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ON A THEOREM OF STUDY CONCERNING CONFORMAL MAPS WITH CONVEX IMAGES II

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1.

The present paper is concerned, as was its predecessor [3], with a question arising from the classical theorem of Study which states that *if the image of a univalent analytic function f on the open unit disk is convex, then for each $r, 0 < r < 1, f(\{|z| < r\})$ is convex* (see[5]). E is to denote throughout this paper a subset of the open unit disk. E will be said to have the *property* S provided that $f(E)$ is convex for each f allowed by the Study theorem. The problem of characterizing the sets E having the property S was proposed by Maxwell Reade to Christian Pommerenke who gave a brief elegant solution of the problem [4]. We may formulate the result of Pommerenke as follows. E will be said to have the *property* P provided that E is convex and that whenever a and b are distinct points of E, the arcs of the two oricycles passing through a and b, which have endpoints a and b and do not contain points of the unit circumference, lie wholly in E. The result of Pommerenke may be stated as follows:

The E having the property S *are exactly those having the property* P.

The requirement of convexity in the definition of the property P may be dropped. We first note that given a and b as above, A, the closure of the bounded region whose frontier consists of those arcs of the oricycles passing through a and b which were specified in the definition of the property P, is convex. Further, an oricycle passing through a given inner point of A is such that its two open arcs with endpoints the given point of $\operatorname{int} A$ and the point where the oricycle is tangent to $\{|z|=1\}$ both contain points of $\operatorname{fr} A$. Since E is assumed to have the property P save for convexity, so that, in particular, $\operatorname{fr} A \subset E$, we conclude, using the observation of the preceding sentence, that $A \subset E$. Since $[a,b] \subset A, [a,b]$ denoting, as usual,

$$\{(1-t)a+tb : 0 \leqq t \leqq 1\},$$

$[a,b] \subset E$. Hence E is *convex*.

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We shall want to have available that part of Pommerenke's theorem which asserts that "P" implies "S". We recall the elegant reasoning of Pommerenke. The set A of the preceding paragraph is the intersection of the relative closures $\Delta_k (k=1,2)$ of bounded disks whose frontiers are oricycles. (Throughout, "relative" is construed in the sense of the relative topology of the open unit disk considered as a subset of the finite complex plane.) For any f allowed by the Study theorem each $f(\Delta_k)$ is convex. Cf. [3; p. 175, line 16, (ii)]. Hence so is $f(A)$. Consequently, $[f(a), f(b)] \subset f(A) \subset f(E)$. Thus if E has the property P, it also has the property S.

In my paper [3], unaware of the work of Pommerenke, I introduced the notion of D-convexity (E is *D-convex* provided that for distinct a and b in E the set A, associated above with a and b, is contained in E) and showed the equivalence of the property S and D-convexity. It is, of course, immediate that the property P and D-convexity are equivalent. While my paper lacked the deftness of Pommerenke's approach in showing that "D" ("P") implies "S", it did yield representation results of Minkowskian type for relatively closed E having the property S: they are intersects of relatively closed disks bounded by oricycles, and, in particular, apart from the trivial case of the open unit disk, such E are *either relatively closed disks bounded by oricycles or else are compact* (*possibly empty*). (The intersection of an empty family is taken as the open unit disk.)

The object of this paper is to continue the study of the sets having the property S. We shall obtain another characterization for these sets (Theorem 2.2) and a canonical representation theorem (Theorem 3.1). The mathematical development of the present paper is independent of its predecessor [3], save for the convenient reference to the simple consequence of Study's theorem used by Pommerenke, which was noted above.

