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FREE SUBGROUPS AND FØLNER'S CONDITIONS

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It has been conjectured [3, p. 9] that any group has a free nonabelian subgroup if and only if it is not amenable. The characterization of the class of nonamenable groups given in Theorem 1 below is due to Følner [2, p. 245]. Here we analyze some possibilities for restricting parameters in Følner's conditions and we show that instances of the conditions identify finitely generated free subgroups.

THEOREM 1. (*Følner.*) *A group G is not amenable if and only if for every k in the interval $0<k<1$ there exist finitely many, not necessarily distinct, elements $a_1,\ldots,a_n$ of G such that for every finite $E \subset G$*

$$n^{-1}\textstyle\sum_{i=1}^{n} N(E \cap Ea_i) \;<\; kN(E)$$

where for any $S, N(S)$ denotes the cardinality of S.

The following theorem shows that Følner's conditions in Theorem 1 can never be satisfied unless $N(\{a_1,\ldots,a_n\}) \geqq 1/k$.

THEOREM 2. *For any group G, for any k in the interval $0<k<1$, and for any finite sequence $a_1,\ldots,a_n$ of not necessarily distinct elements of G, if $N(\{a_1,\ldots,a_n\}) < 1/k$ then there exists a finite subset E of G such that*

$$n^{-1}\textstyle\sum_{i=1}^{n} N(E \cap Ea_i) \;\geqq\; kN(E)\,.$$

PROOF. For finite G we may use $E=G$. Assume G is infinite and let $H=\{a_1,\ldots,a_n\}$. For $m>0$ and $a \in H$ an m-set for a is a set L of the form

$$L \;=\; \{a,a^2,\ldots,a^h,b_1a,b_1a^2,\ldots,b_1a^h,b_2a,b_2a^2,\ldots\}$$

such that $N(L \cap La) \geqq m-1$ and $N(L)=m$. Let E be a union of disjoint m-sets, one for each $a \in H$, with m sufficiently large. Then for every $a \in H$,

$$N(E \cap Ea) \;\geqq\; kN(E)\,.$$

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Since Følner's conditions can only be satisfied if $k \geqq 1/N(\{a_1,\ldots,a_n\})$, cases where the conditions are satisfied with $a_1,\ldots,a_n$ all distinct and $k=1/n$ are *extreme instances* of Følner's conditions. The next theorem shows that these extreme instances of Følner's conditions characterize finite sets of free generators for free subgroups.

THEOREM 3. *For any group G and any finite $F \subset G$ the subgroup generated by F is free on F if and only if for every finite $E \subset G$*

$$\textstyle\sum_{a\in F} N(E\cap Ea) < N(E) .$$

PROOF. First assume the subgroup $\mathscr{G}(F)$ generated by F is not free on F for a finite $F \subset G$. We may assume that $1 \notin F$, since otherwise obviously

$$\textstyle\sum_{a\in F} N(E \cap Ea) \geqq N(E)$$

for every finite $E \subset G$. Let w be a reduced word of minimal length on F with $w=1$ and the right most symbol occurrence in w an element of F, not an indicated inverse of an element of F. Let E be the set of all elements expressible as reduced subwords of w which can be obtained by deleting right terminal substrings of w. For $z \in F \cup F^{-1}$ let

$$E(z) = \{x \in E \mid z \text{ is the right most factor of the expression for } x \text{ as a reduced subword of } w\} .$$

Then for any $a \in F$,

$$E(a^{-1}) \cap E(a)a^{-1} = \emptyset, \quad E(a^{-1}) \cup E(a)a^{-1} \subset E ,$$
$$E \cap Ea = [E(a^{-1}) \cup E(a)a^{-1}]a .$$

So $N(E\cap Ea) = N(E(a^{-1})) + N(E(a))$ and we conclude that

$$\textstyle\sum_{a\in F} N(E\cap Ea) \geqq N(E) .$$

Next we assume that $\mathscr{G}(F)$ is free on F for a finite $F \subset G$. Let L be a set consisting of one element from each left coset of $\mathscr{G}(F)$ in G such that $1 \in L$. For any $y \in G$ the *length* of y is the length of f where f is the unique reduced word on F such that $y = xf$ for some $x \in L$. The proof is completed by induction on the maximum length of the words in E.

A consequence of Theorem 3 is that the finite sequence $a_1,\ldots,a_n$ mentioned in Følner's theorem can be limited to a two term sequence and the number k can be restricted to $\frac{1}{2}$ if and only if the conjecture is true that the class of groups with free nonabelian subgroups is exactly the class of nonamenable groups. Also, since a free nonabelian group has for $n > 0$ a free subgroup on n free generators, cf. [1, p. 259], the conjecture is true if and only if the restriction $1/k = N(\{a_1, \ldots, a_n\})$ may be added to Følner's theorem. Finally we observe that in Theorem 3, even for a fixed value of $N(F)$, no limitation in the form of a finite upper bound for $N(E)$ is possible. This follows from a consequence of the compactness theorem for first-order logic that any elementary class of groups which contains all groups with subgroups free on n-element sets also contains a finite group cf. [4, p. 426].

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