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ON SOME EXTREMAL PROBLEMS IN BANACH SPACES

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1. Introduction.

In studying various extremal problems in Banach spaces it is often good to know if some functions attain their maximum over certain sets. For the case I of linear functionals, there are two classical results. The first one, due to R. C. James says that a weakly closed subset S of a Banach space X is weakly compact iff each continuous linear functional on X attains its supremum on S ([10, p. 139]). The second one, due to E. Bishop and R. R. Phelps states that every Banach space is subreflexive, i.e. for any Banach space X, those elements of X^* which attain their norm on the unit ball of X, are norm dense in X^* (see [4, p. 31]).

In the case II of the norm of linear operators, J. Lindenstrauss proved that if X, Y are Banach spaces, X reflexive, then the set of all bounded linear operators of X into Y which attain their norm on the unit ball of X (i.e. there is an $x \in X$, $\|x\| = 1$, $\|Tx\| = \|T\|$) is norm dense in the Banach space $B(X, Y)$ of all bounded linear operators of X into Y with the usual operator norm.

In the case III of the distance function from a given point, we have the notion of farthest point in a set and the results of E. Asplund and M. Edelstein ([2], [9]) from which we recall here the one of Asplund: If S is a bounded norm closed subset of a reflexive and locally uniformly rotund Banach space X (for definition see the section 2), then except on a set of first Baire catagory, the points in X have farthest points in S.

In the present note, we give some contributions to the second and third case. The author thanks the referee for some suggestions that improved the structure of the paper.

2. Notations and definitions.

We will work in real Banach spaces. If f is a proper convex function on $X(X^*)$, then f^* is the function on $X^*(X)$, conjugate, or dual to f in the Fenchel sense, that is

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$$f^*(y) = \sup_{x \in X}(\langle y,x\rangle - f(x))$$

(cf. [1]). For $a \in X$, $r > 0$,

$$X \supset K_r(a) = \{x \in X; \|x-a\| \leqq r\}.$$

If $\|\cdot\|$ is an equivalent norm on a given Banach space X, then by the dual norm of the norm $\|\cdot\|$ in X^* we mean the usual dual supremum norm on X^* with respect to $\|\cdot\|$ on X. The set of all positive integers is denoted by N. $w(X^*,X)$ respectively $w(X,X^*)$ denotes the weak topology on X^* respectively on X given by all elements of X respectivelyX^*. The first one is sometimes called the w^* topology of X^*. A point x of a convex set C in a Banach space X is called an exposed point of C ([16, p. 140]), if these is an $f \in X^*$ such that $f(y) < f(x)$ for every $y \in C$, $y \neq x$. A point x of a convex set C in a Banach space X is called a strongly exposed point of C if there is an $f \in X^*$ such that $f(y) < f(x)$ for any $y \in C$, $y \neq x$ and moreover, if

$$f(y_n) \to f(x),\ y_n \in C,\ n = 1,2,\ldots, \quad \text{imply} \quad \|y_n - x\| \to 0.$$

A Banach space X is said to be rotund if all points of the norm boundary of its $K_1(0)$ are extreme points of $K_1(0)$. Furthermore, a Banach space X with the norm $\|\cdot\|$ is locally uniformly rotund (LUR) if for $x_n, x \in X$ satisfying $\|x_n\| = \|x\| = 1$, $n = 1,2,\ldots$,

$$\lim_{n\to\infty}\|x_n + x\| = 2 \quad \text{imply} \quad \lim_{n\to\infty}\|x_n - x\| = 0.$$

A Banach space X is WUR if for $x_n, y_n \in X$ satisfying $\|x_n\| = \|y_n\| = 1$,

$$\|x_n + y_n\| \to 2 \quad \text{imply} \quad x_n - y_n \to 0$$

in the $w(X,X^*)$ topology of X.

