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TWISTED MULTIPLICATIONS ON GENERALIZED EILENBERG-MACLANE SPACES

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In this paper we study the generalized Eilenberg-MacLane space

$$K = K(Z_p,n) \times K(Z_p,pn) \times \ldots \times K(Z_p,p^k n) ,$$

where we assume throughout that $p=2$ or n is even. Let $\alpha_r \in H^{np^r}(K;\, Z_p)$ denote the fundamental class of the rth factor. By [3] it is known that $H^*(K;\, Z_p)$ is the free unstable Steenrod algebra on the classes $\alpha_0,\alpha_1,\ldots,\alpha_k$. Also K can be taken to be an abelian topological group, and as such the structure of $H^*(K;\, Z_p)$ is determined as a Hopf algebra over $\mathscr{A}(p)$ by the condition that $\alpha_0,\alpha_1,\ldots,\alpha_k$ are primitive.

In this paper we consider a twisted H structure on K and compute $H^*(K;\, Z_p)$ as a Hopf algebra over $\mathscr{A}(p)$. For $p=2$ and $k=2$, many of these results can be found on [12] and [10]. Also for $p=2$ and k arbitrary, these results were announced in [6].

In section 1, we describe the sub Hopf algebra $A \subset H^*(K;\, Z_p)$ generated by the fundamental classes $\alpha_0,\ldots,\alpha_k$. In section 2, we examine the multiplication of K at the simplicial level. In section 3, we compute $H^*(K;\, Z_p)$ as a Hopf algebra over the mod p Steenrod algebra.

1.

Let $A=A_k=Z_p[\alpha_0,\alpha_1,\ldots,\alpha_k]$ with $\deg \alpha_r = np^r$. By the results of [8] it is possible to put a Hopf algebra structure A so that the dual A^* will resemble a polynomial algebra.

THEOREM 1.1. *Let* $\alpha_. = (\alpha_0,\alpha_1,\ldots,\alpha_k)$. *There are formal polynomials* f_i, *for* $i=0,\ldots,p^{k+1}-1$, *of* $k+1$ *variables and a bicommutative, biassociative Hopf algebra structure on* A *with coproduct* $\psi : A \to A \otimes A$ *satisfying*

1. $f_0(\alpha_.) = 1$,
2. $f_{p^r}(\alpha_.) = \alpha_r$,
3. $\psi f_i(\alpha_.) = \sum_{j=0}^{i} f_j(\alpha_.) \otimes f_{i-j}(\alpha_.)$.

PROOF. See section 1 of [8].

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For example if $p=2$, then using the fact that ψ is an algebra map, it can be checked that the first six polynomials are

$$f_1(\alpha_.) = \alpha_0, \quad f_2(\alpha_.) = \alpha_1,$$
$$f_3(\alpha_.) = \alpha_1\alpha_0+\alpha_0^3, \quad f_4(\alpha_.) = \alpha_2$$
$$f_5(\alpha_.) = \alpha_2\alpha_0+\alpha_1\alpha_0^3, \quad f_6(\alpha_.) = \alpha_2\alpha_1+\alpha_1^3+\alpha_0^6$$

Unfortunately there is no closed form for these polynomials.

Consider A as a Z_p vector space with basis consisting of monomials in the α_r's. Let $a_i \in A^*$, the dual vector space, be dual basis elements to $\alpha_0^{p^i}$. A^* is, of course, a Hopf algebra.

COROLLARY 1.2. *As an algebra over* Z_p,

$$A^* \approx Z_p[a_1, a_2, \ldots, a_m, \ldots,]/(a_1^{p^{k+1}}, \ldots)$$

that is the polynomial algebra on the a_m*'s truncated at height* p^{k+1}.

PROOF. Using the fact that $\psi^* : A^*\otimes A^* \to A^*$ is the multiplication, it follows easily from Theorem 1.1 that $a_n^{p^r}$ is dual to $\alpha_r^{p^n}$. The result follows from the Borel classification theorem [2].

Note that if we let

$$A_\infty = \cup A_k \approx Z_p[\alpha_0, \alpha_1, \ldots, \alpha_k, \ldots]$$

with the polynomials $f_0, f_1 \ldots$ satisfying the conditions of Theorem 1.1, then $A_\infty \approx A_\infty{}^*$ as Hopf algebras over Z_p. Thus A_∞ is a self dual bipolynomial Hopf algebra.

If B is any algebra, and if $b_. = (b_0, \ldots, b_k)$ is a sequence of elements in B, then we can "evaluate" f_i on $b_.$ to get $f_i(b_.) \in B$. In particular $A\otimes A$ is an algebra and $\psi\alpha_. = (\psi\alpha_0, \ldots, \psi\alpha_k)$ is a sequence of elements in $A\otimes A$.

PROPOSITION 1.3. *For all* $\alpha \in A$

$$f_i(\psi\alpha_.) = \textstyle\sum_{j=0}^{i} f_j(\alpha_.)\otimes f_{i-j}(\alpha_.) .$$

PROOF. This formula is immediate from Theorem 1.1 part 3 and the fact that

$$\psi f_i(\alpha_.) = \textstyle\sum_{j=0}^{i} f_j(\alpha_.)\otimes f_{i-j}(\alpha_.)$$

since ψ is a map of algebras.

