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# NON SELF-DETERMINING FACES - AN EXAMPLE

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Alfsen comments [1,p.111] that it is rather hard to find a closed face $F$ of a compact convex set $K$ in a locally convex space which is not self determining, that is, for which the set

$$\{x \in K : a(x) = 0 \quad \text{for every } a \in A(K) \text{ which vanishes on } F\}$$

properly contains $F$. An example of such a face, due to Asimow, is given in Ellis' lecture notes on affine functions and faces of convex sets [2, p.46]. This note gives a very simple example of a $K$ and an $F$ such that any bounded (not assumed continuous) affine function on $K$ that vanishes on $F$ must vanish identically, so that $F$ is rather drastically non self-determining.

The following lemma is proved by a simple computation. (aff $S$, co $S$ denote the affine and convex hull of a set $S$ respectively.)

LEMMA. *Let $F$, $G$ be convex sets in a linear space $E$ such that* aff $F$ *misses $G$. Then $F$ is a face of* co $(F \cup G)$.

Now let $E$ be the real space $L_2[0,1]$ with the usual pointwise ordering. ($L_p, 1 < p < \infty$, will do equally well). Let

$$F = \{x \in E : 0 \leqq x \leqq 1\},$$
$$G = \{x \in E : x \geqq 0 \text{ and } \|x\| \leqq 1\}.$$

Let $a$ be any non-negative element of $E$ which is essentially unbounded on [0,1]. Define

$$K = \mathrm{co}\big(F \cup (a+G)\big),$$

in the weak topology of $E$. Clearly $F$ and $G$ are convex, closed and bounded and hence weakly compact since $E$ is norm-reflexive. Hence $K$ is compact.

Futher, since all the members of aff $F$ are bounded functions, while those in $a+G$ are (essentially) unbounded, the Lemma implies that $F$ is a face of $K$ — clearly a closed face.

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THEOREM. *Let $f$ be a bounded affine function on $K$ such that $f=0$ on $F$. Then $f=0$.*

PROOF. It is clear that aff $G=E$, hence aff $K=E$. Thus the affine function $f$ on $K$ extends uniquely to an affine function, also called $f$, on the whole of $E$. Since $f$ vanishes at $0 \in F$, it must in fact be linear. Now $f$ is bounded on $a+G$, hence on $G-G$: the latter is a norm-neighbourhood of 0 and therefore $f \in E^*$. But the linear span of $F$ (which is $L_\infty$), is norm-dense in $E$, so $F$ is total for $E^*$ and hence $f=0$.

## REFERENCES

1. E. M. Alfsen, *Compact convex sets and boundary integrals*, (Ergebnisse der Mathematik 57) Springer-Verlag, Berlin, Heidelberg, New York, 1971.
2. A. J. Ellis, *Lecture notes on affine functions and faces of convex sets*, Calif. Inst. of Tech. (mimeographed).

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