MATH. SCAND. 32 (1973), 193–213

INFINITE DIMENSIONAL POLYTOPES

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Introduction.

In [10], Phelps defined the class of β-polytopes to be the family of finite codimensional slices of Choquet simplexes. He showed that the family of finite codimensional β-polytopes coincides with the usual finite dimensional polytopes. Thus the class of β-polytopes properly contains both the latter class and the Choquet simplexes, and they share a number of properties of both classes. An infinite dimensional β-polytope cannot, however, be centrally symmetric and this has been shown (in [10]) to be the basis for the fact that a number of "permanence properties" of finite dimensional polytopes are no longer valid for β-polytopes. In what follows, we define a larger class of polytopes: the compact convex sets which are affinely homeomorphic to closed finite codimensional slices of unit balls of the duals of Lindenstrauss spaces (a Banach space whose dual is an $L^1(\mu)$ space). The definition was originally suggested by J. Lindenstrauss and this class of polytopes contains centrally symmetric sets. We call this class of sets the class of L-polytopes. In section 1, we give some results concerning the unit ball of an $L^1(\mu)$ space. In section 2, we characterize the maximal faces of L-polytopes. Extension properties for affine continuous functions on closed faces also hold in the class of L-polytopes. In section 3, we show that every extreme point of an L-polytope is a polyhedral vertex and in section 4, we give examples that some properties for finite dimensional polytopes cannot be generalized.

The author wishes to acknowledge his indebtedness to Professor R. R. Phelps for many fine suggestions and constructive criticism for the preparation of this paper.

1. Basic results.

In this section, our main task is to show that all maximal faces of the unit ball of an $L^1(\mu)$ space are affinely isomorphic. We also give a characterization of the maximal faces of the unit ball of an $L^1(\mu)$ space where μ is σ-finite. For the sake of completeness we include some results which may be known but for which we know of no reference.

Received August 10, 1972.

Lemma 1.1. *Let K be a convex subset of a linear space. Suppose that A is a convex subset of K, then the set*

$$A' = \{x \in K : \lambda x + (1-\lambda)z \in A, \text{ for some } z \in K \text{ and } 0 < \lambda < 1\}$$

is a face of K containing A. Moreover, A' equals the intersection of all faces of K containing A.

Throughout the section, we consider the space $L^1(X,\mu)$ only. We let B denote the unit ball of $L^1(X,\mu)$ and F will denote a maximal face of B. (We make the convention that a maximal face is a maximal *proper* face.) For simplicity, we just take μ to be a positive measure. The propositions proved here can be easily generalized to an arbitrary measure (except Theorem 1.8). From Lemma 1.1 and also from the separation theorem, it follows that if A is a convex subset of $\{x \in B : \|x\| = 1\}$, then there exists a proper face of B containing A.

Proposition 1.2. *Every maximal face of B is norm closed.*

Proof. Let F be a maximal face and let $\bar{F}$ denote the closure of F; then $\bar{F}$ is convex and each element of $\bar{F}$ is of norm 1. Thus, there exists a proper face F_1 of B containing $\bar{F}$. By maximality of F we have $F \subseteq \bar{F} \subseteq F_1 \subseteq F$.

For each $x \in L^1(X,\mu)$ we define $\operatorname{supp} x = \{t \in X : x(t) \neq 0\}$. This is defined to within a set of measure zero.

Lemma 1.3. *Let A be a convex subset of $\{x \in B : \|x\| = 1\}$. Then for any $x, y \in A$, the set $\operatorname{supp} x^+ \cap \operatorname{supp} y^-$ has measure zero.*

Proof. Let $E = \operatorname{supp} x^+ \cap \operatorname{supp} y^-$. If $\mu(E) > 0$, we have $|x+y| < |x| + |y|$ a.e. on E, therefore

$$\int_E |x+y| < \int_E |x| + |y| .$$

Consequently, we have

$$\int \tfrac{1}{2}|x+y| < \int \tfrac{1}{2}(|x| + |y|) = 1 ,$$

contradicting the convexity of A.

Lemma 1.4. *Let F be a maximal face of B and suppose $x \in F$. If y is an integrable function of norm 1 such that*

$$\operatorname{supp} y^+ \subseteqq \operatorname{supp} x^+ \quad \textit{and} \quad \operatorname{supp} y^- \subseteqq \operatorname{supp} x^- ,$$

then $y \in F$.

Proof. From Lemma 1.3, it follows that for any $z \in F$, the sets

$$\operatorname{supp} x^+ \cap \operatorname{supp} z^- \quad \text{and} \quad \operatorname{supp} x^- \cap \operatorname{supp} z^+$$

have measure zero. By hypothesis, we see that the same holds if we replace x by y. Thus, we have for any $\lambda \in [0,1]$,

$$|\lambda y + (1-\lambda)z| = \lambda|y| + (1-\lambda)|z| \quad \text{a.e.}$$

It follows that

$$\int|\lambda y + (1-\lambda)z| = \int(\lambda|y| + (1-\lambda)|z|) = 1 ,$$

so $\operatorname{conv}(\{y\} \cup F)$ is a convex subset of $\{x \in B : \|x\| = 1\}$ and there exists a proper face of B containing $\operatorname{conv}(\{y\} \cup F)$. By the maximality of F, we have $y \in F$.

Lemma 1.5. *Let F be a maximal face of B. Then for any σ-finite measurable set E of positive measure, there exists $y \in F$ such that* $\operatorname{supp} y = E$.

Proof. We assume first that $\mu(E) < \infty$. It suffices to obtain $x \in F$ such that $\operatorname{supp} x \supseteqq E$. Indeed, if such a function x exists, let

$$E_1 = \operatorname{supp} x^+ \cap E, \quad E_2 = \operatorname{supp} x^- \cap E ,$$

then $E_1 \cap E_2$ is a null set and $E_1 \cup E_2 = E$. Let

$$y = (\mu(E))^{-1}(\chi_{E_1} - \chi_{E_2}) .$$

By Lemma 1.4, we see that $y \in F$ and $\operatorname{supp} y = E$.