2.

In 1954, Eilenberg and MacLane introduced an abelian simplicial group model for the $K(\pi, n)$'s. We will use some standard properties of these models and of simplicial complexes. A good reference is [11].

There is an abelian simplicial group, $K(\mathsf{Z}_p, m)$, whose q simplices form the normalized cocycle group $Z_m(\Delta_q; \mathsf{Z}_p)$ of the q simplex. Face and degeneracy maps are homomorphisms induced by certain simplicial inclusions and projections. If X and Y are simplicial complexes, then $X \times Y$ is the simplicial complex with $(X \times Y)_q = X_q \times Y_q$.

These facts mean that we can take a simplicial model for $\boldsymbol{K}$ with

$$\boldsymbol{K}_q = Z^n(\Delta_q; \mathsf{Z}_p) \times \ldots \times Z^{np^k}(\Delta_q; \mathsf{Z}_p) .$$

Thus a q simplex of $\boldsymbol{K}$ is a $k+1$ tuple $a_. = (a_0, \ldots, a_k)$ of cocycles of Δ_q. A q simplex of $\boldsymbol{K} \times \boldsymbol{K}$ will be a pair $(a_., b_.)$ of such $k+1$ triples.

A simplicial map $\varphi : X \to Y$ is a sequence of functions $\varphi_q : X_q \to Y_q$ which commute with the face and degeneracy maps. The projections

$$\pi_r : \boldsymbol{K} \to K(\mathsf{Z}_p, p^r n)$$

are clearly simplicial maps. Furthermore simplicial maps $\varphi : X \to \boldsymbol{K}$ are completely determined by the composites

$$\pi_r \varphi : X \to K(\mathsf{Z}_p, np^r) .$$

Although we will not use the fact, such maps are in one to one correspondence with elements of $Z^{np^r}(X; \mathsf{Z}_p)$.

Our aim is to describe a simplicial map $\mu : \boldsymbol{K} \times \boldsymbol{K} \to \boldsymbol{K}$; which will induce the desired Hopf algebra structure on $H^*(\boldsymbol{K}; \mathsf{Z}_p)$.

Definition 2.1. Let B be a graded connected associative algebra. Then cB is the two sided ideal of B generated by the commutators

$$[x, y] = xy - (-1)^{ab} yx$$

where $x \in B_a$ and $y \in B_b$.

Clearly B/cB is a commutative algebra and a map from B to another commutative algebra factors through B/cB. For example, let

$$T[X_.] = T[X_0, \ldots, X_k] .$$

the tensor algebra on $k+1$ indeterminates, with $\deg X_r = np^r$, then

$$T[X_.]/cT[X_.] = \mathsf{Z}_p[X_.] ,$$

the polynomial algebra on these elements, remembering that $p=2$ or n is even.

The formal polynomials of Theorem 1.1. can be considered to be elements of $\mathsf{Z}_p[X_.]$. We have an evaluation map (isomorphism) $\mathsf{Z}_p[X_.] \to A$ defined by $f_i \mapsto f_i(\alpha_.)$.

Let $\varrho\colon Z_p[X_.] \to T[X_.]$ be a splitting of Z_p modules. Then there are non commutative polynomials ϱf_i for $0 \leq i < p^{k+1}$.

PROPOSITION 2.2. *In* $T[X_.]$,

$$\varrho f_i(\gamma_0, \ldots, \gamma_k) = \sum_{j=0}^{i} \varrho f_j \otimes \varrho f_{i-j} + \Theta$$

where $\gamma_r = \sum_{i+j=p^r} \varrho f_i \otimes \varrho f_j$ *and* $\Theta \in cT[X_.]$.

PROOF. This equation follows from Proposition 1.3 and general algebra.

For any connected simplicial set $X, Z^*(X; Z_p)$ is a connected, associative algebra with the Alexander Whitney cup product. Since $H^*(X; Z_p)$ is commutative, every element of $c(Z^*(X; Z_p))$ is a coboundary.

Let $a_. = (a_0, \ldots, a_k)$ be a q simplex of K. Then for any $g \in T[X_.]$, we get a cocycle $g(a_.)$ obtained by replacing X_r by a_r and $\otimes$ by $\smile$.

DEFINITION 2.3. The ϱ twisted product on $\boldsymbol{K}$ is the simplicial map

$$\mu = \mu(\varrho) : \boldsymbol{K} \times \boldsymbol{K} \to \boldsymbol{K}$$

defined on q simplices by

$$\pi_r \mu(a_., b_.) = \sum_{i+j=p^r} \varrho f_i(a_.) \smile \varrho f_j(b_.)$$

where $(a_., b_.) \in (\boldsymbol{K} \times \boldsymbol{K})_q, \pi_r\colon \boldsymbol{K} \to K(Z_p, np^r)$ is the projection, and $\varrho\colon Z_p[X_.] \to T[X_.]$ is a splitting over Z_p.

Note that since $\smile$ is natural and since face and degeneracy maps are induced by simplicial maps between simplices, μ is a simplicial map. this multiplication is, of course, quite different from the untwisted product

$$m(a_., b_.) = (a_0 + b_0, \ldots, a_k + b_k) .$$

It is interesting to note, however, that the multiplications have the same unit.