G-convexity. We let G denote the group of directly conformal automorphisms of $\{|z|<1\}$ onto itself. E will be said to be *G-convex* provided that $\alpha(E)$ is convex for each $\alpha \in G$. If E has the property S, then E is G-convex (G being a subset of the f allowed by Study's theorem). The exact converse proposition is not true. It suffices to note that an open disk bounded by an oricycle together with two distinct points of the oricycle, neither the point of tangency of the oricycle with $\{|z|=1\}$, is G-convex but does not have the property S. However, the converse holds for open or relatively closed G-convex sets. Cf. the three paragraphs following the statement of Lemma 2.1. Theorem 2.1 yields the result that "modulo a boundary subset the G-convex sets have the property S."

The property I. E will be termed *elementary* provided that it is the union of the bounded open disk bounded by an oricycle and a connected subset of the intersection of the oricycle and the open unit disk. E will be said to have the *property* I provided that it is the intersection of elementary sets. We shall see that the property S and the property I are equivalent (Theorem 2.2).

For the moment we note: (1) an elementary set has the property S, and consequently, (2) the property I implies the property S. Of course, (2) follows at once from (1) and the fact that property S remains preserved under intersection, the intersection of an empty family being understood as the open unit disk. To see that (1) holds we invoke the result of Pommerenke and reduce the problem to showing that *an elementary set has the property* P.

To show this, we use a Möbius transformation mapping $\{|z|<1\}$ onto $\{\operatorname{Im} z>0\}$ and taking the point of tangency of the elementary set into ∞ and prove the following equivalent proposition.

LEMMA 1.1. *Given a positive number c, a set X of the form $\{\operatorname{Im} z>c\} \cup Y$, where Y is a connected subset of $\{\operatorname{Im} z=c\}$, has the property that, with u and v distinct points of X, the arc γ, having endpoints u and v and lying in $\{\operatorname{Im} z>0\}$, of a circle $\varkappa$, passing through u and v and tangent to the real axis, satisfies*

$$\operatorname{Im} z>\min\{\operatorname{Im} u, \operatorname{Im} v\}, \quad z \in \gamma, z \neq u, v . \tag{1.1}$$

This lemma is an immediate consequence of the utterly elementary observation that $\varkappa$ has at most two points in common with a horizontal line and that the points of intersection of $\{\operatorname{Im} z=d\}$, where $d \leqq \min\{\operatorname{Im} u, \operatorname{Im} v\}$, is fully accounted for by that closed arc of $\varkappa$ with endpoints u and v which contains the point where $\varkappa$ is tangent to the real axis.

The fact that an elementary set has the property S is thereby established.

2. G-convex sets.

In this section we obtain a geometric characterization of G-convex sets and apply the result to show that the property S implies the property I.

THEOREM 2.1. *E is G-convex if and only if it is the intersection of a family of sets intermediate to open disks bounded by oricycles and their relative closures, the intersection of an empty family being understood as $\{|z|<1\}$.*

PROOF. The intermediate sets in question are G-convex, as it is verified routinely. Consequently, the intersection of a family of such intermediate sets is G-convex.

The converse proposition, to which we now turn, is less obvious. Let E be a G-convex subset of $\{|z|<1\}$. The cases where E is either empty or $\{|z|<1\}$ are trivial, while the case where E is a singleton is easy for we may suppose that $E=\{0\}$ and note that here

$$E = \{\mathrm{Re}[(1+z)/(1-z)] \geqq 1\} \cap \{\mathrm{Re}[(-1+z)/(-1-z)] \geqq 1\}.$$

In the remaining case, to which we shall restrict our attention for the rest of the proof of Theorem 2.1, E is a proper subset of $\{|z|<1\}$ and $\mathrm{int}\,E \neq \emptyset$. (It is to be noted that E contains the convex hull of a circular arc.) The proof of the converse for the restricted E will be reduced to showing the following lemma.

LEMMA 2.1. *Given E as restricted, $a \in \mathrm{fr}\,E$, $|a|<1$, l a supporting line of $\bar{E}$, such that $a \in l$. Then the oricycle tangent to l at a which lies in the same closed half-plane bounded by l as does E is such that $\bar{E}$ is contained in the closed disk bounded by the oricycle.*

That the proof of the converse proposition for restricted E may be referred to Lemma 2.1 may be seen as follows.