We obtain the function x as follows. Let

$$\alpha = \sup\{\mu(\operatorname{supp} z \cap E) : z \in F\} .$$

We will first find an $x \in F$ such that $\mu(\operatorname{supp} x \cap E) = \alpha$. To this end, for each positive integer n, we choose $x_n \in F$ such that

$$\mu(\operatorname{supp} x_n \cap E) > \alpha - n^{-1}$$

and we let $x = \sum 2^{-n} x_n$. Since F is closed, it contains x and by Lemma 1.3, the set $\operatorname{supp} x_n{}^+ \cap \operatorname{supp} x_m{}^-$ is a zero set for any m,n, hence $\operatorname{supp} x \supseteqq \operatorname{supp} x_n$ for each n. Consequently

$$\alpha \geqq \mu(\operatorname{supp} x \cap E) \geqq \mu(\operatorname{supp} x_n \cap E) > \alpha - n^{-1} .$$

Suppose now that $\mu(E_0)>0$ where $E_0=E\setminus\operatorname{supp}x$. If

$$\mu(\operatorname{supp}z\cap E_0) = 0 \quad \text{for all } z\in F,$$

then for $w=(\mu(E_0))^{-1}\chi_{E_0}$, we would have

$$\|\lambda z+(1-\lambda)w\| = 1 \quad \text{for } \lambda\in[0,1],\ z\in F,$$

hence $\operatorname{conv}(\{w\}\cup F)$ would be a convex subset of $\{x\in B: \|x\|=1\}$ and thus contained in a maximal face. This contradiction shows that there necessarily exists z in F with

$$\mu(\operatorname{supp}z\cap E_0) > 0.$$

By Lemma 1.4, we may assume that $\operatorname{supp}z\subseteq E_0$. Consequently, the function $z_1=\frac{1}{2}(z+x)$ is in F and

$$\operatorname{supp}z_1 = \operatorname{supp}z\cup\operatorname{supp}x.$$

It follows that

$$\mu(\operatorname{supp}z_1\cap E) = \mu(\operatorname{supp}z\cap E_0)+\mu(\operatorname{supp}x\cap E) > \alpha,$$

an impossibility which proves that $E_0=E\setminus\operatorname{supp}x$ has measure zero. Thus $E\subseteq\operatorname{supp}x$, and the proof for the case $\mu(E)<\infty$ is complete.

If $\mu(E)=\infty$, we let $E=\bigcup_{i=1}^{\infty}E_i$ where E_i are disjoint measurable sets of finite positive measure. For each E_i, there exists $y_i\in F$ such that $\operatorname{supp}y_i=E_i$ a.e. Let $y=\sum_{i=1}^{\infty}2^{-i}y_i$. By Proposition 1.2, $y\in F$ and

$$\operatorname{supp}y = \bigcup_{i=1}^{\infty}\operatorname{supp}y_i = \bigcup_{i=1}^{\infty}E_i = E \text{ a.e.}$$

From the above lemma, we see that for any σ-finite measurable set E, there exists y in F, $\operatorname{supp}y^+=E_1$ a.e., $\operatorname{supp}y^-=E_2$ a.e. where $E_1\cup E_2=E$, $E_1\cap E_2=\emptyset$ and by Lemma 1.3, this decomposition is unique within a set of measure zero.

THEOREM 1.6. *Any two maximal faces of the unit ball B are affinely isometric.*

PROOF. Let F be a maximal face and let

$$F_1 = \{x\in B:\ x\geqq 0,\ \|x\|=1\}.$$

We need only show that F and F_1 are affinely isometric.

Define $\varphi\colon F\to F_1$ by $\varphi(x)=|x|$. It is easily seen by Lemma 1.3 that φ is affine and isometric. To show that it is onto, let $x\in F_1$ and let $E=\operatorname{supp}x$. Then there exists a decomposition $E=E_1\cup E_2$ where

$$E_1 = \operatorname{supp}y^+, \quad E_2 = \operatorname{supp}y^-$$

for some y in F. Let $x' = x\cdot(\chi_{E_1} - \chi_{E_2})$. By Lemma 1.4, we have $x' \in F$ and $\varphi(x') = x$.

THEOREM 1.7. *Let F be a maximal face of B then*

$$\operatorname{Aff} F = \{\textstyle\sum_{i=1}^n \lambda_i x_i : \sum_{i=1}^n \lambda_1 = 1,\ x_i \in F,\ i = 1,\ldots,n,\ n \in \mathsf{N}\}$$

is a hyperplane in $L^1(X,\mu)$.

PROOF. We need only show that the linear subspace spanned by F is $L^1(X,\mu)$. Let $x \in L^1(X,\mu)$ and let

$$E_1 = \operatorname{supp} x^+, \quad E_2 = \operatorname{supp} x^- .$$

By Lemma 1.5, we can find measurable sets $\{E_{ij}\}_{i,j=1,2}$ such that

$$E_1 = E_{11} \cup E_{12}, \quad E_2 = E_{21} \cup E_{22}$$

and $y_1, y_2 \in F$ with

$$\operatorname{supp} y_1^+ = E_{11}, \quad \operatorname{supp} y_1^- = E_{12}, \quad \operatorname{supp} y_2^+ = E_{21}, \quad \operatorname{supp} y_2^- = E_{22} .$$

Let for $i,j = 1,2$

$$\begin{aligned} x_{ij} &= x\cdot\chi_{E_{ij}}/\|x\cdot\chi_{E_{ij}}\| \quad && \text{if } \|x\cdot\chi_{E_{ij}}\| \neq 0 , \\ &= 0 && \text{otherwise} . \end{aligned}$$

Then

$$x = \|x\cdot\chi_{E_{11}}\|\cdot x_{11} - \|x\cdot\chi_{E_{12}}\|\cdot(-x_{12}) + \|x\cdot\chi_{E_{21}}\|\cdot x_{12} - \|x\cdot\chi_{E_{22}}\|\cdot(-x_{22})$$

where x_{11}, $-x_{12}$, x_{21}, $-x_{22}$ are in $\operatorname{lin} F$.

REMARK. By the above theorem, the map φ in Theorem 1.6 can be extended to an isometry $\bar\varphi$ of $L^1(X,\mu)$ onto itself. If we let C_1, and C_2 be the cones generated by the maximal faces F and F_1 in Theorem 1.6, then $\bar\varphi$ is an order isomorphism with the orderings induced by C_1 and C_2.

THEOREM 1.8. *Let (X,μ) be a σ-finite measure space. Then every maximal face F of the unit ball B of $L^1(X,\mu)$ is of the form*

$$F_Y = \{x \in B : \operatorname{supp} x^+ \subseteq Y,\ \operatorname{supp} x^- \subseteq X \setminus Y \ \text{ and } \ \|x\| = 1\} ,$$

for some measurable set Y. Conversely, every set of the form F_Y is a maximal proper face of B.

PROOF. It is easy to check that F_Y is a maximal face. On the other hand, if F is a maximal face, by Lemma 1.5, there exists an x in F such that $\operatorname{supp} x = X$. Let $Y = \operatorname{supp} x^+$, we claim that $F_Y \supseteq F$. Indeed, if $y \in F$, then from Lemma 1.3, we see that $\operatorname{supp} y^+ \subseteq Y$ a.e. and $\operatorname{supp} y^- \cap Y$

has zero measure, hence $\operatorname{supp} y^- \subseteqq X \setminus Y$, so $y \in F_Y$. Since F is a maximal face, it follows that $F_Y = F$.

We conclude this section with some properties of the faces of the unit ball B. These properties will be used later on.

LEMMA 1.9. (Decomposition lemma.) *Suppose that V is a vector lattice. If $\{x_j : i \in I\}$ and $\{y_j : j \in J\}$ are finite sequence of nonnegative elements of V and if*

$$\sum_{i\in I} x_i = \sum_{j\in J} y_j ,$$

then there exist $z_{ij} \geqq 0$, $(i,j) \in I \times J$, such that

$$x_i = \sum_{j\in J} x_{ij} \ (i \in I) \quad \text{and} \quad y_j = \sum_{i\in I} z_{ij} \ (j \in J) .$$

PROOF. Cf. [9, p. 61].

