MATH. SCAND. 32 (1973), 258–264

ON AN ASYMPTOTIC FORMULA OF RAMANUJAN

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1. Introduction.

Let $\tau(n)$ denote the number of divisors of a positive integer n. In 1915 S. Ramanujan (cf. [6], (3)) stated without proof the following asymptotic formula:

$$(1.1)\qquad \sum_{n\leq x}\tau^2(n) = Ax\log^3x+Bx\log^2x+Cx\log x+Dx+O(x^{\frac{3}{5}+\varepsilon}),$$

for every $\varepsilon>0$, where $A=\pi^{-2}$, $B=(12\gamma-3)\pi^{-2}-36\pi^{-2}\zeta'(2)$, etc., γ being Euler's constant, $\zeta'(2)$ is the derivative of the Riemann Zeta function $\zeta(s)$ at $s=2$. He also stated that the order of the error term in (1.1) may be improved to $O(x^{\frac{1}{2}+\varepsilon})$, on the assumption of the Riemann hypothesis. In 1922, B. M. Wilson [10] gave a proof of (1.1) with error term $O(x^{\frac{1}{2}+\varepsilon})$ without assuming any hypothesis.

The object of the present paper is to further improve the order of the error term (denoted throughout the paper by $E(x)$) in (1.1).

Let $\tau_4(n)$ denote the number of representations of n in the form $n=d_1d_2d_3d_4$ and let α denote the number which appears in the divisors problem for $\tau_4(n)$, namely

$$(1.2)\qquad \sum_{n\leq x}\tau_4(n) = ax\log^3x+bx\log^2x+cx\log x+dx+O(x^{\alpha}),$$

where $a=\frac{1}{6}$, $b=2\gamma-\frac{1}{2}$, etc.

The formula (1.2) was originally obtained in 1881 by A. Piltz [5] with error term equal to $O(x^{\frac{3}{4}}\log^2x)$. In 1912, E. Landau [4] proved that $\alpha=\frac{3}{5}+\varepsilon$ for every $\varepsilon>0$ and this result was improved further in 1922 by G. H. Hardy and J. E. Littlewood [2] to $\alpha=\frac{1}{2}+\varepsilon$. On the other hand, G. H. Hardy [1] in 1915 proved that $\alpha\geqq\frac{3}{8}$. There is a conjecture (cf. [8, p. 270]) that $\alpha=\frac{3}{8}+\varepsilon$. If this conjecture were true, then it would follow that $\alpha<\frac{1}{2}$. For a discussion about the divisor problem for $\tau_4(n)$, we refer to E. C. Titchmarch (cf. [8, theorem 12.3 and theorem 12.6(B)]).

Through out the paper we assume that the number α appearing in (1.2) is strictly less than $\frac{1}{2}$. With this assumption we prove in this paper that

$$E(x) = O(x^{\frac{1}{2}}\exp\{-A\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\}),$$

Received May 30, 1972.

where A is a positive constant. Further, on the assumption of the Riemann hypothesis, we prove that

$$E(x) = O(x^{(2-\alpha)/(5-4\alpha)} \exp\{A \log x (\log\log x)^{-1}\}) ,$$

where A is a positive constant.

2. Preliminaries.

In this section, we prove some lemmas which are needed in our present discussion. Throughout the following x denotes a real variable $\geqq 3$. We need the following best known estimate concerning the Möbius function $\mu(n)$ obtained by A. Walfisz (cf. [9; Satz 3, p. 191]).

LEMMA 2.1.

$$M(x) = \sum_{n\leqq x}\mu(n) = O(x\delta(x)) , \tag{2.1}$$

where

$$\delta(x) = \exp\{-A \log^{\frac{3}{5}} x (\log\log x)^{-1}\} , \tag{2.2}$$

A being a positive constant.

LEMMA 2.2. *For* $s>1$ *and* $r\geqq 0$,

$$\sum_{n\leqq x} n^{-s}\mu(n)\log^r n = (-1)^r\eta^{(r)}(s) + O(x^{-(s-1)}\delta(x)\log^r x) , \tag{2.3}$$

where

$\eta^{(0)}(s) = \eta(s) = \zeta(s)^{-1}$ *and* $\eta^{(r)}(s)$ *for* $r\geqq 1$ *denotes the rth derivative of* $\eta(s)=\zeta(s)^{-1}$.