A Banach space X is an SDS space (cf. [1, p. 31]) if every continuous convex function is Fréchet differentiable on a dense G_δ subset of its domain of continuity, where by a continuous convex function on a Banach space we mean a function which is defined and convex on all of X, with values in $(-\infty, +\infty\rangle$, and finite valued and continuous at least at some point of X. E. Asplund ([1, p. 32]) proved, that if X is a Banach space which can be given an equivalent norm, such that the corresponding dual norm in X^* is LUR, then X is an SDS space. Thus any Banach space X with X^* separable is an SDS space (see [7], [1, p. 41]). The same is true for any reflexive Banach space (cf. [22]) and any Banach space X, if X and X^* are both weakly compactly generated (see [12]). Furthermore, R. C. James proved in [11, p. 571] that for any $n \in$ N, there is a Banach space B_n such that the nth conjugate space of B_n is the first nonseparable conjugate space of B_n.

In the sequel, we will identify X with its canonical image in X^{**}.

3. The case III of the distance function.

Following E. Asplund ([2, p. 223]), a set $S \subset X$ is called fat in X if it contains a set $G \subset S$ which is a dense G_δ set in X. E. Asplund proved in [2] the following result:

If B is a reflexive locally uniformly rotund Banach space and $S \subset B$ is a norm closed bounded subset of B, then the set

$$a(S) = \{c \in B : \exists s \text{ such that } \|c-s\| \geqq \|c-x\| \ \forall x \in S\}$$

(i.e. the set of all points in B which have farthest points in S), is fat in B.

In the following result, some further information on the set of farthest points is derived.

PROPOSITION 1. *Suppose B is a Banach space such that X^* is* LUR *and* SDS. *Then if S is an arbitrary norm closed bounded subset of B^*, we have that the set $a(S)$ defined above is fat in B^*. Moreover, if we define a set-valued mapping on $a(S)$ into* $\exp B^*$, *by*

$$Ty = \{s \in S \text{ such that } \|s-y\| \geqq \|x-y\|, \forall x \in S\},$$

then there exists a subset $F \subset a(S)$ which is fat in B^* such that the mapping T considered on F is single-valued and norm-norm continuous.

PROOF. Let us follow first the ideas of the proof of E. Asplund in [2, p. 213–216], together with the following considerations. Denote the original norm of B^* by $\|\cdot\|$. Let $r(x)$, for $x \in B^*$ be defined as in [2, p. 214], i.e. $r(x) = \sup_{s \in S} \|x-s\|$. Then, as r is the supremum of w^*-lower semicontinuous functions on B^*, it is w^*-lower semicontinuous on B^*. Suppose S has at least two distinct points. Then r is positive, finite, convex and satisfies the Lipschitz condition with $C=1$ (see [2, p. 214]). Therefore, by another result of E. Asplund [1, p. 32 and 37], r is Fréchet differentiable on a fat subset $E_1 \subset B^*$ with the differentials from B. Denote by F the set $E_1 \cap E$, where E is a fat set from Lemma 1 of [2, p. 213]. Then F is fat in B^*. Take $y \in F$ and denote by p_y the differential of r at y. Then $p_y \in B$, $\|p_y\| = 1$ (by Corollary to Lemma 3 of [2, p. 215]). Thus p_y attains its minimum over $K_{r(y)}(y) \subset B^*$ at a point x. The rest of the first part of the proof is the same as Asplund's one (see [2, p. 216]).

Now we prove that T is single valued on F. Suppose there exist $z_1, z_2 \in T(0)$, $0 \in F$, such that $z_1 \neq z_2$, $r(0)=1$, and $\|z_1\| = \|z_2\| = 1$. If $z_1 = -z_2$, then for t real, $r(tz_1) \geqq 1+|t|$, so as $r(0)=1$ and r convex we get a contradiction with the differentiability of $r(tz_1)$ at $t=0$. Assume now $z_1 \neq -z_2$.