PROPOSITION 1.10. *Let F, G be proper faces of the unit ball B such that $F \cap -G = \emptyset$. Then $\operatorname{conv}(F \cup G)$ is a proper face of B.*

PROOF. We claim that if $x \in \operatorname{conv}(F \cup G)$, then $\|x\| = 1$. We can write

$$x = \lambda x_1 + (1-\lambda)x'$$

where $x_1 \in F$, $x' \in G$ and $\lambda \in (0,1)$. By the remark following Theorem 1.7, we see that all the orderings generated by maximal faces are isomorphic, hence we may assume that F is contained in the maximal face

$$F_1 = \{x \in B : \ x \geqq 0, \|x\| = 1\} .$$

The cone generated by F_1 defines a lattice. Write

$$x' = \alpha x_2 - (1-\alpha)x_3, \quad x_2, x_3, \in F_1, \alpha \in (0,1) .$$

(If $\alpha = 1$, the claim follows trivially, for $\alpha = 0$, the proof is same as below.) Since G is a face of K, we have $x_2, -x_3 \in G$. Let $x = x^+ - x^-$. We then have

$$x^+ - x^- = \lambda x_1 + \alpha(1-\lambda)x_2 - (1-\alpha)(1-\lambda)x_3 ,$$

that is,

$$x^+ + (1-\alpha)(1-\lambda)x_3 = x^- + \lambda x_1 + \alpha(1-\lambda)x_2 .$$

By the decomposition lemma, there exist $\mu_{ij} \geqq 0$, $z_{ij} \in F_1$, $i = 1, 2$, $j = 1, 2, 3$, such that

$$x^+ = \sum_{j=1}^3 \mu_{1j} z_{1j} , \quad (1-\alpha)(1-\lambda)x_3 = \sum_{j=1}^3 \mu_{2j} z_{2j} ,$$

$$x^- = \sum_{i=1}^2 \mu_{i1} z_{i1} , \quad \lambda x_1 = \sum_{i=1}^2 \mu_{i2} z_{i2} ,$$

$$\alpha(1-\lambda)x_2 = \sum_{i=1}^2 \mu_{i3} z_{i3} .$$

Since $x_3 \in -G, x_1 \in F, x_2 \in G,$ by the above equations, we have z_{22}, $z_{23} \in -G$, $z_{22} \in F$ and $z_{23} \in G$. But since $F \cap -G$ and $G \cap -G$ are void sets, we have $\mu_{22}=0$, $\mu_{23}=0$. The second, forth and fifth of the above equations become

$$(1-\lambda)(1-\alpha)x_3 = \mu_{21}z_{21}, \quad \lambda x_1 = \mu_{12}z_{12}, \quad \alpha(1-\lambda)x_2 = \mu_{13}z_{13}.$$

Substituting into the first and third equation, we have

$$x^+ = \mu_{11}z_{11}+\lambda x_1+\alpha(1-\lambda)x_2,$$
$$x^- = \mu_{11}z_{11}+(1-\lambda)(1-\alpha)x_3,$$

so

$$1 \geqq \|x\| = \|x^+\|+\|x^-\| = 2\mu_{11}+1 \geqq 1$$

and hence $\|x\|=1$ as asserted. We may therefore assume that both F and G are contained in the maximal face F_1. To show that $\mathrm{conv}(F \cup G)$ is a face let

$$\lambda x_1+(1-\lambda)x_2 \in \mathrm{conv}(F \cup G),$$

where $x_1, x_2 \in B$, $0<\lambda<1$. Then

$$\lambda x_1+(1-\lambda)x_2 = \alpha y_1+(1-\alpha)y_2,$$

where $y_1 \in F$, $y_2 \in G$, and $0 \leqq \alpha \leqq 1$. If $\alpha=0$ (or $\alpha=1$) then x_1, x_2 are in G (or F) and hence in $\mathrm{conv}(F \cup G)$. If $0<\alpha<1$, by the decomposition lemma, there exist $\mu_{ij} \geqq 0$, $z_{ij} \in F_1$, $i,j=1,2$ such that

$$\lambda x_1 = \mu_{11}z_{11}+\mu_{12}z_{12}, \quad (1-\lambda)x_2 = \mu_{21}z_{21}+\mu_{22}z_{22},$$
$$\alpha y_1 = \mu_{11}z_{11}+\mu_{21}z_{21}, \quad (1-\alpha)y_2 = \mu_{12}z_{12}+\mu_{22}z_{22}.$$

The third and the forth equations imply $z_{11}, z_{21} \in F$ and $z_{12}, z_{22} \in G$. Hence by the first and second equations, we have $x_1, x_2 \in \mathrm{conv}(F \cup G)$.

PROPOSITION 1.11. *Suppose F is a finite dimensional face in a maximal face F_1 of B. Then there exists a face F' in F_1 such that $F \cap F' = \emptyset$ and* $\mathrm{conv}(F \cup F')=F_1$.

Moreover, if $x_1 \in F$ and $x_2 \in F'$, then $\|\alpha x_1+\beta x_2\|=|\alpha|+|\beta|$ for any real α, β.

PROOF. If $F_1=F$, then take F_1 to be the empty set. Hence, assuming that $F_1 \neq F$, we will first show that there exists a face G in F_1 disjoint from F. In fact, let $x \in F_1 \setminus F$ and let $K=F_1 \cap \mathrm{Aff}(F \cup \{x\})$. Then K is a finite dimensional compact convex subset of F_1. Since $K \neq F$, there exists an extreme point x_0 of K which is not in F. Consider

$$G = \{z : \lambda z+(1-\lambda)z' = x_0, \quad 0<\lambda<1, \quad z,z' \in F_1\}.$$

Then G is a face in F_1 disjoint from F. Let F' be the union of all faces in F_1 disjoint from F; it too is a face, in fact, let $x_1, x_2 \in F$, then there exists two faces G_1, G_2 in F_1 disjoint from F such that $x_1 \in G_1$, $x_2 \in G_2$. By Proposition 1.10, the set $\operatorname{conv}(G_1 \cup G_2)$ is a face contained in F_1 disjoint from F, thus

$$\lambda x_1 + (1-\lambda)x_2 \in F \quad \text{for some } 0<\lambda<1 .$$

To show that it is a face, let

$$\lambda x_1 + (1-\lambda)x_2 \in F', \quad 0<\lambda<1 .$$

Then $\lambda x_1 + (1-\lambda)x_2 \in H$ for some face H in F', hence $x_1, x_2 \in H \subseteq F'$. We claim that $\operatorname{conv}(F \cup F') = F_1$, for if this were not true, then there exists $x_1 \in F_1 \setminus \operatorname{conv}(F \cup F')$ and arguing as above, we can find a face containing x_1 disjoint from F and not contained in F'. This is a contradiction.