We first apply to $\bar{E}$ the classical representation of a closed plane convex set as the intersection of a family of closed half-planes (see [2, p. 5]) and shall infer with the aid of Lemma 2.1 that $\bar{E}$ is the intersection of the family of closed disks which are bounded by oricycles and which contain $\bar{E}$.

To that end, we first note that $\mathrm{fr}\,\bar{E}=\mathrm{fr}\,E$ since E is a plane convex set. Consequently, $\mathrm{fr}\,\bar{E}$ contains points of $\{|z|<1\}$ thanks to the assumption on E.

If for two distinct such points the oricycles associated by Lemma 2.1 with the points and supporting lines of $\bar{E}$ through them have *distinct* points in common with $\{|z|=1\}$, then $\bar{E}$ is a *compact* subset of $\{|z|<1\}$ and the asserted representation of $\bar{E}$ follows readily, every point of $\mathrm{fr}\,\bar{E}$ being in $\{|z|<1\}$.

If this assumption is not fulfilled, then the oricycles associated by Lemma 2.1 with the points of $\mathrm{fr}\,\bar{E} \cap \{|z|<1\}$ have one and the same point in common with $\{|z|=1\}$. The argument now proceeds as follows. We note that the entering oricycles are equal. Let $\varkappa$ be the oricycle in question and let γ denote $\varkappa$ less its point on $\{|z|=1\}$. As is readily verified,

$\mathrm{fr}\bar{E} \cap \{|z| < 1\}$ (a subset of γ) is both open and closed in the sense of the relative topology of γ. Consequently, γ being connected

$$\mathrm{fr}\bar{E} \cap \{|z| < 1\} = \gamma .$$

It follows that $\mathrm{fr}\bar{E} = \varkappa$ and that $\bar{E}$ is the bounded closed disk with boundary $\varkappa$.

Hence in each case $\bar{E}$ is the intersection of the family of closed disks δ bounded by oricycles and containing $\bar{E}$.

Continuing, we show that $\bigcap \mathrm{int}\,\delta$ is *open*. To that end, we introduce K, the set consisting of 0 and the centers of the δ. K is a compact subset of $\{|z| < 1\}$. Thanks to this fact and the obvious continuity of the function ϱ on $\{|z| < 1\}$ assigning to 0 the value $1 =$ (radius of $\{|z| = 1\}$) and to $z \neq 0$ the radius of the unique oricycle with center z (so that $\varrho(z) = 1 - |z|$, $|z| < 1$), we see that, with $\zeta(\delta)$ denoting the center of δ, for $a \in \bigcap \mathrm{int}\,\delta$ we have

$$m = \inf_\delta\{\varrho[\zeta(\delta)] - |\zeta(\delta) - a|\} > 0 .$$

(Consider $z \mapsto \varrho(z) - |z-a|, z \in K$.) Hence $\{|z-a| < m\} \subset \bigcap \mathrm{int}\,\delta$. It follows that $\bigcap \mathrm{int}\,\delta$ is open.

We observe that $\mathrm{int}\,E \subset \bigcap \mathrm{int}\,\delta \subset \bar{E}$ and that since E is convex, $\mathrm{int}\,\bar{E} = \mathrm{int}\,E$. Now using the fact that $\bigcap \mathrm{int}\,\delta$ is open, we conclude:

$$\mathrm{int}\,E = \bigcap \mathrm{int}\,\delta .$$

It follows that

$$E = \bigcap (E \cup \mathrm{int}\,\delta) , \tag{2.1}$$

and hence since

$$\mathrm{int}\,\delta \subset E \cup \mathrm{int}\,\delta \subset \delta \cap \{|z| < 1\} ,$$

the asserted reduction of the remaining part of the proof of Theorem 2.1 to Lemma 2.1 is established.