Consider the two dimensional subspace $P \subset B^*$ determined by z_1, z_2. For an arbitrary $h \in P$ satisfying $\|h\| = 1$, the convex functions

$$\varphi_i^h(t) = \|th - z_i\|, \quad i = 1, 2,$$

are differentiable at $t=0$, since $\varphi_i^h(0) = r(0) = 1$ and $\varphi_i^h(t) \leqq r(th)$. Thus the norm of P induced from B^* is differentiable at $-z_i$, $i=1,2$. Let $f_{-z_i} \in P^*, i=1,2$ denote the differentials of the norm of P at $-z_i$. Then $f_{-z_1} \neq f_{-z_2}$ since P is rotund. Take $h_0 \in P$ satisfying $\|h_0\| = 1$ such that $f_{-z_1}(h_0) \neq f_{-z_2}(h_0)$. Then

$$(\varphi_1^{h_0})'(0) \neq (\varphi_2^{h_0})'(0).$$

This easily gives a contradiction with the differentiability of $r(th_0)$ at $t=0$.

Now we prove the continuity of the restriction of T to F on F. The mapping $y \mapsto p_y$ is norm-norm continuous as a mapping considered on F (by [1, Lemma 5, p. 43]). Suppose $y_n \in F$, $y_n \to 0 \in F$ (again without loss of generality), and $r(0)=1$. Each p_{y_n} respectively p_0, $n=1,2,\ldots$, has the norm 1 in B and each of them attains its minimum over $K_1(0) \subset B^*$ at a unique point x_n respectively x, since the norm of B^* is LUR. Moreover, if $p_{y_n} \to p_0$, then $x_n \to x$ in the norm of B^*, since the norm of B is Fréchet differentiable at each $z \neq 0$ (see [9]). It is easy to see that p_{y_n} respectively p_0 attain their minimum over $K_{r(y_n)}(y_n) \subset B^*$ respectively $K_1(0) \subset B^*$ at the point $y_n + r(y_n)x_n$ respectively x. As above,

$$y_n + r(y_n)x_n \in Ty_n, \quad x \in T0$$

and from the continuity of r on B^* we have $y_n + r(y_n)\cdot x_n \to x$.

REMARK. It follows from Proposition 1 and from remarks in Section 2 that the "farthest point property" of the space does not imply its reflexivity.

The following result exhibits a class of Banach spaces which have "farthest point property" with respect to weakly compact subsets.

PROPOSITION 2. *Assume X is a* WUR *Banach space. Let S be a weakly compact subset of X. Then the set $a(S)$ defined as above is fat in X.*

PROOF. Follow the proof of Theorem of E. Asplund [2, p. 216], with the following additional considerations: Take again $y \in E$ and $p \in \partial r(y)$ (the subdifferential of r at y). Suppose $y=0$, $r(0)=1$. We again have $\|p\|=1$. Now take $x_n \in X$, $n=1,2,\ldots$, satisfying $\|x_n\|=1$, such that

$p(x_n) < -1+1/n$. Consider the function r on the closed line segment $\langle 0, -x_n \rangle$. Then for the increase δ_n of r from 0 to $-x_n$ we have from the Lipschitzian property ($C=1$), that $\delta_n \leqq 1$, and since $p \in \partial r(0)$, we also have $\delta_n \geqq 1-1/n$. Hence the ball $K_{\delta_n+1}(-x_n)$ is the smallest closed ball with the center $-x_n$ that contains S, so we may find points $z_n \in S, n=1,2,\ldots,$ such that

$$\|z_n+x_n\|-(\delta_n+1) \to 0 .$$

Of course, since $S \subset K_1(0) \subset X$ $(r(0)=1)$, we have $\|z_n\| \leqq 1$. Thus,

$$\|z_n\| \leqq 1 = \|x_n\|, \quad \|x_n+z_n\| \to 2 .$$

Therefore, by the WUR property, $z_n - x_n \to 0$ in the weak topology of X. From the weak compactnes of S, assume without loss of generality that $z_n \to x \in S$ in $w(X,X^*)$. Then $x_n \to x$ in $w(X,X^*)$ and therefore $p(x_n) \to p(x)$. Thus $1 \geqq \|x\| \geqq |p(x)| = 1$. Hence x is a farthest point to 0 in S.