To show the last assertion, we see that if α and β have the same sign, then it is clear that equality holds since the norm is additive on the cone generated by each maximal face. If $\alpha>0, \beta<0$, say, let $F'' = \operatorname{conv}(F \cup -F')$, it suffices to show that this, too, is a maximal face. As F_1 is of codimension 1 in B, also F'' (which is a face by Proposition 1.10) has codimension 1 in B, hence it is maximal. By the remark following theorem 1.7, we have $B = \operatorname{conv}(F'' \cup -F'')$, and the norm is additive on the cone generated by F'', thus

$$\|\alpha x_1 + (-\beta)(-x_1)\| = |\alpha| + |\beta| .$$

2. Facial properties of L-polytopes.

Let K be a convex subset of a linear space and let H be a convex subset in K. We say that H is of *codimension n* in K if there exists an affinely independent set $\{x_1, \ldots, x_n\}$ in $K \setminus \operatorname{Aff} H$ such that

$$\operatorname{Aff}(H \cup \{x_1, \ldots, x_n\}) = \operatorname{Aff} K .$$

Suppose that $h_1, \ldots, h_n$ are affine functions on K and that

$$M_K = \{x \in K : h_i(x) = 0, \ i=1, \ldots, n\} .$$

Then M_K is called a *finite codimensional slice* of K. If K is a compact convex set and if M_K is closed in K, we call M_K a *closed finite codimensional slice* of K.

DEFINITION 2.1. A compact convex set H is called an *L-polytope* if H is affinely homeomorphic to some M_K where K is an L-ball. (The unit ball of the dual of a Lindenstrauss space with the w^*-topology cf. [7].)

Note that in the definition of L-polytopes, we do not assume M_K to be of finite codimension in K. We will show, however, that an L-polytope is affinely homeomorphic to some M_K such that M_K is of finite codimension in K. We first give two lemmas which will be useful in what follows.

LEMMA 2.2. *Let F be a convex subset of a linear space E, and let*

$$M = \{x \in F : h_i(x)=0,\ i=1,\ldots,n\},$$
$$M_j = \{x \in F : h_i(x)=0,\ i \neq j,\ i \in \{1,\ldots,n\}\}, \quad j=1,\ldots,n,$$

where h_i, $i=1,\ldots,n$, are affine functions on F. Suppose that for each j, there exist $x_j, y_j \in M_j$ such that $h_j(x_j)<0$, $h_j(y_j)>0$; then we have:

(i) *For each $z \in F$, there exists $\lambda>0$, $\lambda_i, \beta_i \geqq 0$, $i=1,\ldots,n$, such that*

$$\lambda+\textstyle\sum_{i=1}^n \lambda_i+\sum_{i=1}^n \beta_i = 1$$

and

$$\lambda z+\textstyle\sum_{i=1}^n \lambda_i x_i+\sum_{i=1}^n \beta_i y_i \in M,$$

(ii) *M is of finite codimension n in F.*

Furthermore, if F is a compact convex set and if h_i, $i=1,\ldots,n$ are continuous affine functions on F, then there exists $k \in \mathbb{R}^+$ such that the λ in (i) *may be chosen greater than k for all $z \in F$.*

PROOF. Without loss of generality, we assume that $0 \in M$. Define the map

$$T:\ \mathbb{R}^n \to \operatorname{lin} F/\operatorname{lin} M$$

such that $T(e_i)=\bar{x}_i$, $i=1,\ldots,n$, where $\bar{x}_i$ is the equivalence class of x_i. For any $z \in F$, we have

$$z-\textstyle\sum_{i=1}^n h_i(z)x_i \in \operatorname{lin} M,$$

thus $\bar{z}=\sum_{i=1}^n h_i(z)x_i$. If we let $\alpha=(\alpha_1,\ldots,\alpha_n)=(h_1(z),\ldots,h_n(z))$, then $T(\alpha)=\bar{z}$. Let I be the subset of $\{1,\ldots,n\}$ such that $\alpha_i<0$ and let $J=\{1,\ldots,n\}\setminus I$. Then

$$\begin{aligned} 0 &= \bar{z}+\textstyle\sum_I (-\alpha_i)T(e_i)-\sum_J (-\alpha_i)T(e_i) \\ &= \bar{z}+\textstyle\sum_I (-\alpha_i)\bar{x}_i+\sum_J \alpha_i(-h_i(x_i)/h_i(y_i))\bar{y}_i . \end{aligned}$$

(Here we use the fact that $\bar{x}_i=(h_i(x_i)/h_i(y_i))\bar{y}_i$.) We let

$$(*) \qquad \lambda = \left(1+\textstyle\sum_I (-\alpha_i)+\sum_J \alpha_i(-h_i(x_i)/h_i(y_i))\right)^{-1}$$

Further for $i=1,\ldots,n$ we let

$$\begin{aligned} \lambda_i &= 0 && \text{if } \alpha \notin I, \\ &= \alpha_i\lambda && \text{if } \alpha \in I, \end{aligned}$$

$$\begin{aligned} \beta_i &= 0 && \text{if} \quad \alpha \notin J\,, \\ &= \alpha_i(-h_i(x_i)/h_i(y_i))\lambda && \text{if} \quad \alpha \in J\,. \end{aligned}$$

Then we have $\lambda > 0$, λ_i, $\beta_i \geqq 0$ for $i = 1, \ldots, n$,

$$\lambda + \sum_{i=1}^n \lambda_i + \sum_{i=1}^n \beta_i = 1$$

and

$$\lambda \bar{z} + \sum_{i=1}^n \lambda_i \bar{x}_i + \sum_{i=1}^n \beta_i \bar{y}_i = 0\,,$$

so

$$\lambda z + \sum_{i=1}^n \lambda_i x_i + \sum_{i=1}^n \beta_i y_i \in F \cap \operatorname{lin} M = M\,.$$

Hence (i) is proved and (ii) follows from this directly. To verify the last assertion, we notice that when F is compact and each h_i, $i = 1, \ldots, n$ is continuous, the set

$$\{h_i(z) : i = 1, \ldots, n,\, z \in F\}$$

is a bounded set in R, hence the equation $(*)$ is uniformly bounded away from 0. That is there exists $k > 0$ such that $\lambda > k > 0$ for all z in F.

LEMMA 2.3. *Let K be a convex set and let M be a finite codimensional slice of K. Suppose that M_0 is a face of M and that F is the smallest face of K containing M_0; then M_0 is of finite codimension in F.*

Suppose K is compact. If M is a closed finite codimensional slice of K and M_0 is closed, then F is compact.