PROPOSITION 2.4. *The* 0 *simplex* $0_. = (0, \ldots, 0) \in \boldsymbol{K}_q$ *is a strict unit for* μ. *That is* $\mu(0_., a_.) = \mu(a_., 0_.) = a_.$.

PROOF. This follows immediately from the fact that $f_i(0_.) = 0$ if $i > 0$ and $f_0(0_.) = 1 \in Z^0(\Delta_q; Z_p)$.

The next proposition states that μ is homotopy commutative and homotopy associative and independent of ϱ up to homotopy. The proof is immediate since the formulas hold after dividing out by $c(Z^*(\Delta_q;\, \mathsf{Z}_p))$.

PROPOSITION 2.5. *For any* $a_.$, $b_.$, *and* $c_.$ *in* K_q, *and any two splittings* ϱ *and* σ *the following cocycles are in* $cZ^*(\Delta_q;\, \mathsf{Z}_p)$:

1. $\mu(\varrho)(a_.,b_.)-\mu(\sigma)(a_.,b_.)$,
2. $\mu(a_.,b_.)-\mu(b_.,a_.)$,
3. $\mu(\mu\times 1)(a_.,b_.,c_.)-\mu(1\times\mu)(a_.,b_.,c_.)$.

Thus μ induces an H structure on $\boldsymbol{K}$ with a strict unit. This means that μ^* induces a Hopf algebra structure on $H^*(\boldsymbol{K};\, \mathsf{Z}_p)$. We will show that under this structure, the sub Hopf algebra

$$A=\mathsf{Z}_p[\alpha_0,\ldots,\alpha_k] \subset H^*(\boldsymbol{K};\, \mathsf{Z}_p)$$

has the structure described in Theorem 1.1.

LEMMA 2.6. *Let*

$$b\in Z^m(\Delta_{m+n};\, \mathsf{Z}_p) = K(\mathsf{Z}_p,m)_{m+n} \quad \textit{and} \quad c\in Z^n(\Delta_{m+n};\, \mathsf{Z}_p) = K(\mathsf{Z}_p,n)_{m+n}.$$

Let $\beta\in Z^m(K(\mathsf{Z}_p,m);\, \mathsf{Z}_p)$, $\gamma\in Z^n(K(\mathsf{Z}_p,n);\, \mathsf{Z}_p)$ *and* $\eta\in Z^{m+n}(K(\mathsf{Z}_p, m+n);\, \mathsf{Z}_p)$ *be fundamental cocycles. Then*

$$\langle\beta\times\gamma,\, b\times c\rangle = \langle\eta,\, b\smile c\rangle$$

and if $m=n$,

$$\langle\beta^2,\, b\rangle = \langle\eta, b\smile b\rangle\, .$$

PROOF. By [5],

$$\langle\eta,\, b\smile c\rangle = \langle b\smile c,\, \Delta_{m+n}\rangle$$

where Δ_q is the standard simplicial q simplex. The evaluation of $\beta\times\gamma$ on $b\times c$ uses the Eilenberg-Zilber formula. The evaluation of $b\smile c$ on Δ_q uses the Alexander-Whitney diagonal formula. Comparison of these two formulae, and a bit of computation, yields the first equation. The second is proved similarly.

THEOREM 2.7. *The coproduct induced by* μ

$$H^*(\boldsymbol{K};\, \mathsf{Z}_p) \xrightarrow{\mu^*} H^*(\boldsymbol{K}\times\boldsymbol{K};\, \mathsf{Z}_p) \xrightarrow{\varkappa} H^*(\boldsymbol{K};\, \mathsf{Z}_p)\otimes H^*(\boldsymbol{K};\, \mathsf{Z}_p)\, ,$$

$\varkappa$ *being an isomorphism, induces a Hopf algebra structure on* $H^*(\boldsymbol{K};\, \mathsf{Z}_p)$ *satisfying*

$$\varkappa^{-1}\mu^* f_i(\alpha_.) = \textstyle\sum_{j=0}^{i} f_j(\alpha_.)\otimes f_{i-j}(\alpha_.)\, .$$

PROOF. Let $\eta_0, \ldots, \eta_k \in Z^*(K; Z_p)$ be the fundamental classes. Clearly it suffices to show that

$$\mu^\#(\eta_r) = \sum \varrho f_i(\eta_.) \times \varrho f_j(\eta_.)$$

in $Z^{p^r n}(K \times K; Z_p)$. Let $(a_., b_.) \in (K \times K)_q$. Then this will follow from

$$\begin{aligned}\langle \mu^\# \eta_r, (a_., b_.)\rangle &= \langle \eta_r, \textstyle\sum \varrho f_i(a_.) \smile \varrho f_j(b_.)\rangle \\ &= \textstyle\sum \langle \varrho f_i(\eta_.) \times \varrho f_j(\eta_.), (a_., b_.)\rangle\end{aligned}$$

These equations in turn follow from the linearity properties of the Kronecker product and repeated use of Lemma 2.5.

Thus we have an explicit H structure defined on the simplicial level of K. This H structure has a strict unit and is associative and commutative up to homotopy.

It is not clear from the above that K has a classifying space. In fact, as we will show in [9] (see also the appendix of [6]), K is at least a $2p-2$ fold loop space for odd primes p.