In the following, $\operatorname{ext}\overline{\operatorname{conv}}\, S$ will mean the set of all extreme points of the closed convex hull of the set S. Moreover, $\operatorname{wcl} S$ denotes the weak closure of the set S.

COROLLARY. *Assume a Banach space X is* WUR *and its norm is Fréchet differentiable at all nonzero points. Let S be a weakly compact subset of X. Denote by M_S the set of all $s \in S$ for which an element c of X exists such that $\|s-c\| = \sup_{x \in S}\|x-c\|$ (i.e. the set of all farthest points in S).*

Then the closed convex hull of M_S is equal to that of S and thus $\operatorname{ext}\overline{\operatorname{conv}}\, S \subset \operatorname{wcl} M_S$.

PROOF. We will use Šmuljan's theorem on duality of Fréchet differentiability and strong exposedness (see[20], [21], or [1, p. 35]), together with the Bishop-Phelps theorem on subreflexivity of Banach spaces. From these results it follows that those points which are strongly exposed of $K_1(0) \subset X^*$, by elements of X are norm dense on the boundary of $K_1(0) \subset X^*$. Then by the result of S. Mazur and R. R. Phelps (cf. [18, p. 976]), every closed convex bounded subset of X can be representated as the intersection of all closed balls that contain it. This fact may be used to prove our statement (cf.[9, p. 175]) as follows: Denote the closed convex hull of S by T. Clearly, $M_S \subset T$. Suppose there is an $x \in T$, $x \notin M_S$. Then there is, by the Mazur-Phelps theorem a $c_0 \in X$ and $r>0$ such that

$$x \notin K_r(c_0), \quad K_r(c_0) \supset M_S .$$

Take an $\varepsilon > 0$ such that $r+3\varepsilon \leqq \|x-c_0\|$. Then there is a $c \in a(S)$ such that $\|c-c_0\| \leqq \varepsilon$. Let $s \in S$ be the farthest point to c in S. We have for any $y \in S$

$$\|y-c_0\| \leqq \|y-c\|+\|c-c_0\| \leqq \|s-c\|+\|c-c_0\| \leqq \|s-c_0\|+2\|c-c_0\| \leqq r+2\varepsilon .$$

Thus $S \subset K_{r+2\varepsilon}(c_0)$ and therefore $T \subset K_{r+2\varepsilon}(c_0)$, a contradiction. The rest of our statement follows from the famous Krejn-Milman theorems (cf. [15, p. 325, 332]).

REMARK. The WUR spaces are not uncommon. For instance, any Banach space with separable dual can be easily equivalently renormed to be WUR (cf. [24, p. 200]). $l_2(B_n)$ is WUR if all B_n, $n=1,2,\ldots$, are (cf. [25, p. 22]).

The following result says that also reflexive Banach spaces with Fréchet differentiable norm at all nonzero points have "farthest point property" for weakly closed bounded subsets.

PROPOSITION 3. *Assume X is a reflexive Banach space with Fréchet differentiable norm at all nonzero points.*

Then for any weakly closed bounded subset S of X, the set $a(S)$ defined as above (before Proposition 1), is fat in X and if we use the notations from the preceeding Corollary, the closed convex hull of the set M_S is equal to that of S and thus $\operatorname{ext}\overline{\operatorname{conv}}\, S \subset \operatorname{wcl} M_S$.

PROOF. Follow again the ideas of the proof of the Theorem in [2, p. 216]. Take $y \in E$ and $p \in \partial r(y)$. Suppose $y=0$ and $r(0)=1$. Let p attains its minimum over $K_1(0) \subset X$ at $x \in K_1(0)$. Then as it is shown in the above mentioned proof, for any $l>0$, the smallest closed ball with the center $-lx$ which contain S is $K_{l+1}(-lx)$. Take the $f \in K_1(0) \subset X^*$ such that $f(x)=1$. Then $\sup_{u \in S} f(u)=1$, since $S \subset K_1(0) \subset X$ and since