PROOF. Let

$$M = \{x \in K : h_i(x) = 0, \quad i = 1, \ldots, n\}\,,$$

where $h_i, i = 1, \ldots, n$ are affine functions on K. Since F contains M_0, we have

$$M_0 = \{x \in F : h_i(x) = 0, \quad i = 1, \ldots, n\}\,.$$

We may assume that n is the smallest integer such that the above equality holds. Let

$$M_j = \{x \in F : h_i(x) = 0, \quad i \neq j,\, i = 1, \ldots, n\}\,.$$

Then $M_0 \subsetneqq M_j \subsetneqq F$. We claim that for each $j \in \{1, \ldots, n\}$, there exist x_j, y_j such that $h_j(x_j) > 0$, $h_j(y_j) < 0$. Indeed, let

$$F' = \{x \in F : \lambda x + (1 - \lambda) y \in M \text{ where } y \in F \text{ and } 0 < \lambda < 1\}\,.$$

Then F' is a face of F containing M_0 and thus $F' = F$. Since $M_j \neq F$, there exists $x_j \in F \setminus M_j$ and we have $h_j(x_j) > 0$ (or < 0). There also exist $y_j \in F$ and $0 < \lambda < 1$ such that

$$\lambda x_j + (1-\lambda) y_j \in M_0 .$$

It follows that $h(y_j) < 0$. Hence we have found $\{x_i\}_{i=1}^n$, $\{y_i\}_{i=1}^n$ which satisfy the conditions of Lemma 2.2, and therefore M_0 is of finite codimension in F.

To show the last part, let $\{z_\alpha\}_{\alpha \in I}$ be a net in F. By the above lemma, we have $k > 0$, $\lambda_\alpha > k > 0$, $\lambda_{i\alpha}$, $\beta_{i\alpha} \geqq 0$, $\alpha \in I$, $i = 1, \ldots, n$, such that

$$\lambda_\alpha + \sum_{i=1}^n \lambda_{i\alpha} + \sum_{i=1}^n \beta_{i\alpha} = 1$$

and

$$\lambda_\alpha z_\alpha + \sum_{i=1}^n \lambda_{i\alpha} x_i + \sum_{i=1}^n \beta_{i\alpha} y_i \in M_0 .$$

By compactness, we may assume that $\{\lambda_\alpha\}$ converges to $\lambda > 0$, $\{\lambda_{i\alpha}\}$ converges to λ_i, $\{\beta_{i\alpha}\}$ converges to β_i, $i = 1, \ldots, n$ and $\{z_\alpha\}$ converges to z. Hence

$$\lambda z + \sum_{i=1}^n \lambda_i x_i + \sum_{i=1}^n \beta_i y_i \in M_0 \subseteqq F$$

and

$$\lambda + \sum_{i=1}^n \lambda_i + \sum_{i=1}^n \beta_i = 1 .$$

Since F is a face and $\lambda > 0$, we have $z \in F$ which shows that F is compact.

Corollary 2.4. (*Lazar*). *Suppose K is a compact convex set and that M is a closed finite codimensional slice in K. If F is a closed face of M, then there exists a closed face F_1 of K such that $F = M \cap F_1$. If F is a G_δ set in $H_1 \cap K$, then F is a G_δ set in K.*

Proof. The first part follows directly from Lemma 2.3. The second part follows from the last part of [10, Lemma 3.4].

Proposition 2.5. *Let H be an L-polytope. Then there exists a closed finite codimensional slice M_K of an L-ball K such that H is affinely homeomorphic to M_K and M_K is of finite codimension in K.*

Proof. Without loss of generality, we assume that

$$H = \{x \in K' : k_i(x) = 0, i = 1, \ldots, n\}$$

where $k_1, \ldots, k_n \in A(K')$ and K' is an L-ball. Let F be the smallest face of K' containing H. If $F = K'$, then by Lemma 2.3, the proof is complete. If $F \neq K'$, then F is a proper face of K'. Again, by the same lemma, it is compact, thus it is a Choquet simplex. Let $K = \operatorname{conv} F \cup -F)$. Then K is an L-ball and

$$H = \{x \in K : h_i(x) = 0, \ i = 1, \ldots, n\},$$

where h_i is the restriction of k_i to K, $i = 1, \ldots, n$. Lemma 2.3 shows that H is of finite codimension in F, hence H is of finite codimension in K.

Let H be an L-polytope. We call an L-ball K with the property of Proposition 2.5 to be an *envelope* of H. Our next three propositions are concerned with maximal faces of L-polytopes. We make the convention that *maximal face* shall mean maximal proper face.

LEMMA 2.6. *Let X be a Banach space isometric to an $L^1(\mu)$ space. Suppose that F is a maximal face of the unit ball $B(X)$. Then every maximal face of F is of codimension* 1 *in F.*

PROOF. Let F_1 be a maximal face of F. If F_1 is not of codimension 1 in F, there exist x, y in F such that F_1, x and y are affinely independent. Since F is a linearly closed and linearly bounded, we may assume that x, y are such that $\operatorname{Aff}\{x, y\} \cap F = [x, y]$. Let

$$F_2 = \{z \in F : \lambda z + (1-\lambda)z' = x \text{ for some } z' \in F \text{ and } 0 < \lambda < 1\}.$$

Then F_2 is a face which is not equal to F_1 and does not contain the point y. By Proposition 1.10, $\operatorname{conv}(F_1 \cup F_2)$ is a proper face of F. This contradicts the fact that F_1 is a maximal face of F.

PROPOSITION 2.7. *Suppose that H_1 is a face of an L-polytope H. Then H_1 is a maximal face of H if and only if H_1 is of codimension* 1 *in H. (We assume that H is not a single point.)*

PROOF. We need only prove the necessity. Let K be an envelope of H. Let K_0 be the smallest face of K containing H. Then K_0 is either an L-ball or a Choquet simplex (Lemma 2.3). In the latter case it is a maximal face of an L-ball. Suppose now that H is of codimension n in K_0. Let K_1 be a maximal proper face of K_0 containing H_1. By Lemma 2.6, it is of codimension 1 in K_0. We can find $y_1, \ldots, y_n \in K_1$ such that $H, y_1, \ldots, y_n$ are affinely independent and

$$\operatorname{Aff}(H \cup \{y_1, \ldots, y_n\}) = \operatorname{Aff} K_0 .$$

If we can show that

$$\operatorname{Aff}(H_1 \cup \{y_1, \ldots, y_n\}) = \operatorname{Aff} K_1,$$

then H_1 is of codimension $(n+1)$ in K_0 hence of codimension 1 in H. In fact, for $x \in K_1$, we have

$$x = \lambda y + \sum_{i=1}^{n} \lambda_i y_i, \quad \text{where } \lambda + \sum_{i=1}^{n} \lambda_i = 1 \text{ and } y \in H .$$

If $\lambda = 0$, then x is in $\mathrm{Aff}(H_1 \cup \{y_1, \ldots, y_n\})$. If $\lambda \neq 0$, then

$$\lambda^{-1} - \lambda^{-1} \sum_{i=1}^{n} \lambda_i = 1$$

and

$$\lambda^{-1} - \lambda^{-1} \sum_{i=1}^{n} \lambda_i y_i \in \mathrm{Aff}\, K_1 ,$$

which implies that y is in $H \cap \mathrm{Aff}\, K_1$. But $H \cap \mathrm{Aff}\, K_1$ is a proper face of H containing H_1, hence $H \cap \mathrm{Aff}\, K_1 = H_1$. This shows that

$$y \in H_1 \quad \text{and} \quad x \in \mathrm{Aff}(H \cup \{y_1, \ldots, y_n\}) .$$

The reverse inclusion is obvious, so we have

$$\mathrm{Aff}(\mathrm{H}_1 \cup \{y_1, \ldots, y_n\}) = \mathrm{Aff}\, K_1$$

and the proof is complete.