PROOF. From the well-known formula (cf. [3, theorem 287]),

$$\sum_{n=1}^{\infty} n^{-s}\mu(n) = \zeta(s)^{-1} = \eta(s) ,$$

we have

$$\sum_{n=1}^{\infty} n^{-s}\mu(n)\log^r n = (-1)^r\eta^{(r)}(s) ,$$

so that

$$\sum_{n\leqq x} n^{-s}\mu(n)\log^r n = (-1)^r\eta^{(r)}(s) - \sum_{n>x} n^{-s}\mu(n)\log^r n .$$

Putting $f(n)=n^{-s}\log^r n$, it can be easily shown that

$$f(n+1)-f(n) = O(n^{-(s+1)}\log^r n) .$$

Therefore by partial summation and (2.1),

$$\begin{aligned}\sum_{n>x}\mu(n)f(n) &= -M(x)f([x]+1)-\sum_{n>x}M(n)\{f(n+1)-f(n)\}\\ &= O\big(x^{-(s-1)}\delta(x)\log^r x\big)+O\big(\textstyle\sum_{n>x}n^{-s}\delta(n)\log^r n\big)\\ &= O\big(x^{-(s-1)}\delta(x)\log^r x\big)+O\big(\delta(x)\textstyle\sum_{n>x}n^{-s}\log^r n\big)\\ &= O\big(x^{-(s-1)}\delta(x)\log^r x\big)+O\big(x^{-(s-1)}\delta(x)\log^r x\big)\\ &= O\big(x^{-(s-1)}\delta(x)\log^r x\big)\,.\end{aligned}$$

Hence the lemma follows.

LEMMA 2.3. (*Cf.* [8, *theorem* 14–26(*A*), *p.* 316]). *If the Riemann hypothesis is true, then*

(2.4) $$M(x) = \sum_{n\leq x}\mu(n) = O(x^{\frac12}\omega(x))\,,$$

where

(2.5) $$\omega(x) = \exp\{A\log x(\log\log x)^{-1}\}\,,$$

A being a positive constant.

LEMMA 2.4. *If the Riemann hypothesis is true, then for $s>1$,*

(2.6) $$\sum_{n\leq x}n^{-s}\mu(n)\log^r n = (-1)^r\eta^{(r)}(s)+O\big(x^{\frac12-s}\omega(x)\log^r x\big)\,.$$

PROOF. Following the same argument adopted in lemma 2.2, we get this lemma by making use of (2.4) instead of (2.1). In fact, we have only to replace $\delta(x)$ in lemma 2.2 by $x^{-\frac12}\omega(x)$.

3. Main results.

In this section, we first prove a lemma and then prove the results mentioned in the introduction.

LEMMA 3.1. $\tau^2(n) = \sum_{d^2|n}\mu(d)\tau_4(n/d^2)\,.$

PROOF. Since $\mu(n)$ and $\tau_4(n)$ are multiplicative it follows (cf. [7, lemma 2.4]) that the function on the right is a multiplicative function of n. Since $\tau^2(n)$ is also multiplicative, it is enough, if we verify the identity for $n=p^a$, where p is a prime and $a\geq 1$. We note that

$$\tau_4(p^a) = (a+1)(a+2)(a+3)/6$$

(cf. [8, (1.2.6), p. 5]). We have

$$\sum_{d^2|p}\mu(d)\tau_4(p/d^2) = \mu(1)\tau_4(p) = 4\,,$$

and for $a \geqq 2$,

$$\sum_{d^2|p^a}\mu(d)\tau_4(p^a/d^2) = \mu(1)\tau_4(p^a)+\mu(p)\tau_4(p^{a-2}) = (a+1)(a+2)(a+3)/6-(a-1)a(a+1)/6 = (a+1)^2 .$$

Hence the lemma follows.

THEOREM 3.1. *For* $x \geqq 3$,

$$(3.1)\quad \sum_{n\leqq x}\tau^2(n) = ax\log^3 x/\zeta(2)+(b/\zeta(2)+6a\eta^{(1)}(2))x\log^2 x+ +(c/\zeta(2)+4b\eta^{(1)}(2)+12a\eta^{(2)}(2))x\log x+ +(d/\zeta(2)+2c\eta^{(1)}(2)+4b\eta^{(2)}(2)+8a\eta^{(3)}(2))x+ +E(x) ,$$

where $E(x)=O(x^{\frac{1}{2}}\delta(x))$, $\delta(x)$ *being given by* (2.2), a,b,c,d *are constants in the asymptotic formula* (1.2) *and* $\eta^{(r)}(s)$ *is the rth derivative of* $\eta(s)=\zeta(s)^{-1}$ *at* $s=2$ *for* $r=1,2,3$.

