) is the G.N.S.-construction of the state φ_e on M_e. To check that ξ_1 is cyclic for M_1 in H_e it is enough to prove that $x \in M$ implies $eJeJx\xi_0 \in M_1\xi_1$ which follows from the equality

$$eJeJx\xi_0 = exJeJ\xi_0 = exe\xi_0 .$$

Now $e\Delta^{it} = \Delta^{it}e$ for all $t \in \mathsf{R}$ and similarly JeJ commutes with Δ^{it} for all t, so Δ leaves H_e invariant and its restriction to H_e is a closed positive operator.

Let $x \in M_1$, then there exists an X in M_e such that $\pi(X) = x$, in particular

$$\|e\xi_0\| x\xi_1 = xe\xi_0 = X\xi_0$$

and

$$\|e\xi_0\| x^*\xi_1 = x^*e\xi_0 = X^*\xi_0 ,$$

hence $Sx\xi_1 = x^*\xi_1$ and the involution S_e corresponding to (M_1, H_e, ξ_1) coincides with S on $M_1\xi_1$. Similarly we get the coincidence of F_e with F on $M_1'\xi_1$. It follows that $S_e = J_R\Delta_R^{\frac{1}{2}}$ where J_R is the restriction of J to H_e and Δ_R the restriction of Δ to H_e. By the uniqueness of the polar decomposition of closed operators we get the equality $\Delta_e = \Delta_R$. Hence the modular operator of the state φ_e on M_e is the restriction of the modular operator of φ on M to the invariant subspace $eJeJH$.

DEFINITION 4. For a faithful normal state φ on M put

$$\mathfrak{S}_\varphi = \bigcap \{\text{spectrum of the modular operator of } \varphi_e \text{ on } M_e\}$$

where e runs throuph all non zero projections of the center of M_φ.

LEMMA 5. *Let $\lambda_1 > 0$, $\lambda_1 \in \mathfrak{S}_\varphi$ and $\lambda_2 > 0$, $\lambda_2 \in \text{Sp}\Delta$ then $\lambda_1\lambda_2 \in \text{Sp}\Delta$.*

Proof. (a) We first show that if a bounded open interval V of $]0,\infty[$ intersects $\operatorname{Sp}\Delta$ there exists a non zero $x \in M$ with

$$\chi(\Delta)x\xi_0 = x\xi_0 ,$$

χ being the characteristic function of V. By hypothesis $\chi(\Delta) \neq 0$, so there is a $y \in M$ with $\chi(\Delta)y\xi_0 \neq 0$. Let χ_n be a sequence of C^∞ functions on $]0,\infty[$ with $0 \leqq \chi_n \leqq \chi$ and $\chi_n(\Delta) \to \chi(\Delta)$ strongly when $n \to \infty$. Then there exists an n with $\chi_n(\Delta)y\xi_0 \neq 0$. Further by [2] one has

$$\chi_n(\Delta)y\xi_0 \in M\xi_0 ,$$

and obviously

$$\chi(\Delta)\chi_n(\Delta)y\xi_0 = \chi_n(\Delta)y\xi_0 .$$

(b) Let V_1 be a compact interval of $]0,\infty[$ with λ_1 in its interior, then let e be a non zero projection of the center of M_φ. Since the interior of V_1 intersects $\operatorname{Sp}\Delta_e$ there exists by (a) an element $x \neq 0$ of the reduced induced algebra M_1 of M in $eJeJH$ such that

$$x\xi_1 = \chi_1(\Delta_e)x\xi_1$$

where χ_1 is the characteristic function of V_1. Now $x\xi_1 \in H_e$, hence

$$\chi_1(\Delta_e)x\xi_1 = \chi_1(\Delta)x\xi_1 .$$

Since $x \in M_1$ there exists an X in M_e with $x\xi_1 = X\xi_0$, so

$$\chi_1(\Delta)X\xi_0 = X\xi_0, \quad X \neq 0, \ X \text{ in } M_e .$$

We claim that for such V_1 the supremum $\bigvee \operatorname{Supp} x$, where x runs over all elements in M with

$$\chi_1(\Delta)x\xi_0 = x\xi_0 ,$$

is equal to one. In fact it is a certain projection k with for all $t \in \mathsf{R}$, $\Delta^{it}k\Delta^{-it} = k$ because

$$\chi_1(\Delta)\Delta^{it}x\Delta^{-it}\xi_0 = \Delta^{it}x\Delta^{-it}\xi_0$$

if $\chi_1(\Delta)x\xi_0 = x\xi_0$. Also for all unitary $u \in M_\varphi$, $uku^* = k$ because

$$\chi_1(\Delta)uxu^*\xi_0 = \chi_1(\Delta)uJuJx\xi_0 = uJuJ\chi_1(\Delta)x\xi_0$$

since u and JuJ commute with Δ. So we know that k belongs to the center of M_φ hence $1-k$ is a projection e in the center of M_φ. If $e \neq 0$, there exists an $X \in M$ with

$$X\xi_0 = \chi_1(\Delta)X\xi_0, \quad X \neq 0, \quad eX = Xe = X ,$$

so $\operatorname{Supp} X \leqq e$ which contradicts $\operatorname{Supp} X \leqq k$ if $X \neq 0$.

(c) Now let W be any neighbourhood of $\lambda_1\lambda_2$ in $]0,\infty[$, choose V_1 and V_2 compact intervals containing respectively λ_1 and λ_2 in their interior and such that $V_1 \cdot V_2 \subset W$. Let χ_1, χ_2 and χ be the respective characteristic functions of V_1, V_2 and V. By (a) there exists $x \in M$ with $x \neq 0$ and

$$x\xi_0 = \chi_2(\Delta)x\xi_0 ,$$

by (b) there exists $y \in M$ with

$$y\xi_0 = \chi_1(\Delta)y\xi_0$$

and $yx \neq 0$ because $1 = \bigvee \mathrm{Supp}\, y$, when y runs over all elements in M satisfying

$$\chi_1(\Delta)y\xi_0 = y\xi_0 .$$

If we apply lemma 3 to the left generalised Hilbert algebra $\mathscr{A} = M\xi_0$ we get

$$\chi(\Delta)yx\xi_0 = yx\xi_0$$

hence V intersects the spectrum of Δ. It then follows that $\lambda_1\lambda_2 \in \mathrm{Sp}\,\Delta$ as far as W was arbitrary.

Proof of the theorem. Since the theorem is obvious in the semifinite case we assume M is type III. It is enough to prove b). Let φ be a faithful normal state on M, let $\lambda_2 > 0$, $\lambda_2 \in \mathrm{Sp}\,\Delta_\varphi$, let $\lambda_1 > 0$, $\lambda_1 \in S(M)$, then $\lambda_1\lambda_2 \in \mathrm{Sp}\,\Delta_\varphi$ will follow from the inclusion $S(M) \subset \mathfrak{S}_\varphi$. This inclusion is true because for each non zero projection e in the center of M_φ, M_e is isomorphic to M and hence $\mathrm{Sp}\,\Delta_{\varphi_e} \supset S(M)$ because φ_e is a faithful normal state on M_e.

This result will be used later to improve the classification of type III factors.

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