LEMMA 2.8. (*Dubins* [3]) *Let K be a linearly closed, linearly bounded convex set and let M be a finite codimensional slice in K. Let x be an extreme point in M, then x is a finite convex combination of extreme points of K.*

PROOF. (This proof differs from that of Dubins.) Let F be the smallest face of K containing M. Then M is of finite codimension in F. Let

$$F' = \{y \in F : \lambda y + (1-\lambda)z = x \text{ where } z \in F \text{ and } 0 < \lambda < 1\} ,$$

Then F' is a face of F. Since x is an extreme point of M, we have

$$\mathrm{Aff}\, F' \cap \mathrm{Aff}\, M = \{x\}$$

and since M is of finite codimension in F, this implies that $\mathrm{Aff}\, F'$ is finite dimensional. Since F' is linearly closed and linearly bounded, it is compact. Hence x is a convex combination of finitely many extreme points of F' and these are also extreme points of K.

PROPOSITION 2.9. *If H is an infinite dimensional L-polytope, then every maximal face H_1 of H contains infinitely many extreme points.*

PROOF. Let K_0, K_1 be the faces containing H, H_1 respectively defined as in Proposition 2.7. We see that K_0 is an infinite dimensional L-ball or a Choquet simplex and K_1 is a maximal face of K_0. Hence it contains infinitely many extreme points. We will let $\partial_e C$ denote the set of extreme points on a convex set C. Suppose $\partial_e H_1$ were finite. By Dubins' lemma, there exists a finite set A contained in $\partial_e K_1$ such that each point of $\partial_e H_1$ is a convex combination of points in A. Let $B \subseteqq \partial_e K_1 \setminus A$ such that B is a finite set and $\operatorname{conv} B$ has dimension greater than n where n is the codimension of H in K. It is obvious that $\operatorname{conv} B \cap H$ is nonempty, compact and is a face of H hence contains an extreme point of H. Furthermore,

$$\operatorname{conv} B \cap H \subseteqq K_1 \cap H = H_1,$$

so we can find an extreme point in H_1 which is not in $\partial_e H_1$ which is a contradiction.

PROPOSITION 2.10. *A maximal face of an infinite dimensional L-polytope cannot be centrally symmetric.*

PROOF. Let H, H_1, K_0, K_1 be defined as in Proposition 2.7. If $\bar{H}_1 \subseteqq K_1$ then by a proof similar to Lemma 2.3, the face K_1 is compact and hence it is a Choquet simplex. Suppose H_1 is symmetric about a. $\bar{H}_1$ is also symmetric about a. Let $x+a$ be an extreme point of $\bar{H}_1$, then $-x+a$ is also an extreme point and both of them are finite convex combinations of extreme points of K_1, hence the same is true for

$$a = \tfrac{1}{2}(x+a) + \tfrac{1}{2}(-x+a) .$$

Let μ be a probability measure representing a and supported by extreme points $x_1, \ldots, x_n$, $y_1, \ldots, y_m$ of K_1 such that $x+a$ and $-x+a$ are convex combinations of $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ respectively. Since $\bar{H}_1$ is infinite dimensional, we can find another extreme point $y \in \bar{H}_1$ which is not in the affine variety generated by the above extreme points. Hence we can find another boundary probability representation for a. This contradicts the existence of a unique boundary probability measure representing each point of a Choquet simplex.

Next, consider the case where $\bar{H}_1 \nsubseteq K_1$. Then there exists $x \in \bar{H}_1 \setminus K_1 \subseteqq K \setminus K_1$. If H_1 has a symmetric center a, than $\bar{H}_1$ also has a as a center of symmetry. Hence x, $-x+2a$ are symmetric with respect to a and

$$\tfrac{1}{2}x + \tfrac{1}{2}(-x+2a) = a \in H_1 \subseteqq K_1 .$$

If $-x+2a \in K$, then since K_1 is a face, x will be in K_1 which is impossible. Hence $-x+2a \notin K$, which also contradicts the fact that $\bar{H}_1 \subseteqq K$. We conclude that $\bar{K}_1$ cannot be centrally symmetric.

In [10], Phelps showed that the β-polytopes have certain extension properties and he also characterized the G_δ face of such polytopes. These results can be generalized to the class of L-polytopes.

LEMMA 2.11. (*Phelps* [10].) *Suppose that K is a compact convex set and M is a closed finite codimensional slice of K. If M is contained in no proper face of K, then any continuous affine function on M can be extended to a continuous affine function on K.*

THEOREM 2.12. *Suppose that H is an L-polytope and that F is a closed face of H. If g is a continuous affine function on F, then g admits an extension to a continuous affine functional f on H.*

Furthermore, there is a uniform bound on the norm of the extension.

PROOF. Let $H = M_K$, where K is an envelope of H and M is a closed finite codimensional slice of K. By Corollary 2.4, there exists a closed face F of K such that $F_1 \cap H = F$ and by Lemma 2.11, we can extend g to g' on F_1. By [7, Proposition 2.5], we can extend g' to g'' on K. Let f be the restriction of g'' to H; f has the required property.

The last assertion follows from Alfsen [2, p. 114].

THEOREM 2.13. *If H is an L-polytope and if F is a closed face of H which is G_δ in H, then there exists a continuous affine function $f \geqq 0$ on H such that*

$$F = \{x \in H : f(x) = 0\} .$$

PROOF. Let $H = M_K$, where K is an L-ball and M is a closed finite codimensional slice of K. By Corollary 2.4, we can find a closed G_δ face F_1 in H such that $F = H \cap F_1$ and by [7, Proposition 2.7] there exists a continuous affine function $g \geqq 0$ on K such that

$$F_1 = \{x \in K : g(x) = 0\} .$$

Let the restriction of g to H be denoted by f; then f is the required function.

3. Polyhedral vertices of L-polytopes.

DEFINITION 3.1. Let K be a compact convex subset of a locally convex space and define

$$\operatorname{cone}(x, K) = x + \bigcup_{\lambda \geqq 0} \lambda(K - x) .$$

A point x in K is called a *polyhedral vertex* of K if $\operatorname{cone}(x, K)$ is closed and proper.

The definition was introduced by Alfsen and Nordseth [1], who proved that every extreme point of a Choquet simplex is a polyhedral vertex.

Hall-Pedersen [6] proved that this is also true for an α-polytope. In what follows, we show that it is the case for an L-polytope.

