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HUREWICZ THEOREMS FOR PAIRS AND SQUARES

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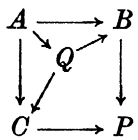
In [3] (with which we must assume familiarity) we proved a relative Hurewicz theorem (Theorem 17) in the form that the fibre of a map is like the co-fibre. Here we prove a more detailed result, and prove a similar result for squares.

We assume all given spaces to have the homotopy types of CW-complexes.

LEMMA 1. *Let $A \to B$, $A \to C$ be m-,n-connected with $m \geqq 2, n \geqq 2$, and form the diagram*



by taking a push-out and a pull-back. Then $A \to Q$ is $(m+n-1)$-connected.

PROOF. $C \to P$ and $B \to P$ are m-,n-connected. (This may be seen by considering B to be obtained from A by adding cells of dimension $\geqq m+1$, and C by cells $\geqq n+1$. Then P is obtained by adding both lots of cells.) And so $A \to P$ is $\min(m,n)$-connected.

By mapping a 1-point space into P and taking all pull-backs, we may assume that $P = *$ and, therefore, $Q = B \times C$. It also follows that all spaces are simply-connected. Now an examination of homology shows that $A \to Q$ is $(m+n-1)$-connected.

THEOREM 2. *Let $A \to B$ have fibre F and co-fibre K. Let the connectivities of F,A,B be f,a,b respectively, and assume $a \geqq 1$ and $b \geqq 1$. Then the fibre of $\Sigma F \to K$ can be obtained from $F * \Omega B$ by attaching cells of dimension $\geqq a+b+f+3$.*

PROOF. By taking one pull-back and three push-outs construct the following diagram:

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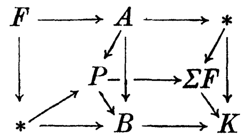


By [2, Theorem 1.1] or [3, Theorem 14] the fibre of $P \to B$ is $F * \Omega B$, so $P \to B$ is $(f+b+2)$-connected. The map $P \to \Sigma F$ is $(a+1)$-connected.

Now

$$\begin{array}{ccc} P & \longrightarrow & \Sigma F \\ \downarrow & & \downarrow \\ B & \longrightarrow & K \end{array}$$

is a push-out. Let Q be the pull-back. Then, by Lemma 1, $P \to Q$ is $(a+b+f+2)$-connected. The fibre of $Q \to B$ is the same as that of $\Sigma F \to K$, so the result follows.

Notation. An arrow $\Rightarrow$ between two similar diagrams will denote that, in the complete diagram, maps have been omitted between corresponding objects, for clarity.

Lemma 3. *Let*

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \\ C' & & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

be homotopy-commutative, and take two push-outs and four co-fibres to form

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ C' & \longrightarrow & P' \end{array} \quad \Rightarrow \quad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & P \end{array} \quad \Rightarrow \quad \begin{array}{ccc} A'' & \longrightarrow & B'' \\ \downarrow & & \downarrow \\ C'' & \longrightarrow & P'' . \end{array}$$

Then the last square is a push-out.

Proof. We may assume that the original diagram is strictly commutative and the maps are co-fibrations. Then the push-outs and co-fibres may be taken in the topological category. The result is then obvious.

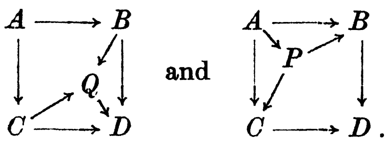
Pre-amble. Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & D \end{array}$$

be a (homotopy-commutative) square. Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & Q \end{array} \qquad \text{and} \qquad \begin{array}{ccc} P & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & D \end{array}$$

be the push-out and pull-back. We get diagrams



Let $F,G,H,$ be the fibres of $A \to P$, $B \to D$, $C \to D$ respectively, and K the co-fibre of $Q \to D$. By definition, the homotopy groups of the original square are those, in two dimensions lower, of F, and the homology groups are those of K. Let D,F,G,H,P,Q be d,f,g,h,p,q-connected.

Theorem 4. *There is a homotopy-commutative square*

$$\begin{array}{ccc} G * H & \longrightarrow & \Sigma\overline{F} \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & K \end{array}$$

in which:

(i) *$\overline{F}$ is obtained from ΣF by adding cells in dimensions $\geqq f+p+3$*

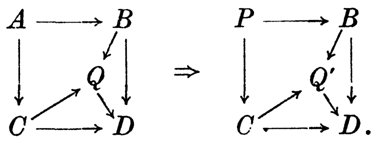
(i) *K is obtained from the push-out by adding cells in dimensions $\geqq g+h+d+5$*

(iii) *the pull-back is obtained from $G*H$ by adding cells in dimensions $\geqq g+h+q+4$.*

PROOF. Let

$$\begin{array}{ccc} P & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & Q' \end{array}$$

be the push-out, and form the diagram



Let $\bar{F}$ be the co-fibre of $A \to P$. Then Lemma 3 shows that the co-fibre of $Q \to Q'$ is $\Sigma\bar{F}$.

From the sequence $Q \to Q' \to D$ we obtain

$$\begin{array}{ccccc} & & G*H & \longrightarrow & * \\ & & \downarrow & & \downarrow \\ Q & \longrightarrow & Q' & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma\bar{F} & \longrightarrow & K \end{array}$$

where the top square is a pull-back and the bottom squares are push-outs. The result follows from Lemma 1 and [3, Theorem 17 and Theorem 14].

COROLLARY 5. *$\Sigma^2 F \to K$ is* $\min(f+p+5, g+h+3)$*-connected.*

This generalises a result of Blakers-Massey [1, Theorem 1] and a result of Namioka [4, Theorem 2.4].

REFERENCES

1. A. L. Blakers and W. S. Massey, *The homotopy groups of a triad*, III. Ann. of Math. 58 (1953), 409–417.
2. T. Ganea, *A generalisation of the homology and homotopy suspension*. Comment. Math. Helv. 39 (1965), 295–322.
3. M. Mather, *Pull-backs in homotopy theory*, to appear in Canad. J. Math.
4. I. Namioka, *Maps of pairs in homotopy theory*. Proc. Lond. Math. Soc. 12 (1962), 725–738.

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