PROOF. In virtue of lemma 3.1 above, we have

$$(3.2)\quad \sum_{n\leqq x}\tau^2(n) = \sum_{n\leqq x}\sum_{d^2\delta=n}\mu(d)\tau_4(\delta) = \sum_{d^2\delta\leqq x}\mu(d)\tau_4(\delta) ,$$

the summation being extended over all ordered pairs (d,δ) such that $d^2\delta \leqq x$.

Let $z=x^{\frac{1}{2}}$. Further, let $0<\varrho=\varrho(x)<1$, where the function ϱ will be suitably chosen later. From (3.2), we have

$$\sum_{n\leqq x}\tau^2(n) = \sum_{n^2r\leqq x}\mu(n)\tau_4(r) .$$

If $n^2r\leqq x$, then both $n\geqq \varrho z$ and $r\geqq \varrho^{-2}$ can not simultaneously hold good and so we have

$$\sum_{n\leqq x}\tau^2(n) = \sum_{\substack{n^2r\leqq x\\ n\leqq \varrho z}}\mu(n)\tau_4(r)+\sum_{\substack{n^2r\leqq x\\ r\leqq \varrho^{-2}}}\mu(n)\tau_4(r)-\sum_{\substack{n\leqq \varrho z\\ r\leqq \varrho^{-2}}}\mu(n)\tau_4(r)$$

$$(3.3)\quad = S_1+S_2-S_3 ,$$

say. Now, by (1.2),

$$S_1 = \sum_{\substack{n^2r\leqq x\\ n\leqq \varrho z}}\mu(n)\tau_4(r) = \sum_{n\leqq \varrho z}\mu(n)\sum_{r\leqq xn^{-2}}\tau_4(r)$$
$$= \sum_{n\leqq \varrho z}\mu(n)\{axn^{-2}\log^3(xn^{-2})+bxn^{-2}\log^2(xn^{-2})+cxn^{-2}\log(xn^{-2})+ +dxn^{-2}+O((xn^{-2})^{\alpha})\}$$
$$= (ax\log^3 x+bx\log^2 x+cx\log x+dx)\sum_{n\leqq \varrho z}n^{-2}\mu(n)- -2x(3a\log^2 x+2b\log x+c)\sum_{n\leqq \varrho z}n^{-2}\mu(n)\log n+ +4x(3a\log x+b)\sum_{n\leqq \varrho z}n^{-2}\mu(n)\log^2 n- -8ax\sum_{n\leqq \varrho z}n^{-2}\mu(n)\log^3 n+O(x^{\alpha}\sum_{n\leqq \varrho z}n^{-2\alpha}) .$$

Since $0 < 2\alpha < 1$, by our assumption, we have

$$x^{\alpha} \sum_{n \leqq \varrho z} n^{-2\alpha} = O\left(x^{\alpha}(\varrho z)^{1-2\alpha}\right) = O(\varrho^{1-2\alpha} z) .$$

Hence applying lemma 2.2 for $r = 0, 1, 2, 3$ and $s = 2$, we get that

$$\begin{aligned} S_1 = {} & (ax\log^3 x + bx\log^2 x + cx\log x + dx)\{\zeta(2)^{-1} + O(\delta(\varrho z)/\varrho z)\} - \\ & - 2x(3a\log^2 x + 2b\log x + c)\{-\eta^{(1)}(2) + O\left(\delta(\varrho z)\log(\varrho z)/\varrho z\right)\} + \\ & + 4x(3a\log x + b)\{\eta^{(2)}(2) + O\left(\delta(\varrho z)\log^2(\varrho z)/\varrho z\right)\} - \\ & - 8ax\{-\eta^{(3)}(2) + O\left(\delta(\varrho z)\log^3(\varrho z)/\varrho z\right)\} + \\ & + O(\varrho^{1-2\alpha} z) . \end{aligned}$$

$$\begin{aligned} (3.4) \qquad = {} & ax\log^3 x/\zeta(2) + \left(b/\zeta(2) + 6a\eta^{(1)}(2)\right)x\log^2 x + \\ & + \left(c/\zeta(2) + 4b\eta^{(1)}(2) + 12a\eta^{(2)}(2)\right)x\log x + \\ & + \left(d/\zeta(2) + 2c\eta^{(1)}(2) + 4b\eta^{(2)}(2) + 8a\eta^{(3)}(2)\right)x + \\ & + O\left(\varrho^{-1} z\delta(\varrho z)\log^3 z\right) + O(\varrho^{1-2\alpha} z) . \end{aligned}$$