LEMMA 3.2. *Every extreme point of an L-ball is a polyhedral vertex.*

PROOF. Let K be the L-ball embedded into $A_0(K)^*$. Suppose a is an extreme point of K, and suppose that $C = \text{cone}(0, K-a)$. We want to show that C is w^*-closed in $A_0(K)^*$. By [4, Theorem 3.2, Theorem 4.1], we see that

$$K = (a-C) \cap (C-a)$$

for each extreme point a of K. Hence

$$\begin{aligned} C \cap K &= C \cap (a-C) \cap (C-a) \\ &= C \cap (a-C) \cap (a-C) \cap (C-a) \\ &= [(\tfrac{1}{2}C - \tfrac{1}{2}a) + \tfrac{1}{2}a] \cap [(\tfrac{1}{2}a - \tfrac{1}{2}C) + \tfrac{1}{2}a] \cap K \\ &= \tfrac{1}{2}[(C-a) \cap (a-C) + \mathrm{a}] \cap K \\ &= \tfrac{1}{2}(K+a) \cap K \end{aligned}$$

which is w^*-compact. By the Krein-Smulyan theorem, the set C is w^*-closed. That the cone is proper follows from the fact that x is an extreme point of K. Thus, it is a polyhedral vertex of K.

LEMMA 3.3. *Let C be a closed cone in a locally convex space and let F be a finite dimensional subspace. Then $F+C$ is a closed cone.*

PROOF. Cf. [5, Proposition 7.5].

PROPOSITION 3.4. *If H is an L-polytope, then every extreme point of H is a polyhedral vertex.*

PROOF. Let K be an envelope of H, so that

$$H = K \cap \{x \in A_0(K)^* : h_i(x) = 0, \ i = 1, \ldots, n\},$$

$h_1, \ldots, h_n \in A(K)$. Let a be an extreme point of H. First we claim that

$$\text{cone}(a, H) = M \cap \text{cone}(a, K)$$

where M is the affine variety generated by H. Indeed, let

$$x \in M \cap \text{cone}(a, K)$$

and write $x = a + \lambda(y-a)$, where $\lambda \geqq 0$ and $y \in K$. If $\lambda = 0$, then

$$x = a \in \text{cone}(a, H).$$

If $\lambda \neq 0$, then since $y \in M \cap K$, we have $y \in H$ and $x \in \text{cone}(a, H)$.

If a is an extreme point of H, then by Dubins' lemma, it is a convex combination of extreme points $\{x_1,\ldots,x_n\}$ of K. Let F be the affine variety generated by $\{x_1,\ldots,x_n\}$; then

$$\operatorname{cone}(a,K) = \operatorname{cone}(x_1, K)+F .$$

Indeed, by translation, we may let $a=0$, so that F is a linear subspace. For $\lambda k \in \operatorname{cone}(0,K)$, $\lambda \geqq 0$ and $k \in K$, we have

$$\lambda k = x_1+\lambda(k-x_1)-(1-\lambda)x_1 \in \operatorname{cone}(x_1, K)+F .$$

Conversely, suppose $z=x_1+\lambda(k-x_1)+y \in \operatorname{cone}(x_1,k)+F$ where $\lambda \geqq 0$, $k \in K$ and $y \in F$. Since (x_1+y) and $\lambda(k-x_1)$ are in $\operatorname{cone}(0,K)$, $\lambda \geqq 0$ and $k \in K$, we have

$$\operatorname{cone}(a, K) = \operatorname{cone}(x_1, K)+F .$$

By Lemma 3.1 and Lemma 3.3, the set $\operatorname{cone}(a,K)$ is closed and by the first part of the proof, we see that $\operatorname{cone}(a,H)$ is closed. It is also a proper cone since a is an extreme point of H. Thus a is a polyhedral vertex of H.

The following result was proved for Choquet simplexes by Alfsen and Nordseth [1]. We use a similar technique.

Proposition 3.5. *If H is an L-polytope such that the set* $\operatorname{cone}(x,H)$ *is closed for each $x \in H$, then H is finite dimensional.*

Proof. Let $H=M_K$, where K is an envelope of H and M is a closed finite codimensional slice of K. We will prove the proposition by the following steps:

(i) Let

$$F_x = \{y \in H :\ \lambda y+(1-\lambda)z = x \text{ for some } z \in H \text{ and } 0<\lambda<1\} .$$

We claim that

$$F_x = H \cap (2x-\operatorname{cone}(x, H))$$

which will show that every F_x is closed. Now, $y \in F_x$ if and only if there exist $z \in H$ and $0<\lambda<1$ such that

$$\lambda y+(1-\lambda)z = x .$$

Equivalently,

$$y = \lambda^{-1}x-(\lambda^{-1}-1)z$$

where $z \in H$ and $0<\lambda<1$, that is,

$$y = 2x-((\lambda^{-1}-1)(z-x)+x), \quad z \in H, 0<\lambda<1 ,$$

which means $y \in H \cap (2x - \text{cone}(x,H))$.

(ii) If H_1 is a maximal face of H which contains infinitely many extreme points, then there exists a point x in H_1 such that $F_x \cap \partial_e H_1$ is a countable set. Indeed, let $\{x_i\}_{i=1}^{\infty}$ be a countable set of extreme point in $\partial_e H_1$. Let

$$x = \sum_{i=1}^{\infty} \lambda_i x_i, \quad \text{where } \lambda_i>0 \text{ and } \sum_{i=1}^{\infty} \lambda_i = 1 .$$

Let B be the set of extreme points in the envelope K such that each member of B is a convex component of the x_i. By Dubins' lemma, the set B is countable. Let G_x be the smallest face on K containing F_x. By Lemma 2.3 and by (i), we see that G_x is compact and hence it is a Choquet simplex. We claim that

$$G_x \cap \partial_e K = B .$$

It is clear that $G_x \cap \partial_e K \supset B$. On the other hand, if

$$y \in (G_x \cap \partial_e K) \setminus B ,$$

then $x=\lambda y+(1-\lambda)z$ for some $z \in K$ and $0<\lambda<1$. Let μ_z be a boundary probability measure representing z. Then $\lambda\varepsilon_y+(1-\lambda)\mu_z$ is a boundary probability measure representing x. We thus have two boundary probability measures representing x and supported by B. This contradicts the fact that G_x is a Choquet simplex. Hence $G_x \cap \partial_e H = B$. It follows from Dubins' lemma and $G_x \cap H = F_x$ that F_x contains only countably many extreme points.

(iii) Give $\partial_e H_1$ a topology generated by the family of subbasic closed sets of I consisting of $\partial_e H_1$ and sets of the form $F \cap \partial_e H_1$ where F is closed face of H_1. We claim that $\partial_e H_1$ is compact under this topology. Indeed, let $\{A_\alpha\}$ be a family of subbasic closed sets in I with the finite intersection property, then there exists a family of closed faces $\{G_\alpha\}$ of H_1 such that $G_\alpha \cap \partial_e H_1 = A_\alpha$ for each α. We known that $\bigcap G_\alpha$ is a nonvoid compact face in H_1, so it contains extreme points of $\partial_e H_1$. This shows that $\bigcap A_\alpha$ is nonempty and thus $\partial_e H_1$ is compact.