$$\sup_{u \in S} f(u) \leqq 1-\varepsilon, \quad \varepsilon>0$$

would imply $\sup_{u \in S_1} f(u) \leqq 1-\varepsilon$ where S_1 is the closed convex hull of S. Then from the proof of Mazur-Phelps result we find that there would be a closed ball K with the center $-l_0 x$ for some $l_0>0$ such that $S \subset K$ and

$$\sup_{u \in K} f(u) \leqq 1-\tfrac{1}{2}\varepsilon$$

(for details see [27, Proposition 1]). The radius of K is evidently smaller than l_0+1, a contradiction. Thus there are $z_n \in S$, $n=1,2,\ldots$, such that $f(z_n) \to 1$. From the weak compactness of S suppose without loss of generality that $z_n \to z \in S$ in $w(X,X^*)$. Then $f(z)=1$ and thus $\|z\|=1$ and z is a farthest point to 0 in S. Therefore we have proved that $a(S)$ is fat in X. The proof now proceeds as in Corollary to Proposition 2.

The assumption of weak closednees of subsets in Proposition 3 is dropped in the following.

COROLLARY. *Suppose a reflexive Banach space X has Fréchet differentiable norm at all nonzero points and satisfies the following condition:*

(H) *If* $\|x_n\| = \|x\| = 1$ *and* $x_n \to x$ *in* $w(X,X^*)$, *then* $\|x_n - x\| \to 0$.

Then for any norm closed bounded subset $S \subset X$, the set $a(S)$ is fat in X and the closed convex hull of the set M_S (with the notations as above) is equal to the closed convex hull of the set S. Thus $\operatorname{ext}\overline{\operatorname{conv}}\, S \subset \operatorname{wcl} M_S$.

PROOF. Follow the proof of Proposition 3. From the reflexivity of X, assume (without loss of generality) $z_n \to z \in X$ in $w(X,X^*)$. Since $\|z_n\| \leqq 1$ and $f(z_n) \to 1$, we have $\|z\| = 1$. Thus, by the property (H), $\|z_n - z\| \to 0$. Therefore $z \in S$ and is a farthest to 0 in S.

REMARK. Clearly, LUR implies (H). D. Wulbert proved in [23] that $l_2(B_n)$ satisfy (H) if all B_n, $n = 1, 2, \ldots$, does.

4. The case II of linear operators.

In this Section we state a w^*-analog of the result of J. Lindenstrauss mentioned in the Introduction. First, it is easy to see that Lemma 1 of J. Lindenstrauss [16, p. 140] has the following variant for dual spaces:

LEMMA 1. *Suppose T is a linear bounded operator of X^* into Y^*, which is also w^*–w^* continuous. Then there is an $x \in X^*$, $\|x\| = 1$ such that $\|Tx\| = \|T\|$, iff the following assertion is valid:*

There exist $x_k \in X^*$, $f_k \in Y$, $k = 1, 2, \ldots$, such that $\|x_k\| = \|f_k\| = 1$, and

$$|f_j(Tx_k)| \geqq \|T\| - 1/j \quad \textit{for } \mathsf{N} \ni j \leqq k,\ k = 1, 2, \ldots .$$

PROOF. If the condition holds, take x a $w(X^*,X)$-limit point of the sequence $\{x_k\}$, $k = 1, 2, \ldots$. Then $x \in K_1(0) \subset X^*$ and there exists a subnet x_{k_ν}, $\nu \in A$ of the sequence $\{x_k\}$, $k = 1, 2, \ldots$, such that $x_{k_\nu} \to x$, in $w(X^*,X)$. Now, since T is $w(X^*,X)$ -$w(X^*,X)$ continuous on X^*, we have for any $j \in \mathsf{N}$

$$|f_j(Tx)| = \lim_{\nu \in A} |f_j(Tx_{k_\nu})| .$$

Thus, since for any $j \in \mathsf{N}$,

$$|f_j(Tx_{k_\nu})| \geqq \|T\| - 1/j \quad \text{for } \nu \geqq \gamma_0^j,\ \nu \in A ,$$