The deviation from strict associativity and commutativity of K is contained in the ideal $c(Z^*(\Delta_q); \mathrm{Z}_p)$. Thus it should be possible to show that K is a homotopy everything H space in the sense of [1] by showing that the cocycle cup product is strongly homotopy commutative. This will imply that K is an infinite loop space. The author has significant partial results in this direction.

3.

We now compute $H^*(K; \mathrm{Z}_p)$ as a Hopf algebra over $\mathscr{A}(p)$. In constrast with the results in [6], we will work entirely with the Milnor basis for $\mathscr{A}(p)$. Unless explicitly stated to the contrary, we will assume that p is an odd prime and that n is even.

NOTATION 3.1. Let $E = (\varepsilon_0, \varepsilon_1, \ldots)$ and $R = (r_1, r_2, \ldots)$ be sequences of non-negative integers almost all 0 with $\varepsilon_i = 0$ or 1. Then $\mathscr{P}(E, R)$ will be the element in the Milnor basis of $\mathscr{A}(p)$ dual to

$$\tau_0^{\varepsilon_0} \tau_1^{\varepsilon_1} \ldots \xi_1^{r_1} \xi_2^{r_2} \ldots .$$

We write $\mathscr{P}(R)$ for $\mathscr{P}(0, R)$ and $\mathscr{Q}_i$ for $\mathscr{P}(\Delta_{i+1}, 0)$ where $\Delta_{i+1} = (0, 0, \ldots, 1, 0, \ldots)$ with the 1 in the $i+1$'st place. Furthermore set

$$\mathscr{P}_i(r) = \mathscr{P}(r\Delta_j) = \mathscr{P}(0, \ldots, r, 0, \ldots)$$

and $\mathscr{P}_j = \mathscr{P}_j(1)$. Thus $\mathscr{P}_j(r)$ is dual to ξ_j^r.

By [7], the excess of $\mathscr{P}(E,R)$ is $|E,R| = \sum\varepsilon_i + 2\sum r_i$. Thus we can restate Cartan's theorem (see [3]) on the cohomology of $\boldsymbol{K}$ as an algebra over $\mathscr{A}(p)$ as follows.

PROPOSITION 3.2. *$H^*(\boldsymbol{K};\, Z_p)$ is the free commutative Z_p algebra on elements $\mathscr{P}(E,R)\alpha_r$ with $|E,R| < \deg\alpha_r = np^r$.*

Since the sub Hopf algebra $A \subset H^*(\boldsymbol{K};\, Z_p)$ is biassociative and bicommutative, and since $\mathscr{A}(p)$ is biassociative and cocommutative, it is easy to check that $H^*(\boldsymbol{K};\, Z_p)$ is a biassociative and bicommutative Hopf algebra.

By [14, Proposition 4.23], there is an exact sequence with $H^* = H^*(\boldsymbol{K};\, Z_p)$,

$$PH^* \xrightarrow{\xi} PH^* \xrightarrow{\nu} QH^* \xrightarrow{\lambda} QH^*$$

where $\xi(c) = c^p$ is the Frobenius homomorphism, ν is the composite

$$PH^* \to H^* \to QH^*,$$

and λ is the dual of the homology Frobenius homomorphism. Thus $\lambda(x) = y$ if x is the pth divided power of y. For dimension reasons, $\nu : PH^q \to QH^q$ is an isomorphism if $q \not\equiv 0 \pmod{2p}$.

PROPOSITION 3.3. *$\lambda(\alpha_r) = \alpha_{r-1}$ with the convention that $\alpha_{-1} = 0$.*

PROOF. This is immediate from Corollary 1.2 and the definition of λ.

DEFINITION 3.4. If $q \not\equiv 0 \pmod{2p}$ and $c \in H^q$, then $\langle c\rangle \in PH^q$ denotes the unique primitive class such that $\langle c\rangle - c$ is decomposable.

PROPOSITION 3.5. *Let $\Theta \in \mathscr{A}(p)$ and assume that $2p$ does not divide* $\dim c$ *or* $\dim \Theta c$. *Then $\Theta\langle c\rangle = \langle \Theta c\rangle$.*

This follows since H^* is a Hopf algebra over $\mathscr{A}(p)$ and so Θ sends primitives to primitives. If $\dim \Theta c \equiv 0 \pmod{2p}$, then $\Theta\langle c\rangle$ is still primitive but may be a pth power.

Recall that the primitive elements of the Hopf algebra $\mathscr{A}(p)$ are generated as a Z_p module by $\mathscr{Q}_i$ and $\mathscr{P}_j$ for $i \geqq 0$ and $j \geqq 1$. These operations act as derivatives on $H^*(\boldsymbol{K};\, Z_p)$ (see [13]).

PROPOSITION 3.6. *Let $\Theta \in P\mathscr{A}(p)$. Then*

$$\langle \Theta\alpha_r\rangle = \Theta X_r + X_{r-1}^{p-1}\Theta X_{r-1} + \ldots + X_0^{p^r-1}\Theta X_0$$

where $X_t = \alpha_t + g(\alpha_0, \ldots, \alpha_{t-1})$ is the polynomial in A described in (14) [8].

PROOF. See Theorem 11 in [8].