(iv) The set $\partial_e H_1$ is a finite set. Indeed, suppose it is an infinite set, by (ii) we can find a closed face $F_0 \subsetneqq H_1$ with countably many extreme points, say, $\partial_e F_0 = \{x_i\}_{i=1}^{\infty}$. Let F_n be the face generated by $\{x_i\}_{i=n}^{\infty}$ as in (ii) and let A_n denote the set of extreme points of F_n. We see that $\{A_n\}_{n=1}^{\infty}$ is a family of compact sets in $\partial_e H$ and has the finite intersection property, so $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$ which contradicts the fact that the family $\{A_n\}_{n=1}^{\infty}$ has void intersection.

(v) By (iv) and Proposition 1.9, we conclude that H is a finite dimensional polytope.

4. Some examples.

In this section, we are going to give some examples which show that some properties of a finite dimensional polytope do not hold in the class of L-polytopes.

EXAMPLE 4.1. The product of two L-polytopes need not be an L-polytope.

Let F be an infinite dimensional Choquet simplex, considered as a subset of $A(F)^*$. Let

$$K = \operatorname{conv}(F \cup -F),$$

that is K = unit ball of $A(F)^*$. Then K is an L-polytope. We show that $K \times K$ is not an L-polytope. If $K \times K$ were an L-polytope, we would have $K \times K = M_{K'}$, where K' is an envelope of $K \times K$ and $M_{K'}$ is a closed finite codimensional slice of K_1. The set $K \times F$ is a maximal face of $K \times K$ and is compact. Let F_1 be a maximal face of K' containing $K \times F$. Then since $K \times F$ is finite codimensional in F_1, we see that F_1 is compact and is a Choquet simplex.

Let x_1, x_2 be two extreme points of K such that $x_2 \in F$, so (x_1, x_2), $(-x_1, x_2)$ are extreme points of $K \times F$. We can write

$$(x_1, x_2) = \textstyle\sum_{i=1}^{n} \lambda_i y_i, \quad (-x_1, x_2) = \sum_{j=1}^{m} \beta_j z_j,$$

where $\lambda_i, \beta_j \geqq 0$, $\sum_{i=1}^{n} \lambda_i = 1$, $\sum_{j=1}^{m} \beta_j = 1$ and y_i, z_j are extreme points of $F_1, i = 1, \ldots, n, j = 1, \ldots, m$, and hence

$$(0, x_2) = \tfrac{1}{2}\left(\textstyle\sum_{i=1}^{n} \lambda_i y_i + \sum_{j=1}^{m} \beta_j z_j\right).$$

Since K is infinite dimensional, there exists an extreme point x_3 of K such that (x_3, x_2) is not in the linear subspace generated by y_i, z_j, $i = 1, \ldots, n$, $j = 1, \ldots, m$. Similarly, we have

$$(x_3, x_2) = \textstyle\sum_{i=1}^{h} \lambda_i' y_i', \quad (-x_3, x_2) = \sum_{j=1}^{k} \beta_i' z_j'$$

and

$$(0, x_2) = \tfrac{1}{2}\left(\textstyle\sum_{i=1}^{h} \lambda_i' y_i' + \sum_{j=1}^{m} \beta_j' z_j'\right),$$

where $\sum_{i=1}^{h} \lambda_i' = 1$, $\sum_{j=1}^{k} \beta_i' = 1$, $\lambda_i', \beta_j' \geqq 0$ and y_i', z_j' are extreme points of F_1, thus we can find two boundary measures representing the point

$(0,x_2)$ of the Choquet simplex F_1. This contradication shows that $K\times K$ is not an L-polytope.

EXAMPLE 4.2. The intersection of two L-polytopes is not necessarily an L-polytope.

Let c be the set of real sequences of the form $y=(y_n)_{n=1}^{\infty}$ such that

$$y_1 = \lim_{n\to\infty} y_n ,$$

with supremum norm. Then l_1 is the dual of c and we let B denote the unit ball of l_1. If

$$S = \{x : x_n \geqq 0 \text{ for each } n, \textstyle\sum_{n=1}^{\infty} x_n = 1\} ,$$

then S is w^*-closed and is a Choquet simplex. Take $x\in S$ to be the sequence

$$x = (\tfrac{1}{2}+2^{-2}, 2^{-3}, 2^{-4}, \ldots) .$$

Consider the set $x-B$. Each $y\in x-B$ can be written uniquely as $y=z+\alpha x$ where $z\in \operatorname{lin}(x-S)$. We define

$$B_1 = \{z-\alpha x : z+\alpha x\in x-B, \ z\in \operatorname{lin}(x-S)\}$$

and let $B_2=B-x$. Both B_1 and B_2 are L-polytopes, but we claim that $B_1\cap B_2$ is not an L-polytope. Since $S-x$ and $x-S$ are maximal faces of B_1,B_2 respectively, $(S-x)\cap(x-S)$ is a face of $B_1\cap B_2$. It is a proper face because

$$-x\in (B_1\cap B_2)\setminus((S-x)\cap(x-S)) .$$

To show that it is maximal, we need only show that it is of codimension 1. We let $\delta_n=(x_i)_{i=1}^{\infty}$ be the points in S such that $x_i=0$ for $i\neq n$ and $x_n=1$. Note that

$$2^{-(n-1)}(\delta_n-\delta_1)\in (S-x)\cap(x-S)$$

for $n>1$ and $\operatorname{lin}(S-x)$ is the w^*-closed subspace generated by $\{\delta_n-x\}_{n=1}^{\infty}$. Furthermore, $\operatorname{lin}((S-x)\cap(x-S))$ is the w^*-closed subspace generated by $\{\delta_n-\delta_1\}_{n=2}^{\infty}$. Now

$$\begin{aligned}\delta_1-x &= (2^{-2}, -2^{-3}, -2^{-4}, \ldots)\\ &= \textstyle\sum_{n=2}^{\infty} 2^{-(n+1)}(\delta_1-\delta_n) .\end{aligned}$$

This implies that (δ_1-x) is in $\operatorname{lin}((S-x)\cap(x-S))$ and

$$\delta_n-x = (\delta_n-\delta_1)+(\delta_1-x)\in \operatorname{lin}((S-x)\cap(x-S)) .$$

Hence

$$\operatorname{lin}((S-x)\cap(x-S)) = \operatorname{lin}(S-x) .$$

Thus, we conclude that $(S-x)\cap(x-S)$ is of codimension 1 in $B_1\cap B_2$ and hence it is a maximal face of $B_1\cap B_2$. Notice that $(S-x)\cap(x-S)$ is a symmetric set with center of symmetry 0. By Proposition 2.10, the set $B_1\cap B_2$ cannot be an L-polytope.

EXAMPLE 4.3. There exists a compact convex set K such that each extreme point is a polyhedral vertex, but K is not an L-polytope.

We first observe that if K_1, K_2 are two compact convex sets such that $x_1 \in K_1$, $x_2 \in K_2$ are polyhedral vertexes, then $(x_1, x_2) \in K_1 \times K_2$ is a polyhedral vertex of $K_1 \times K_2$. Now, we can choose K_1, K_2 to be two L-polytopes such that $K_1 \times K_2$ is not an L-polytope (Example 4.1) but every extreme point of $K_1 \times K_2$ is a polyhedral vertex.

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