we have

$$\|T\| \geqq |f_j(Tx)| \geqq \|T\| - 1/j$$

for any $j \in \mathsf{N}$. Therefore

$$\|Tx\| = \sup_{f \in K_1(0) \subset Y} |f(Tx)| = \|T\| .$$

If there is an $x \in X^*$, $\|x\| = 1$ such that $\|Tx\| = \|T\|$, we may simply put $x_k = x, k = 1, 2, \ldots$ and take $f_j \in Y$ satisfying $\|f_j\| = 1$ such that

$$|f_j(Tx)| \geqq \|Tx\| - 1/j , \quad j \in \mathsf{N} .$$

Next we obtain:

PROPOSITION 4. *Let X, Y be arbitrary Banach spaces. Denote by $B^*(X^*, Y^*)$ the set of all linear bounded operators of X^* into Y^* which are w^*-w^* continuous.*

Then there exist a norm dense set $D \subset B^*(X^*, Y^*)$ formed by operators which attain their norms on $K_1(0) \subset X^*$.

PROOF. Following the proof of J. Lindenstrauss for generally nondual spaces, we make only a few changes (see [16, p. 141]).

Let $T \in B^*(X^*, Y^*)$ satisfying $\|T\| = 1$ and $\varepsilon \in (0, \frac{1}{3})$ be given. Choose a decreasing sequence $\{\varepsilon_k\}, k = 1, 2, \ldots$ of positive numbers such that

$$2\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon, \quad 2\sum_{i=k+1}^{\infty} \varepsilon_i < \varepsilon_k^2, \quad \varepsilon_k < 1/10k, \quad k = 1, 2, \ldots$$

Next choose inductively a sequence $\{T_k\}$ of linear bounded operators of X^* into Y^* and sequences $\{x_k\} \subset K_1(0) \subset X^*$ and $\{f_k\} \subset K_1(0) \subset Y$ such that

$$T_1 = T$$

$$\|T_k x_k\| \geqq \|T_k\| - \varepsilon_k^2, \quad \|x_k\| = 1, \quad k = 1, 2, \ldots ,$$

$$f_k(T_k x_k) \geqq \|T_k x_k\| - \varepsilon_k^2, \quad \|f_k\| = 1 , \quad k = 1, 2, \ldots ,$$

$$T_{k+1} x = T_k x + \varepsilon_k f_k(T_k x) T_k x_k, \quad x \in X^*, k = 1, 2, \ldots .$$

We may verify that similarly as in the above mentioned work of J. Lindenstrauss,

$$\|T_j - T_k\| \leqq 2\sum_{i=j}^{k-1} \varepsilon_i, \quad \tfrac{2}{3} \leqq \|T_k\| \leqq \tfrac{4}{3}, \quad j < k, k = 1, 2, \ldots ,$$

$$\|T_{k+1}\| \geqq \|T_k\| + \varepsilon_k \|T_k\|^2 - 4\varepsilon_k^2, \quad k = 1, 2, \ldots ,$$

$$\|T_k\| \geqq \|T_j\| \geqq 1, \quad j < k, k = 1, 2, \ldots ,$$

$$|f_j(T_j x_k)| \geqq \|T_j\| - 6\varepsilon_j, \quad j < k, k = 1, 2, \ldots .$$

Then T_j converges in the norm of $B(X^*, Y^*)$ (= the Banach space of all bounded linear operators of X^* into Y^*) to a linear bounded operator $\hat{T}$ of X^* into Y^* such that $\|\hat{T} - T_j\| \leqq \varepsilon_j^2$ and $\|\hat{T} - T\| \leqq \varepsilon$. Furthermore, it is easy to see that the partializations $T_j/K_1(0)$ are continuous with respect to the relativized w^* topology on $K_1(0) \subset X^*$ and the w^* topology of Y^*. Thus the same property is shared also by $\hat{T}/K_1(0)$ — their uniform limit with respect to the norm (and thus a fortiori with respect to the w^*) topology of Y^*. Therefore $\hat{T} \in B^*(X^*, Y^*)$, by the Banach-Dieudonné theorem (see [5, p. 265]). Now, as in Lindenstrauss' work,

$$|f_j(\hat{T}x_k)| \geqq \|\hat{T}\| - 1/j, \quad j < k, \ k = 1, 2, \ldots$$

(cf. 16, p. 142]). Using now Lemma 1, we see $\hat{T}$ attains its norm on $K_1(0) \subset X^*$.