THEOREM 3.7. *$PH^*(K; Z_p)$ is generated as a left $\mathscr{A}(p)$ module by α_0, $\langle \mathscr{Q}_i\alpha_r\rangle$ and $\langle \mathscr{P}_j\alpha_r\rangle$ for $i \geqq 0, j \geqq 1$ and $r = 1, \ldots, k$.*

PROOF. These classes are surely primitive. Assume $x \in H^*(K; Z_p)$ satisfies $\lambda(x) = 0$. We know that λ is a map of $\mathscr{A}(p)$ modules so

$$\lambda(\mathscr{P}(E,R)\alpha_r) = (\lambda\mathscr{P}(E,R))\alpha_{r-1}.$$

Also $\lambda(\mathscr{P}(E,R)) = 0$ unless $E = 0$ and $r_i \equiv 0 \pmod p$ for all i. Furthermore, if $R = (pr_1', pr_2', \ldots)$, then $\lambda\mathscr{P}(0,R) = \mathscr{P}(0,R')$ [15, Proposition 4.3]. Thus $\operatorname{Ker}\lambda$ is the left ideal generated by the $\mathscr{Q}_i$ and $\mathscr{P}_j$'s.

Since the elements $\mathscr{P}(E,R)\alpha_r$ form a Z_p basis for $H^*(K; Z_p)$, it follows that $\langle x\rangle$ is in the left ideal generated by α_0, $\langle \mathscr{Q}_i\alpha_r\rangle$ and $\langle \mathscr{P}_i\alpha_r\rangle$ if $\dim x \not\equiv 0 \pmod{2p}$. If x is a decomposable primitive, then $x = y^p = \mathscr{P}^s y$ for some primitive $y \in H^{2s}(K; Z_p)$. Thus the result holds for all x.

These generators do not generate $PH^*(K; Z_p)$ freely. For example, it is easy to see that the relation $\mathscr{Q}_i\mathscr{Q}_j = -\mathscr{Q}_j\mathscr{Q}_i$ implies $\mathscr{Q}_i\langle \mathscr{Q}_j\alpha_r\rangle = -\mathscr{Q}_j\langle \mathscr{Q}_i\alpha_r\rangle$. To completely describe these relations, we must first examine the structure of the Steenrod algebra more closely.

THEOREM 3.8. *The kernel $\mathscr{A}_\lambda$ of $\lambda: \mathscr{A}(p) \to \mathscr{A}(p)$ is generated as a left $\mathscr{A}(p)$ ideal by $P\mathscr{A}(p)$. A generating set of relations is given by*

1. $\mathscr{Q}_i\mathscr{Q}_j = -\mathscr{Q}_j\mathscr{Q}_i$ *and* $\mathscr{Q}_i^2 = 0$,
2. $\mathscr{P}_i\mathscr{P}_j = \mathscr{P}_j\mathscr{P}_i$,
3. $\mathscr{Q}_i\mathscr{P}_j = \mathscr{P}_j\mathscr{Q}_i$ *if* $i > 0$,
   $\mathscr{Q}_0\mathscr{P}_j = \mathscr{P}_j\mathscr{Q}_0 - \mathscr{Q}_j$,
4. $(\mathscr{P}_j)^p = 0$.

PROOF. By Theorem 4 in [13], it is easy to check that the above generators are indeed relations in $\mathscr{A}(p)$. We must show these relations generate all others.

A basic set of generators for the left ideal $\mathscr{A}_\lambda$ corresponds to a Z_p basis of $\operatorname{Tor}^1_{\mathscr{A}(p)}(\mathscr{A}_\lambda, Z_p)$. A generating set of relations corresponds to $\operatorname{Tor}^2_{\mathscr{A}(p)}(\mathscr{A}_\lambda; Z_p)$. Following section 2 of [12], consider the Hopf algebra

$$\Gamma = E(\mathscr{Q}_0, \mathscr{Q}_1, \ldots) \otimes Z_p[\mathscr{P}_1, \mathscr{P}_2, \ldots]/(\mathscr{P}_1^p, \ldots)$$

Then by basic algebra, the dual Hopf algebra

$$\Gamma^* = E(\tau_0, \tau_1, \ldots) \otimes Z_p[\xi_1, \xi_2, \ldots]/(\xi_1^p, \ldots)$$

Clearly

$$\Gamma^* = \operatorname{Coker}(\lambda^* : \mathscr{A}(p)^* \to \mathscr{A}(p)^*)$$

where $\lambda^*(x) = x^p$. Thus, dually we have $\mathscr{A}_\lambda = \mathscr{A}(p)\bar{\Gamma}$.

Let R be a resolution of Γ over Z_p. Then $\bar{\mathscr{A}}(p) \otimes R$ is a resolution of $A(p)$ over $\mathscr{A}_\lambda$ and so

$$\operatorname{Tor}^q{}_{\mathscr{A}(p)}(\mathscr{A}_\lambda, \mathsf{Z}_p) \approx \operatorname{Tor}^q{}_\Gamma(\mathsf{Z}_p, \mathsf{Z}_p) .$$

By basic homological algebra [4],

$$\operatorname{Tor}_\Gamma(\mathsf{Z}_p, \mathsf{Z}_p) = \mathsf{Z}_p[s\mathscr{Q}_0, s\mathscr{Q}_1, \ldots] \otimes E(s\mathscr{P}_1 s\mathscr{P}_2, \ldots) \otimes \mathsf{Z}_p[t\mathscr{P}_1, t\mathscr{P}_2, \ldots].$$

where $\operatorname{bideg} sx = (1, \deg x)$ and $\operatorname{bideg} tx = (2, p \deg x)$.