Proof of Lemma 2.1. We apply the rotation σ about 0 which takes l onto a vertical line v and $\bar{E}$ onto a set of the closed right half-plane bounded by v, the term 'rotation' being understood to allow the identity map. We thereupon apply the Möbius transformation τ which keeps fixed the points of intersection of v with $\{|z| = 1\}$ and maps $\sigma(a)$ onto the point of v on the real axis. We note that τ maps $\{|z| < 1\}$ onto itself and the open right half-plane bounded by v onto itself. The set $\tau \circ \sigma(E)$ is G-convex. If it is contained in the closed circular disk which has $[\tau \circ \sigma(a), 1]$ as a diameter, E has the property stated in Lemma 2.1.

We confine our attention therefore, as we may, to the case where l is vertical, E lies in the closed right half-plane bounded by l, and a is real.

We introduce the Möbius transformation β, depending on the parameter $b, 0<b<1$, given by

$$z \mapsto (z-b)/(1-bz),\ z \in \mathbf{C}, \quad \infty \mapsto -b^{-1}.$$

The map β carries l onto a circle which is orthogonal to the real line less the point of this circle furthest left $(=-b^{-1})$. The point of this circle furthest right is $\beta(a)$. Now

$$\beta(\bar{E}) \subset \{\operatorname{Re} z \geqq \beta(a)\}. \tag{2.2}$$

Otherwise, $\beta(\bar{E})$, which is convex by virtue of the G-convexity of E, would contain a segment $[\beta(a), c], \operatorname{Re} c<\beta(a)$, and hence a point arbitrarily close to $\beta(a)$ of the bounded open circular disk D with frontier $\beta(l) \cup\{\beta(\infty)\}$. On applying $\operatorname{inv} \beta$ to $\beta(\bar{E}) \cap D$ we conclude that $E \cap\{\operatorname{Re} z<a\} \neq \emptyset$. This is a contradiction. (An argument of this type is used in[3].)

From (2.2) it follows that

$$\bar{E}=\operatorname{inv} \beta[\beta(\bar{E})] \subset \operatorname{inv} \beta(\{\operatorname{Re} z \geqq \beta(a)\}). \tag{2.3}$$

That is, $\bar{E}$ is contained in the closed circular disk with diameter $[a, b^{-1}]$. Since b is restricted only by the condition $0<b<1$, it is immediate that $\bar{E}$ is contained in the closed circular disk with diameter $[a, 1]$.

The proof of Lemma 2.1 is completed.

It is clear from the developments of this section that if E is G-convex, then $\operatorname{int} E$ and $\bar{E} \cap\{|z|<1\}$ both have the property S. Consequently, in the case where E contains more than one point, E is intermediate to an open set having the property S (namely $\operatorname{int} E$) and its relative closure.

Theorem 2.1 gives an easy access to the next theorem, which is of central interest in the investigation of Study sets.

Theorem 2.2. *E has the property* S *if and only if it has the property* I.

Proof. We need show only "S" $\rightarrow$ "I". The converse proposition is just observation (2) of section 1. Let E be given having the property S. Then E is G-convex. Putting aside the trivial cases (E empty, a singleton or the open unit disk) we consider the representation (2.1) of E and let D be an intermediate set of the constructed type (i.e. of form $E \cup \operatorname{int} \delta$). $D-\operatorname{int} D$ is connected since E has the property P. Hence D is elementary. Theorem 2.2 follows.

The proof of the following theorem is left to the reader. It may be based on Theorem 2.1 and the definition of A.

THEOREM 2.3. *E is G-convex if and only if* $\{a,b\} \cup \operatorname{int} A \subset E$, *when* $a, b \in E$, *with* $a \neq b$, *A being the set associated with the points a and b in section* 1.

3. A canonical representation theorem.

This section develops a representation theorem for sets having the property S which draws its inspiration from the Minkowskian notion of a support function. For convenience we put aside the trivial cases of the empty set and $\{|z| < 1\}$. Any other set having the property S whose closure contains a point of $\{|z| = 1\}$ is necessarily *elementary* as we see by appeal to Theorem 2.2. We put this case aside as well.