We have

$$\begin{aligned} S_2 &= \sum_{\substack{n^2 r \leqq x \\ r \leqq \varrho^{-2}}} \mu(n)\tau_4(r) = \sum_{r \leqq \varrho^{-2}} \tau_4(r) \sum_{n \leqq (x/r)^{\frac{1}{2}}} \mu(n) = \sum_{r \leqq \varrho^{-2}} \tau_4(r) M\left((x/r)^{\frac{1}{2}}\right) \\ &= O\left(x^{\frac{1}{2}} \sum_{r \leqq \varrho^{-2}} \tau_4(r) r^{-\frac{1}{2}} \delta\left((x/r)^{\frac{1}{2}}\right)\right) , \end{aligned}$$

by (2.1). Since $\delta(x)$ is monotonic decreasing and $(x/r)^{\frac{1}{2}} > \varrho z$, we have $\delta\left((x/r)^{\frac{1}{2}}\right) \leqq \delta(\varrho z)$. Also, by (1.2),

$$\sum_{r \leqq \varrho^{-2}} \tau_4(r) r^{-\frac{1}{2}} = O\left(\varrho^{-1} \log^3(\varrho^{-2})\right) .$$

Hence

$$(3.5) \qquad S_2 = O\left(\varrho^{-1} z\delta(\varrho z)\log^3(1/\varrho)\right) .$$

Also, we have by (2.1) and (1.2),

$$\begin{aligned} S_3 &= \sum_{\substack{n \leqq \varrho z \\ r \leqq \varrho^{-2}}} \mu(n)\tau_4(r) = \sum_{r \leqq \varrho^{-2}} \tau_4(r) M(\varrho z) \\ &= O\left(\varrho^{-2} \log^3(\varrho^{-2}) \varrho z \delta(\varrho z)\right) \\ (3.6) \qquad &= O\left(\varrho^{-1} z\delta(\varrho z)\log^3(\varrho^{-1})\right) . \end{aligned}$$

Hence by (3.3), (3.4), (3.5) and (3.6), we have

$$\begin{aligned} (3.7) \qquad \sum_{n \leqq x} \tau^2(n) = {} & ax\log^3 x/\zeta(2) + \left(b/\zeta(2) + 6a\eta^{(1)}(2)\right)x\log^2 x + \\ & + \left(c/\zeta(2) + 4b\eta^{(1)}(2) + 12a\eta^{(2)}(2)\right)x\log x + \\ & + \left(d/\zeta(2) + 2c\eta^{(1)}(2) + 4b\eta^{(2)}(2) + 8a\eta^{(3)}(2)\right)x + \\ & + O\left(\varrho^{-1} z\delta(\varrho z)\log^3 z\right) + O\left(\varrho^{-1} z\delta(\varrho z)\log^3(\varrho^{-1})\right) + \\ & + O(\varrho^{1-2\alpha} z) . \end{aligned}$$

Now, we choose

(3.8) $$\varrho = \varrho(x) = \{\delta(x^{\frac{1}{4}})\}^{\frac{1}{2}},$$

and write

(3.9) $$f(x) = \log^{\frac{3}{5}}(x^{\frac{1}{4}})\{\log\log(x^{\frac{1}{4}})\}^{-\frac{1}{5}} = (\tfrac{1}{4})^{\frac{3}{5}}u^{\frac{3}{5}}(v-\log 4)^{-\frac{1}{5}},$$

where $u = \log x$ and $v = \log\log x$.

(3.10) For $v \geqq 2\log 4$, that is, $u \geqq 16$, $x \geqq e^{16}$,

we have

$$v^{-\frac{1}{5}} \leqq (u-\log 4)^{-\frac{1}{5}} \leqq (\tfrac{1}{2}v)^{-\frac{1}{5}},$$

so that

(3.11) $$\tfrac{1}{2}(\tfrac{1}{2})^{\frac{3}{5}}u^{\frac{3}{5}}v^{-\frac{1}{5}} \leqq f(x) \leqq (\tfrac{1}{2})^{\frac{3}{5}}u^{\frac{3}{5}}v^{-\frac{1}{5}}.$$

(3.12) We assume without loss of generality that the constant A in (2.2) is less than 1.