A simple counting argument shows that the set of generators and relations given in the Theorem suffice.

This Theorem will induce all the relations in $PH^*(K; \mathsf{Z}_p)$ arising from stable relations in $\mathscr{A}(p)$. There are, however, more relations which arise from excess considerations.

THEOREM 3.9. *If* $\Theta \in \mathscr{A}_\lambda \subset \mathscr{A}(p)$ *has excess* e, *then there are operations* $\beta_i, \gamma_j, \eta_i$ *and* ζ_j *in* $\mathscr{A}(p)$ *for* $i \geqq 0$ *and* $j \geqq 1$ *satisfying*

$$\Theta = \sum \beta_i \mathscr{Q}_0 \mathscr{P}^{b_i} + \sum \gamma_j \mathscr{P}^1 \mathscr{P}^{c_j} + \sum \eta_i \mathscr{Q}_i + \sum \zeta_j \mathscr{P}_j$$

where $b_i \geqq e-1$ *and* $c_j \geqq e-2$, $\operatorname{exc}(\eta_i) \geqq \deg \mathscr{Q}_i + e$, *and* $\operatorname{exc}(\zeta_j) \geqq \deg \mathscr{P}_i + e$.

PROOF. We must first introduce an order on the set of sequences $R = (r_1, r_2, \ldots)$ of non-negative integers, almost all 0. Recall that $|R| = 2\sum r_i = \operatorname{exc}(\mathscr{P}(R))$ [7].

We say $R < R'$ if $|R| < |R'|$ or if $|R| = |R'|$ and R is less than R' in the lexiographic order from the right (compare [7]). For example

$$(2, 1, 1, 1, 0, \ldots) < (2, 11, 1, 0, \ldots) < (11, 2, 0, 1, 0, \ldots) .$$

If $R < R'$ we say $\mathscr{P}(R')$ has higher order than $\mathscr{P}(R)$.

The following equations can be checked without much difficulty using Theorem 4 [12].

1) $\mathscr{P}(r_1, r_2, \ldots) = \mathscr{P}(pr_2, pr_3, \ldots)\mathscr{P}^{|R|}$ + terms of higher order.
2) If $\varphi \in \mathscr{A}(p)$, then there are $\omega_j \in \mathscr{A}(p)$ such that $\mathscr{Q}_i \varphi = \sum_{j \geqq i} \omega_j \mathscr{Q}_j$.
3) $\mathscr{Q}_{i+1} \mathscr{P}^s = \sum_{j=0}^{i} (-1)^{i-j} \mathscr{P}^{s+p^i+\cdots+p^j} \mathscr{Q}_j + (-1)^{i+1} \mathscr{Q}_0 \mathscr{P}^{s+p^i+\cdots+1}$.
4) $r_{i+1} \mathscr{P}(r_1, r_2, \ldots) = \mathscr{P}(r_1 + p^i, r_2, \ldots, r_{i+1} - 1, \ldots) \mathscr{P}_i$ + terms of higher order.

By [7], the operation Θ can be written as a linear combination of Milnor basis elements $\mathscr{P}(E,R)$ satisfying

$$|E, R| = \sum \varepsilon_i + 2\sum r_i \geqq e .$$

Consider the summand $\mathscr{P}(E,R)$ with least R. If $E \neq 0$, then equations 1 and 2 show that we can write

$$\mathscr{P}(E,R) = \sum \omega_t \mathscr{P}(E_t, 0)\mathscr{P}^{|R|} + \text{terms of higher order}$$

where E_t are certain sequences with $|E_t| = |E|$.

If $E_t = (\varepsilon_0, \varepsilon_1 \ldots)$ contains a non zero entry ε_i for $i > 0$, then we can write

$$\omega_t \mathscr{P}(E_t, 0)\mathscr{P}^{|R|} = \omega_t \mathscr{P}(E_t - \Delta_i, 0)\mathscr{Q}_i \mathscr{P}^{|R|}$$

and iterated use of equation 3 enables us to write this term in the form

$$\sum \beta_i \mathscr{Q}_0 \mathscr{P}^{b_i} + \sum \eta_i \mathscr{Q}_i .$$

If $E_t = (1,0,0,\ldots)$, then $\omega_t \mathscr{P}(E_t,0)\mathscr{P}^{|R|} = \omega_t \mathscr{Q}_0 \mathscr{P}^{|R|}$ is already in the desired form.

If $E = 0$, then we know that $r_{i+1} \not\equiv 0 \pmod p$ for some $i \geqq 0$, since $\mathscr{P}(0,R)$ is in the kernel of λ. If this is satisfied for some $i \geqq 1$, then equation 4 reduces $\mathscr{P}(0,R)$ to the form $\zeta \mathscr{P}_i +$ terms of higher order. If r_1 is the only entry not divisible by p, then $|R| \not\equiv 0 \pmod p$ and so

$$\mathscr{P}(0,R) = \gamma \mathscr{P}^1 \mathscr{P}^{|R|-1} + \text{terms of higher order}$$

by equation 1.