Given, then, a set X having the property S, which is not $\emptyset, \{|z| < 1\}$, nor elementary, and a point η of the unit circumference, we note that there is a (necessarily unique) elementary set $H(\eta)$ satisfying:

(i) it contains X,

(ii) its frontier is tangent to $\{|z| = 1\}$ at η,

(iii) it is contained in all elementary sets satisfying (i) and (ii).

Indeed, introducing for each η of the unit circumference u_η, the normalized minimal positive harmonic function on $\{|z| < 1\}$ given by

$$u_\eta(z) = \operatorname{Re}[(\eta + z)/(\eta - z)], \tag{3.1}$$

and μ defined on the unit circumference by

$$\mu(\eta) = \inf u_\eta(X), \tag{3.2}$$

we verify that $\mu(\eta) > 0$ and

$$H(\eta) = \{u_\eta(z) > \mu(\eta)\} \cup [X \cap \{u_\eta(z) = \mu(\eta)\}]. \tag{3.3}$$

Further

$$X = \bigcap_{|\eta|=1} H(\eta), \tag{3.4}$$

as we see with the aid of Theorem 2.2. The representation (3.4) is of Minkowskian kind. The sets $\{u_\eta(z) \geqq c(>0)\}$ take over the role of the half-planes of the Minkowskian theory. We remark that the function μ is continuous thanks to the uniform continuity of $(z,\eta) \mapsto u_\eta(z)$, $z \in X$, $|\eta| = 1$. We now consider the following problem in the Minkowskian spirit for sets having the property S:

Characterize the maps $\eta \mapsto \mu(\eta), \eta \mapsto H(\eta)$, $|\eta| = 1$.

We proceed to obtain necessary conditions on μ and H which will be seen to be characteristic.

CONDITION ON μ. Let $\eta_1, \ldots, \eta_n$ be $n(\geqq 2)$ distinct points of $\{|z|=1\}$. Let

$$Y = \bigcap_{1 \leqq k \leqq n} \{u_{\eta_k}(z) \geqq \mu(\eta_k)\} . \tag{3.5}$$

Then Y is compact and contains X. Further

$$\min u_\eta(Y) \leqq \mu(\eta) \leqq \max u_\eta(Y) , \tag{3.6}$$

as follows at once from the definition of μ and the noted properties of Y.

NOTATION. ∂A will denote $A - \operatorname{int} A$.

CONDITION ON H. The following condition on H is obvious.

$$\partial H(\eta) \subset H(\zeta) , \tag{3.7}$$

where η and ζ are unrestricted points of the unit circumference.

We shall now see that the condition that $Y \neq \emptyset$ and (3.6) hold is characteristic for μ, and that the conditions $\bigcap H(\eta) \neq \emptyset$ and (3.7) together with the characterizing condition for μ are characteristic for H.

DEFINITION OF Φ. We introduce the class Φ consisting of the positive finite-valued functions φ on $\{|z|=1\}$ satisfying the conditions that the set Z obtained by replacing μ by φ in (3.5) is not empty and that (3.6) holds with Y and μ replaced respectively by Z and φ. Clearly

$$\bigcap_{|\eta|=1} \{u_\eta(z) \geqq \varphi(\eta)\} \tag{3.8}$$

is compact and has the property S. It is *non-empty* by virtue of the Heine-Borel-Lebesgue theorem since each Z is not empty. We shall see that φ is the μ associated with (3.8). In particular, by the remarked continuity of the μ it will follow that the function φ is continuous.

DEFINITION OF $\Lambda(\varphi)$. Given $\varphi \in \Phi$, we introduce $\Lambda(\varphi)$, the set of maps λ from $\{|z|=1\}$ into the family of elementary sets satisfying:

(a) For each $\eta, |\eta|=1, \lambda(\eta)$ is an elementary set intermediate to $\{u_\eta(z) > \varphi(\eta)\}$ and $\{u_\eta(z) \geqq \varphi(\eta)\}$.

