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SOME APPLICATIONS OF CONVEXITY THEORY TO BANACH ALGEBRAS

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1. Introduction.

Let A be a complex unital Banach algebra with state space S and with $Z = \mathrm{co}(S \cup -iS)$. We study the representation of A as the real Banach space $A(Z)$, consisting of all real-valued continuous affine functions on Z.

In section 2, we show how the fact that A is a B^*-algebra if and only if A^{**} is a B^*-algebra is related to a convexity theorem, and we deduce the fact that if A is a complex Lindenstrauss space then A is a $C(X)$ space. We show also that this latter conclusion is valid if A is a complex Lindenstrauss space for its supremum norm over S.

It is well-known that if A is a B^*-algebra then $A(S)^*$ has the unique minimal decomposition property, that is every φ in $A(S)^*$ has a unique decomposition $\varphi = \varphi_1 - \varphi_2$ with $\varphi_1, \varphi_2 \geqq 0$ and $\|\varphi\| = \|\varphi_1\| + \|\varphi_2\|$. We prove in section 3, that A is a B^*-algebra if and only if $A(Z)^*$ has the unique minimal decomposition property.

In section 4, we show that if K is a compact convex set and if the extreme boundary of a closed face F is the union of the extreme boundaries of a sequence $\{F_n\}$ of closed split faces of K then F is a split face of K. If, moreover, each F_n is a simplex then K is also a simplex. A consequence of this result is that if the Choquet boundary of a function algebra A on X is covered by a sequence of generalized peak interpolation sets for A, then $A = C(X)$.

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2.

Throughout this section we will let A denote a complex unital Banach algebra with identity e. Let S denote the state space of A and let $Z = \mathrm{co}(S \cup -iS)$, in the Banach dual space A^*. It was observed in [3] that the Bohnenblust–Karlin theorem implies that there is a real-linear homeomorphism θ of A onto $A(Z)$, the Banach space of all real-valued

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continuous affine functions on Z with the supremum norm, where θ is defined by $\theta a(\varphi) = \mathrm{re}\,\varphi(a)$ for all a in A and φ in Z. In the same paper the Vidav–Palmer theorem was interpreted to show that A is a B^*-algebra if and only if S is a split face of Z.

A complex Banach space whose dual space is isometrically isomorphic to a complex L^1-space is called a *complex Lindenstrauss space.* If L is a uniformly closed linear subspace of the continuous complex-valued functions on a compact Hausdorff space, such that L contains constants and separates points, then Hirsberg and Lazar [9] have proved that L is a complex Lindenstrauss space if and only if $\mathrm{co}(T \cup -iT)$ is a simplex, where

$$T = \{F \in L^* : F(1) = 1 = \|F\|\}$$

is the state space of L.

Let A_1 denote A equipped with the equivalent norm, $\|a\|_1 = \sup\{|\varphi(a)| : \varphi \in S\}$ for all a in A. Then A_1 is a closed linear subspace of $C(S)$ containing constants and separating points, and it is easy to verify that S is also the state space of A_1. If Z is a simplex then S is necessarily a split face of Z.

Combining these results, together with the fact that the state space of a B^*-algebra with identity is a simplex if and only if the algebra is commutative, we immediately obtain the following result.

THEOREM 1. *A_1 is a complex Lindenstrauss space if and only if $A = C(X)$, for some compact Hausdorff space X.*

For the case of a function algebra A the theorem was first proved in [7] and in [9]. We shall show below that Theorem 1 remains true if A replaces A_1. First, however, we prove a convexity theorem.

We recall (cf. [1]) that if K is a compact convex subset of a locally convex Hausdorff space then K may be identified with the state space of $A(K)$. The second dual space $A(K)^{**}$ can be represented as a space $A(K^{**})$ for some compact convex set K^{**}. Alternatively $A(K)^{**}$ coincides with the space $A^{\mathrm{b}}(K)$ of all bounded real-valued affine functions of K, with the supremum norm.

If E is a subset of a Banach space V then $\hat{E}$ will denote the canonical embedding of E in X^{**} and $\bar{E}$ will denote the w^*-closure of $\hat{E}$ in X^{**}.

THEOREM 2. *Suppose that $K = \mathrm{co}(F \cup G)$ where $F = u^{-1}(0)$ and $G = u^{-1}(1)$ for some u in $A(K)$. Then $\bar{F} = \hat{u}^{-1}(0)$ and $\bar{G} = \hat{u}^{-1}(1)$, and $K^{**} = \mathrm{co}(\bar{F} \cup \bar{G})$. Moreover, F is a split face of K if and only if $\bar{F}$ is a split face of K^{**}.*

PROOF. We have $0 \leqq u \leqq 1$ and hence, because the orderings of $A(K)$ and $A(K)^{**}$ are compatible, it follows that $0 \leqq \hat{u} \leqq 1$. For similar reasons $\hat{K}$ is w^*-dense in K^{**}, and so $K^{**} = \overline{\mathrm{co}}(F \cup G) = \mathrm{co}(\bar{F} \cup \bar{G})$. If $\tilde{F} = \hat{u}^{-1}(0)$ and $\tilde{G} = \hat{u}^{-1}(1)$ then $\bar{F} \subseteq \tilde{F}$ and $\bar{G} \subseteq \tilde{G}$. Suppose that φ belongs to $\tilde{F} \setminus \bar{F}$. Then $\varphi = \lambda \bar{f} + (1-\lambda)\bar{g}$ for some $0 \leqq \lambda < 1, \bar{f}$ in $\bar{F}$ and $\bar{g}$ in $\bar{G}$. Therefore, $\bar{g}$ belongs to the face $\tilde{F}$ as well as to $\tilde{G}$, which is impossible. Hence we have $\bar{F} = \tilde{F}$ and similarly $\bar{G} = \tilde{G}$.