To finish the proof, simply note that the number of basis elements $\mathscr{P}(E,R)$ of a fixed degree is finite. Therefore we simply iterate our procedure to reduce Θ to the desired form.

The following Proposition can be checked in a straight forward manner using Theorem 4 [13].

PROPOSITION 3.10.

$$\mathscr{Q}_0 \mathscr{P}^s = \sum (-1)^i \mathscr{P}^{s-p_i} \mathscr{Q}_i$$

and

$$s\mathscr{P}^s = \mathscr{P}^1 \mathscr{P}^{s-1} = \sum (-1)^{j+1} \mathscr{P}^{s-p_j} \mathscr{P}_j ,$$

where $p_i = 1 + p + \ldots + p^{i-1}$ *if* $i > 0$ *and* $p_0 = 0$.

THEOREM 3.11. *The relations among the generators of* $PH^*(K; Z_p)$ *as an unstable* $\mathscr{A}(p)$ *module given in Theorem* 3.7 *are generated by the following equations.*

1. $\mathcal{Q}_i\langle\mathcal{Q}_j\alpha_r\rangle = -\mathcal{Q}_j\langle\mathcal{Q}_i\alpha_r\rangle, \quad \mathcal{Q}_i\langle\mathcal{Q}_i\alpha_r\rangle = 0,$
2. $\mathcal{P}_i\langle\mathcal{P}_j\alpha_r\rangle = \mathcal{P}_j\langle\mathcal{P}_i\alpha_r\rangle,$
3. $\mathcal{Q}_i\langle\mathcal{P}_j\alpha_r\rangle = \mathcal{P}_j\langle\mathcal{Q}_i\alpha_r\rangle$ *if* $i>0$ *and* $\mathcal{Q}_0\langle\mathcal{P}_j\alpha_r\rangle = \mathcal{P}_j\langle\mathcal{Q}_0\alpha_r\rangle + \langle\mathcal{Q}_j\alpha_r\rangle,$
4. $(\mathcal{P}_j)^{p-1}\langle\mathcal{P}_j\alpha_r\rangle = \langle\mathcal{P}_j\alpha_{r-1}\rangle^p,$
5. $\sum(-1)^i\mathcal{P}^{s-p^i}\langle\mathcal{Q}_i\alpha_r\rangle = 0$ *if* $2s \geqq np^r$
6. $\sum(-1)^{j+1}\mathcal{P}^{s-p^j}\langle\mathcal{P}_j\alpha_r\rangle = 0$ *if* $2s > np^r$ *and* $s \not\equiv 0 (mod\, p)$.

PROOF. With the exception of equation 4, all the equations occur in dimensions not divisible by $2p$. Thus they are relations in $PH^*(K; Z_p)$ by Proposition 3.5 and Theorems 3.8 and 3.9. To check equation 4, we must show

$$(\mathcal{P}_j)^{p-1}(\mathcal{P}_jX_r + \ldots + X_0^{p^r-1}\mathcal{P}_jX_0) = (\mathcal{P}_jX_{r-1} + \ldots + X_0^{p^{r-1}-1}\mathcal{P}_jX_0)^p.$$

This follows from the Leibnitz formula on iterated derivations taken mod p, since $\mathcal{P}_j$ is a derivation of Z_p algebras.

Now assume that x is a relation in $PH^*(K; Z_p)$. This means that x is an element of F, the free left $\mathcal{A}(p)$ module generated by α_0, $\langle\mathcal{Q}_i\alpha_r\rangle$ and $\langle\mathcal{P}_j\alpha_r\rangle$, $i \geqq 0, j \geqq 1$ and $r = 1,\ldots,k$, such that the natural projection onto $PH^*(K; Z_p)$ sends x to 0. Let F_t be the sub module of F where we restrict r to run from 1 to t if $t \geqq 1$ and set $F_0 \approx \mathcal{A}(p)$ with generator α_0.

Let t be the least number such that $x \in F_t$. We shall show that the equations 1 through 6 suffice to transform x to $y \in F_{t-1}$. The Theorem will follow.

We can write $x = \sum\Theta_i\mathcal{Q}_i\alpha_t + \sum\varphi_j\mathcal{P}_j\alpha_t$ modulo F_{t-1} by the definition of $\langle\mathcal{Q}_i\alpha_t\rangle$ and $\langle\mathcal{P}_j\alpha_t\rangle$. This implies that

$$\text{exc} \sum \Theta_i\mathcal{Q}_i + \sum \varphi_j\mathcal{P}_j > \dim\alpha_t = np^t.$$

By Theorem 3.8 and 3.9 we can write

$$x = \sum \beta_i\mathcal{Q}_0\mathcal{P}^{b_i}\alpha_t + \sum \gamma_j\mathcal{P}^1\mathcal{P}^{c_j}\alpha_t + \sum \eta_i\mathcal{Q}_i\alpha_t + \sum \zeta_j\mathcal{P}_j\alpha_t$$

modulo F_{t-1}, using only the relations 1 through 4 above. From excess considerations, $\eta_i\mathcal{Q}_i\alpha_t$ and $\zeta_j\mathcal{P}_j\alpha_t$ are 0 in $PH^*(K; Z_p)$.

The identities of Proposition 3.10 use only relations in the stable Steenrod algebra. Thus relations 5 and 6 will finish the reduction of x to an element of F_{t-1}.