(b) $\bigcap_{|\eta|=1} \lambda(\eta) \neq \emptyset$.

(c) $\partial[\lambda(\eta)] \subset \lambda(\zeta)$, where η and ζ are unrestricted points of $\{|z|=1\}$.

Let $\mathfrak{S}$ ("$\mathfrak{S}$" for "Study") denote the family of sets having the property S which are non-empty subsets of compact subsets of $\{|z|<1\}$, that is, the subfamily of the family of sets having the property S to which attention was confined at the beginning of this section. Let μ_X denote the function μ with domain $\{|z|=1\}$ associated with $X\in\mathfrak{S}$ by (3.2) and let H_X denote the function H with domain $\{|z|=1\}$ associated with $X\in\mathfrak{S}$ by (3.3). We have (canonical representation theorem):

THEOREM 3.1. *$X\mapsto H_X$ is a bijective map of $\mathfrak{S}$ onto $\bigcup_{\varphi\in\Phi}\Lambda(\varphi)$ whose inverse is given by*

$$\lambda\mapsto\bigcap_{|\eta|=1}\lambda(\eta)\,.$$

COROLLARY 3.1. (*Characterization of the H.*)

$$\{H_X:\ X\in\mathfrak{S}\}=\bigcup_{\varphi\in\Phi}\Lambda(\varphi)\,.$$

That is, the H are the maps λ of the unit circumference into the family of elementary sets which satisfy (b) *and* (c) *and for which there exists $\varphi\in\Phi$ such that* (a) *is satisfied.*

PROOF OF THEOREM 3.1. It is clear that $X\mapsto H_X$ maps $\mathfrak{S}$ *into* $\bigcup_{\varphi\in\Phi}\Lambda(\varphi)$. That it is *injective* then follows from the observation that if $H_X=H_Y$, then

$$X=\bigcap_{|\eta|=1}H_X(\eta)=\bigcap_{|\eta|=1}H_Y(\eta)=Y\,.$$

The following considerations yield the *surjectivity* of $X\mapsto H_X$. We consider $\lambda\in\bigcup_{\varphi\in\Phi}\Lambda(\varphi)$ and let ψ denote the unique member of Φ such that $\lambda\in\Lambda(\psi)$. From the observations made when the class Φ was defined we see that

$$X_0=\bigcap_{|\eta|=1}\{u_\eta(z)\geqq\psi(\eta)\}\in\mathfrak{S}\,.$$

Further, as we shall see, $\mu_{X_0}=\psi$. Indeed, it is evident that $\mu_{X_0}\geqq\psi$. To show that $\mu_{X_0}\leqq\psi$, we fix $\zeta, |\zeta|=1$, and show that $\mu_{X_0}(\zeta)\leqq\psi(\zeta)$. Now

$$\{u_\zeta(z)=\psi(\zeta)\}\cap\bigcap_{\eta\neq\zeta}\{u_\eta(z)\geqq\psi(\eta)\}\neq\emptyset\,.$$

Otherwise, using the compactness of $\{|z|\leqq 1\}$, we would conclude that (3.6) with Y and μ replaced respectively by Z and ψ would not be true for all allowed Z. Hence

$$X_0\cap\{u_\zeta(z)=\psi(\zeta)\}\neq\emptyset$$

and so $\mu_{X_0}(\zeta)\leqq\psi(\zeta)$ as we wished to show. It now follows that $\mu_{X_0}=\psi$.

We pause to note that the result of the preceding paragraph concerning ψ is valid for an arbitrary member φ of Φ. That is, $\mu_{X_0}=\varphi$ with φ replacing ψ in the definition of X_0. Hence we conclude (1) *membership in Φ is characteristic for the μ*, and (2) *the members of Φ are continuous*, as was noted when Φ was introduced.