By (3.8), (2.2) and (3.9), we have

(3.13) $$\varrho = \exp\{-\tfrac{1}{2}Af(x)\}.$$

By (3.10), we have $(\frac{1}{2})^{\frac{8}{5}}u^{\frac{3}{5}}v^{-\frac{1}{5}} \leqq \frac{1}{4}u$.

Hence by (3.11), (3.12), (3.13) and the above,

$$\varrho \geqq \exp\{-A(\tfrac{1}{2})^{\frac{8}{5}}u^{\frac{3}{5}}v^{-\frac{1}{5}}\} \geqq \exp\{-(\tfrac{1}{2})^{\frac{8}{5}}u^{\frac{3}{5}}v^{-\frac{1}{5}}\}$$
$$\geqq \exp\{-\tfrac{1}{4}u\} = \exp\{-\tfrac{1}{4}\log x\},$$

so that $\varrho \geqq x^{-\frac{1}{4}}$. Hence

(3.14) $$\log(\varrho^{-1}) \leqq \log(x^{\frac{1}{4}}) = O(\log x) \quad \text{and} \quad \varrho z \geqq x^{\frac{1}{4}}.$$

Since $\delta(x)$ is monotonic decreasing, $\delta(\varrho z) \leqq \delta(x^{\frac{1}{4}})$, by (3.8) and so by (3.11) and (3.13), we have

(3.15) $$\varrho^{-1}\delta(\varrho z) \leqq \varrho \leqq \exp\{-\tfrac{1}{2}A(\tfrac{1}{2})^{\frac{8}{5}}u^{\frac{3}{5}}v^{-\frac{1}{5}}\}.$$

Hence by (3.14) and (3.15), the first and second O-terms of (3.7) are each equal to

$$O(x^{\frac{1}{2}}\exp\{-\tfrac{1}{2}A(\tfrac{1}{2})^{\frac{8}{5}}u^{\frac{3}{5}}v^{-\frac{1}{5}}\}\log^3 x)$$

which is

$$O(x^{\frac{1}{2}}\exp\{-\tfrac{1}{2}A(1-2\alpha)(\tfrac{1}{2})^{\frac{8}{5}}u^{\frac{3}{5}}v^{-\frac{1}{5}}\}),$$

since $0 < 1-2\alpha < 1$, by our assumption.

By (3.13) and (3.11), we see that the third O-term in (3.7) is also of the above order. Thus, if $E(x)$ denotes the sum of the three error terms in (3.7), we have

(3.16) $$E(x) = O(x^{\frac{1}{2}}\exp\{-B\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\}),$$

where B is a positive constant.

THEOREM 3.2. *If the Riemann hypothesis is true, then the error term* $E(x)$ *in the asymptotic formula for* $\sum_{n\leq x}\tau^2(n)$ *is*

$$O\big(x^{(2-\alpha)/(5-4\alpha)}\omega(x)\big),$$

where α *is the number given by* (1.2) *and* $\omega(x)$ *is given by* (2.5).

PROOF. Following the same procedure adopted in theorem 3.1 and making use of lemma 2.4 for $r=0,\ 1,2,3$ and $s=2$ instead of lemma 2.2 for $r=0,\ 1,2,3$ and $s=2$, we get that

$$(3.17)\quad E(x) = O(\varrho^{-\frac{3}{2}}z^{\frac{1}{2}}\log^3 z\,\omega(\varrho z)) + O(\varrho^{-\frac{3}{2}}z^{\frac{1}{2}}\log^3(\varrho^{-1})\omega(\varrho z)) + O(\varrho^{1-2\alpha}z).$$

Now, choosing $\varrho = z^{-(5-4\alpha)^{-1}}$, we see that $0<\varrho<1, \varrho^{-1}<z$, so that $\log(\varrho^{-1})<\log z$ and

$$\varrho^{-\frac{3}{2}}z^{\frac{1}{2}} = \varrho^{1-2\alpha}z = x^{(2-\alpha)/(5-4\alpha)}.$$

Since $\omega(x)$ is monotonic increasing and $\varrho z<z$, we have $\omega(\varrho z)<\omega(z)$. Hence by (3.17) and the above, we have

$$\begin{aligned} E(x) &= O\big(x^{(2-\alpha)/(5-4\alpha)}\omega(x^{\frac{1}{2}})\log^3 x\big) \\ &= O\big(x^{(2-\alpha)/(5-4\alpha)}\omega(x)\big). \end{aligned}$$

Hence theorem 3.2 follows.

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