5. Applications.

In this Section we give some applications of the notions studied in this note to the structure of Banach spaces.

PROPOSITION 5. *Assume X is a Banach space. Then*

(i) *If X is an* SDS *space, then the norm of X^{**} is Fréchet differentiable on a $w(X^{**}, X^*)$ dense set in X^{**}.*

(ii) *If X^* is an* SDS *space, then $K_1(0) \subset X^{**}$ is the $w(X^{**}, X^*)$ closed convex hull of those of its points lying in X that are strongly exposed by elements of X^*.*

PROOF. V. L. Šmuljan proved in [20], [21] (see also [7, p. 296]) that the norm of a Banach space X (X^*) is Fréchet differentiable at x, $\|x\| = 1$ iff whenever $f_n \in X^*$ (X) satisfy $\|f_n\| = 1$ and $f_n(x) \to 1$, then $\{f_n\}$ is a norm convergent sequence in X^* (X). From this and the $w(X^{**}, X^*)$ density of the canonical image of X in X^{**}, (i) easily follows.

If X^* is an SDS space, then $K_1(0) \subset X$ is the closed convex hull of its strongly exposed points—the result following from the ones in Asplund's paper [1](see [26, p. 452]). From this and the first part of this proof (ii) easily follows.

COROLLARY 1. *If for a Banach space X, X^{**} is separable, then the norm of X^{**} is Fréchet differentiable on a $w(X^{**}, X^*)$ dense set in X^{**} and $K_1(0) \subset X^{**}$ is the w (X^{**}, X^*) closed convex hull of those of its points from X that are strongly exposed by elements of X^*.*

PROOF. Use the remarks in Section 2.

COROLLARY 2. *$l_1(\mathsf{N})$ is not isometrically isomorphic to any bidual of Banach space.*

PROOF. The norm of $l_1(\mathsf{N})$ is nowhere Fréchet differentiable, as it is remarked in [16, p. 145]. This is easily seen, since otherwise there would be a point $x \in c_0{}^*(\mathsf{N})$, $\|x\| = 1$ at which the norm of $c_0{}^*(\mathsf{N})$ is Fréchet differentiable with the differential y, lying in $c_0(\mathsf{N})$ by Asplund's result ([1, p. 37]). Then y would be a strongly exposed point of $K_1(0) \subset c_0(\mathsf{N})$, a contradiction.

PROPOSITION 6. *Assume X is a Banach space such that X^* is an* SDS *space. Then*

(i) *Every w^* compact set in X^* is the intersection of finite unions of closed balls in X^*.*

(ii) *Suppose $\{A_n\}$, $n = 1, 2, \ldots$ is an arbitrary countable family of closed convex bounded sets in X. Define the set $M \subset X^*$ as follows:*

$$M = \{f \in X^* \text{ such that } \forall n \in N, f \text{ attains its maximum on } A_n\}.$$

Then M is fat in X^*.

PROOF. (i) follows exactly as Theorem 3 of [6, p. 411], using the w^* compactness of $K_1(0) \subset X^*$.

(ii). For any $n \in \mathsf{N}$, let F_n be a function on X defined as follows: $F_n(x) = 0$ on A_n, $F_n(x) = +\infty$ for $x \notin A_n$. Then the Fenchel dual function $F_n{}^*$ on X^* is continuous, finite, convex and thus Fréchet differentiable on a dense G_δ subset $G_n \subset X^*$ with the differentials lying in X ([1] p. 37). Furthermore, if $F_n{}^*$ is Fréchet differentiable at f with the differential $x\,(\in X)$ then it follows from the results of [1, Proposition 5 on p. 46] that $x \in A_n$ and $f(x) = \sup_{u \in A_n} f(u)$.