We return to the main argument and show

$$\operatorname{int} X_0 = \bigcap_{|\eta|=1}\{u_\eta(z) > \psi(\eta)\}.$$

It suffices to show that the set of the right side is *open*. The proof can be referred to that given in the seventh paragraph after the statement of Lemma 2.1 with appropriate modifications. The following demonstration is direct. If a is a member of the set in question, then using the continuity of ψ, we see that

$$\eta \mapsto u_\eta(a) - \psi(\eta)$$

is a continuous, positive-valued function on the unit circumference, and hence that it has a positive minimum. Consequently, by uniform continuity, there exists a neighborhood of a such that for z in this neighborhood and $|\eta|=1, u_\eta(z)>\psi(\eta)$, so that a is an inner point of

$$\bigcap_{|\eta|=1}\{u_\eta(z) > \psi(\eta)\},$$

which is therefore *open*, a being an arbitrary member.

The set $\bigcap_{|\eta|=1}\lambda(\eta)$ will be denoted by X_1. We shall show that $H_{X_1}=\lambda$. The surjectivity of $X \mapsto H_X$ and the fact that the preimage of λ is $\bigcap_{|\eta|=1}\lambda(\eta)$ follow. It is evident that $\operatorname{int} X_0 \subset X_1 \subset X_0$. Further $X_1 \neq \emptyset$ by condition (b) on λ. We conclude that $X_1 \in \mathfrak{S}$. If $\operatorname{int} X_0=\emptyset, X_0$ reduces to a singleton thanks to the property P and so $X_1=X_0$. In this case $\mu_{X_1}=\psi$. If $\operatorname{int} X_0 \neq \emptyset$, then $\mu_{\operatorname{int} X_0}=\mu_{X_0}$. This follows from the inclusions

$$\operatorname{int} X_0 \subset X_0 \subset \overline{\operatorname{int} X_0},$$

the right inclusion being a consequence of the fact that X_0 is convex and $\operatorname{int} X_0 \neq \emptyset$. We now infer that $\mu_{X_1}=\mu_{X_0}=\psi$.

To complete the proof we proceed as follows. At all events,

$$\operatorname{int}\lambda(\eta) \subset H_{X_1}(\eta) \subset \lambda(\eta), \quad |\eta| = 1. \tag{3.9}$$

Now $\partial\lambda(\eta) \subset \lambda(\zeta)$ by condition (c) on the λ and hence $\partial\lambda(\eta) \subset X_1$. Obviously

$$\partial\lambda(\eta) \subset \{u_\eta(z) = \psi(\eta)\}.$$

These facts and (3.9) yield $\lambda=H_{X_1}$.

The proof of Theorem 3.1 is complete.

4. A property of H_X.

Given $X \in \mathfrak{S}$, the associated set H_X enjoys the property stated in the following theorem.

THEOREM 4.1. *Given* $|a|<1$, *the set*

$$K = \{\eta : a \in \partial H_X(\eta)\} \tag{4.1}$$

is a closed connected subset of the unit circumference.

Before turning to the proof we make two remarks. First, $K \neq \emptyset$ if and only if $a \in \partial X$. Indeed, if $K \neq \emptyset$, $a \in \partial H_X(\eta)$ for some $\eta, |\eta|=1$, and so $a \in \partial X$, while the converse follows from the fact that if $a \in \partial X$, then

$$a \notin \operatorname{int} X = \bigcap_{|\eta|=1} \operatorname{int} H_X(\eta) ,$$

so that for some η we have

$$a \in H_X(\eta) - \operatorname{int} H_X(\eta) = \partial H_X(\eta) ,$$

and thus $K \neq \emptyset$. Second, as is readily verified, K is the full circumference if and only if $X=\{a\}$.

PROOF OF THEOREM 4.1. *K connected.* We show that if η_1 and η_2 are two distinct points of K, then they belong to an arc of the unit circumference lying in K. The connectedness of K is then immediate. Under the hypothesis just made K is not empty.