In [9] the cohomology of the classifying space of K will be computed as a Hopf algebra over $\mathcal{A}(p)$. The relations of the previous Theorem will imply similar relations in the cohomology of the classifying space. For $k=2$ and $p=2$, Milgram [12] and Kristensen and Pedersen [10], using different techniques arrived at apparently different answers. Theorem 3.11 will imply that these differences are not real.

Let $s=p_{j+1}=p^j+\ldots+1>np^r$. Then relation 5 implies that

$$\sum_{i=0}^{j+1}(-1)^{i+1}\mathscr{P}^{p_{j+1}-p_i}\langle\mathscr{Q}_i\alpha_r\rangle = 0$$

or

$$\langle\mathscr{Q}_{j+1}\alpha_r\rangle = \sum_{i=0}^{j}(-1)^{j-i}\mathscr{P}^{p^j+\ldots+p^i}\langle\mathscr{Q}_i\alpha_r\rangle.$$

In Theorem 3 [6], relations were given for the case were $p=2$. This equation shows that relation c follows from relation d in that theorem.

As another example of relations in $PH^*(\boldsymbol{K};\, \mathsf{Z}_p)$, it is easy to check inductively that

$$\mathscr{P}_j(p^s-p^{s-1})\ldots\mathscr{P}_j(p^2-p)\mathscr{P}_j(p-1)\langle\mathscr{P}_j\alpha_r\rangle = (-1)^s\langle\mathscr{P}_j\alpha_{r-s}\rangle^{p^s}.$$

Here we use the facts $\mathscr{P}_j(p-1)=-(\mathscr{P}_j)^{p-1}$ and $\mathscr{P}(pR)x^p=(\mathscr{P}(R)x)^p$.

Let $L\subset PH^*(\boldsymbol{K};\, \mathsf{Z}_p)$ be the unstable $\mathscr{A}(p)$ module generated by $\langle\mathscr{Q}_i\alpha_r\rangle$ for $i\geqq 0$ and $r=0,1,\ldots,k$, where $\langle\mathscr{Q}_i\alpha_0\rangle=\mathscr{Q}_i\alpha_0$.Let G_0 be the Z_p submodule of $PH^*(\boldsymbol{K};\, \mathsf{Z}_p)$ generated by $\mathscr{P}(R)\alpha_0$ and inductively define G_t to be the union of G_{t-1} and the Z_p submodule of $PH^*(\boldsymbol{K};\, \mathsf{Z}_p)$ generated by $\mathscr{P}(R)\langle\mathscr{P}_j\alpha_t\rangle$ for all R and $j\geqq 1$. Then clearly

$$PH^*(\boldsymbol{K};\, \mathsf{Z}_p) \approx L\oplus G_k\,,$$

although the splitting is not as $\mathscr{A}(p)$ modules. Finally set $M_t=G_t/G_{t-1}$ for $t=0,1,\ldots,k$ where $G_{-1}=0$.

THEOREM 3.12. *As a coalgebra*

$$H^*(\boldsymbol{K};\, \mathsf{Z}_p) \approx E(L^-)\otimes\Gamma_1(L^+)\otimes\otimes_{t=0}^{k}\Gamma_{k-t+1}(M_t)$$

where L^+ and L^- are the even and odd dimensional submodules of L and $\Gamma_s(B)$ is the divided power coalgebra on the Z_p module B truncated at height p^s.

PROOF. Let $x=\mathscr{P}(E,R)\alpha_r$ be an arbitrary Z_p basis element of $H^*(\boldsymbol{K};\, \mathsf{Z}_p)$. If $E\neq 0$, then $x\in L$ modulo decomposables since $\lambda x=0$.

Assume $E=0$ and write $R=(p^m r_1',\ p^m r_2',\ldots)=p^m R'$ for some $m\geqq 0$ where $r_i\not\equiv 0 (\mathrm{mod}\ p)$ for some $i\geqq 1$. If $m\leqq r$, then

$$\lambda^m\mathscr{P}(p^m R')\alpha_r = \mathscr{P}(R)\alpha_{r-m}$$

which lies in M_{r-m} modulo decomposables. Similarly if $m>r$, then $\lambda^r(\mathscr{P}(p^m R')\alpha_r$ lies in M_0 modulo decomposables. Thus x is a p^mth or p^rth divided power of a primitive modulo decomposables. The Theorem is now easily deduced from Borel's classification of commutative Hopf algebras [2].

We have inclusions of H spaces $K(\mathsf{Z}_p,n)=\boldsymbol{K}_0 \subset \ldots \boldsymbol{K}_k \subset \ldots$ where $\boldsymbol{K}_k$ is what we have called $\boldsymbol{K}$. Let $\boldsymbol{K}_\infty = \bigcup_{k\geqq 0}\boldsymbol{K}_k$.

COROLLARY 3.13. *As a coalgebra*

$$H^*(\boldsymbol{K}_\infty;\, \mathsf{Z}_p) \cong E(L^-)\otimes\Gamma_1(L^+)\otimes\Gamma(G)$$

where $G=\bigcup G_t$ *and* Γ *is the untruncated divided power Hopf algebra.*

The analogous result for $p=2$ can be read off by setting all of $L=0$ (see [6, Theorem 4]).

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