REFERENCES

1. E. Asplund, *Fréchet differentiability of convex functions*, Acta Math. 121 1968, 31–48.
2. E. Asplund, *Farthest points in reflexive locally uniformly rotund Banach spaces*, Israel J. Math. 4 (1966), 213–216.
3. E. Asplund and R. T. Rockafellar, *Gradients of convex functions*, Trans. Amer. Math. Soc. 139 (1968), 443–467.
4. E. Bishop and R. R. Phelps, *The support functionals of convex sets*, in Proceedings of Symposia in Pure Math. 7, *Convexity*, 27–36, Amer. Math. Soc. Providence R.I. 1961.
5. N. Bourbaki, *Elements de Mathématique*, Livre V, *Espaces Vectoriels Topologiques*, Russian translation, Moskva 1959. (Act. Sci. Ind. 1189, 1229, Hermann, Paris 1953, 1955).
6. H. Corson and J. Lindenstrauss, *On weakly compact subsets of Banach spaces*, Proc. Amer. Math. Soc. 17 (1966), 407–412.
7. D. F. Cudia, *The geometry of Banach spaces. Smoothness*, Trans. Amer. Math. Soc. 110 (1964), 284–314.

8. N. Dunford and J. Schwarts, *Linear operators I*, Russian translation, Moskva 1962. (Pure and applied math. 7, Interscience Publishers, New York · London 1967.)
9. M. Edelstein, *Farthest points of sets in uniformly convex Banach spaces*, Israel J. Math. 4 (1966), 171–176.
10. R. C. James, *Weakly compact sets*, Trans. Amer. Math. Soc. 113 (1964), 129–140.
11. R. C. James, *Separable conjugate spaces*, Pacific J. Math. 10 (1960), 563–571.
12. K. John and V. Zizler, *A renorming of dual spaces*, Israel J. Math. 12 (1972), 331–336.
13. M. I. Kadec, *The conditions for the differentiability of the norm of Banach space*, Uspehi Mat. Nauk 20 (1965), 183–187 (in Russian).
14. J. L. Kelley and I. Namioka, *Linear topological spaces*, D. van Nostrand, Princeton, 1963.
15. G. Köthe, *Topological vector space* I, English translation, (Grundlehren Math. Wissensch. 159), Springer-Verlag, Berlin · Heidelberg · New York, 1969.
16. J. Lindenstrauss, *On operators which attain their norm*, Israel J. Math. 1 (1963), 139–148.
17. A. R. Lovaglia, *Locally uniformly convex Banach spaces*, Trans. Amer. Math. Soc. 78 (1955), 225–238.
18. R. R. Phelps, *A representation theorem for bounded convex sets*, Proc. Amer. Math. Soc. 11 (1960), 976–983.
19. R. T. Rockafellar, *Convex analysis*, (Princeton Math. Series 28), Princeton University Press, Princeton 1970.
20. V. L. Šmuljan, *On some geometrical properties of the unit ball of a Banach space*, Mat. Sb. 6 (1939), 77–94 (Russian).
21. V. L. Šmuljan, *On the differentiability of the norm of a Banach space*, Dokl. Akad. Nauk SSSR, 27 (1940), 643–648.
22. S. Trojanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math. 37 (1971), 173–180.
23. D. Wulbert, *Differential theory for nonlinear approximation*, to appear.
24. V. Zizler, *Some notes on various rotundity and smoothness properties of separable Banach spaces*, Comment. Math. Univ. Carolinae 10 (1969), 195–206.
25. V. Zizler, *On some rotundity and smoothness properties of Banach spaces*, Dissertationes Math. Rozprawy Mat. 87 (1971), 1–37.
26. V. Zizler, *Remark on extremal structure of convex sets in Banach spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 19 (1971), 451–455.
27. V. Zizler, *A note on seperation of convex sets*, Czechoslovak Math. J. 21 (1971), 340–343.

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