If the sets $\{u_{\eta_k}(z)=\mu(\eta_k)\}, k=1,2$, have only a in common, then $X=\{a\}$ and, consequently, K is the unit circumference. There is no need of further consideration of this case.

Otherwise, let b denote the other point of intersection of $\{u_{\eta_k}(z)= \mu(\eta_k)\}$, $k=1,2$. Let ω denote the open arc of $\{|z|=1\}$ with endpoints η_1 and η_2 which abuts on the component of

$$\{|z| < 1\} - \bigcup_{k=1}^{2} \{u_{\eta_k}(z) = \mu(\eta_k)\} \tag{4.2}$$

which does not have a as a frontier point. Let Y denote

$$\bigcap_{k=1}^{2} \{u_{\eta_k}(z) \geqq \mu(\eta_k)\} .$$

We start by showing

$$\min u_\eta(Y) = u_\eta(a), \quad \eta \in \omega . \tag{4.3}$$

To that end, we introduce

$$\alpha(z) = \eta(z-1)/(z+1), \quad \operatorname{Re} z > 0 ,$$

and note that $u_\eta \circ \alpha(z) = \operatorname{Re} z$. Hence

$$\begin{aligned}\min u_\eta(Y) &= \min\{\operatorname{Re} z : z \in \alpha^{-1}(Y)\} \\ &= \min\{\operatorname{Re}[\operatorname{inv}\alpha(a)], \operatorname{Re}[\operatorname{inv}\alpha(b)]\}\end{aligned}$$

(minimum principle applied to $\operatorname{Re} z$ and argument used in the proof of Lemma 1.1.). Now thanks to the definition of α we have

$$\operatorname{Re}[\operatorname{inv}\alpha(a)] \neq \operatorname{Re}[\operatorname{inv}\alpha(b)] .$$

Otherwise, three distinct oricycles would contain a and b. If

$$\operatorname{Re}[\operatorname{inv}\alpha(a)] > \operatorname{Re}[\operatorname{inv}\alpha(b)] ,$$

then μ_η would be unbounded on the component Ω of (4.2) which does not have b as a frontier point since $u_\eta \circ \alpha(z) = \operatorname{Re} z$ and the image with respect to α of

$$\{\operatorname{inv}\alpha(a) + t[\operatorname{inv}\alpha(a) - \operatorname{inv}\alpha(b)] : t > 0\}$$

would lie in Ω. But u_η is bounded on Ω. We conclude that

$$\operatorname{Re}[\operatorname{inv}\alpha(a)] < \operatorname{Re}[\operatorname{inv}\alpha(b)] .$$

Thus we obtain (4.3). It is routine to conclude

$$u_\eta(a) = \min u_\eta(Y) \leqq \inf u_\eta(X) = \mu(\eta) .$$

Since $a \in X$, $u_\eta(a) \geqq \mu(\eta)$. Consequently, $u_\eta(a) = \mu(\eta)$. Hence $\eta \in K$. We conclude that $\omega \subset K$. The asserted property of η_1, η_2 follows.

K closed. We may put aside the case where $a \notin \partial X$ for then K is empty. With $a \in \partial X$,

$$K = \{\eta : u_\eta(a) = \mu(\eta)\} ,$$

which is closed by the continuity of $\eta \mapsto u_\eta(a) - \mu(\eta)$.

The proof of Theorem 4.1 is completed.

Acknowledgement.

I wish to thank Professor Maxwell Reade for bringing the historical facts in question to my attention and to express my regrets that I was unaware of the work of Pommerenke when my paper [3] was being prepared. It should also be noted that Aumann considered a cognate question in his paper [1], as was kindly pointed out to me by Dr. Herold of the University of Würzburg (cf. my Zentralblatt Autorreferat of [3]). Although Aumann did not treat the problem of characterizing sets having the property S, his result is relevant and can be made to bear on the question.

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