If $\bar{F}$ is split in K^{**} then, since $\hat{F} = \bar{F} \cap \hat{K}$, it follows that F is split in K. Conversely, suppose that F is a split face of K. For each p in $A^{\mathrm{b}}(K)$ define p^* in $A^{\mathrm{b}}(K)$ by

$$p^*(\lambda f + (1-\lambda)g) = p(\lambda f - (1-\lambda)g) \quad \text{for all } f \text{ in } F, g \text{ in } G\,.$$

Then p^* is well defined and $(p+p^*)$ is zero on G and equals $2p$ on F. Consequently $(p+p^*)$ is zero on $\bar{G}$ and equals $2p$ on $\bar{F}$. Therefore, if

$$\lambda_1 \bar{f}_1 + (1-\lambda_1)\bar{g}_1 = \lambda_2 \bar{f}_2 + (1-\lambda_2)\bar{g}_2$$

for some $0 < \lambda_i < 1$, $\bar{f}_i$ in $\bar{F}$, $\bar{g}_i$ in $\bar{G}$, then $\lambda_1 p(\bar{f}_1) = \lambda_2 p(\bar{f}_2)$ for all p in $A(K)^{**}$. It follows that $\lambda_1 = \lambda_2$ and $\bar{f}_1 = \bar{f}_2$, and hence $\bar{F}$ is split in K^{**}.

Suppose that A^{**} is also a Banach algebra with identity $\hat{e}$. Then the mapping θ represents A as $A(Z)$ and, using the result [4,12.3], we get that the natural analogue of θ represents A^{**} as $A(Z)^{**} = A(Z^{**})$. Moreover $S = (\theta e)^{-1}(1)$, $-iS = (\theta e)^{-1}(0)$ and so Theorem 2 shows that the state space of A^{**} coincides with $\bar{S}$.

It is well-known that if A is a B^*-algebra with identity e then, for the Arens multiplication, A^{**} is a B^*-algebra with identity $\hat{e}$ (cf. [4]). A proof of that result is included in the proof of the more general result in Corollary 3 below, where we assume no relationship between the algebraic structures of A and A^{**}, or their identities.

COROLLARY 3. *Let A and A^{**} be complex unital Banach algebras. Then A is a B^*-algebra if and only if A^{**} is a B^*-algebra.*

PROOF. Let A be a B^*-algebra with identity e. Then for the Arens multiplication A^{**} is a Banach algebra with identity $\hat{e}$. We know that S is a split face of Z and hence, by Theorem 2, S^{**} is a split face of Z^{**}. Therefore, for the Arens multiplication, A^{**} is a B^*-algebra with identity $\hat{e}$ and (cf. [11]) the identity u of the given algebra A^{**} is unitary.

Let S_1 denote the state space of the given algebra A^{**} and for each φ in A^{***} let $\varphi_u(a) = \varphi(au^*)$ for all a in A^{**} (where au^* is the product in the B^*-algebra A^{**}). If $\tau(\varphi) = \varphi_u$ then τ is a linear map from A^{***} into itself which is injective since $\varphi_u = \varphi'_u$ implies that $\varphi(auu^*) = \varphi'(auu^*)$, that is $\varphi(a)=\varphi'(a)$. It is easy to verify that τ maps S^{**} into S_1; in fact, since each ψ in S_1 has the form φ_u, where $\varphi(a)=\psi(au)$, τ maps S^{**} onto S_1. Therefore, the restriction of τ to $Z^{**}=\mathrm{co}(S^{**}\cup -iS^{**})$ is an affine isomorphism onto $Z_1=\mathrm{co}(S_1\cup -iS_1)$. Since S^{**} is a split face of Z^{**} it follows that S_1 is a split face of Z_1, and hence the given algebra A^{**} is a B^*-algebra.

Conversely, let A^{**} be a B^*-algebra with identity u and let A be a Banach algebra with identity e. By repeating the argument just given, we see that for the Arens multiplication A^{**} is a B^*-algebra with identity $\hat{e}$. Therefore, S^{**} is a split face of Z^{**}, and Theorem 2 shows that S is a split face of Z. Hence A is a B^*-algebra.

Corollary 4. *A complex unital Banach algebra A is a complex Lindenstrauss space if and only if $A=C(X)$ for some compact Hausdorff space X.*

Proof. If A is a complex Lindenstrauss space then A^{**} is isometrically isomorphic to a commutative B^*-algebra $C(Y)$. By Corollary 3, A is a B^*-algebra and so is A^{**} for the Arens multiplication. The proof of Corollary 3, shows that the state space S^{**} of A^{**} for the Arens multiplication is affinely isomorphic to the state space of $C(Y)$, and hence S^{**} is a simplex. Therefore, the Arens multiplication is commutative and so A is a commutative B^*-algebra.

3.

As in the previous section, let K be a compact convex set. We recall [5, Theorem 2] that $A(K)^*$ has the unique minimal decomposition property if and only if every y in $A(K)^*$ with $y(1)=0$ and $\|y\|=1$ has a unique decomposition $y=\frac{1}{2}y_1-\frac{1}{2}y_2$, where y_1 and y_2 belong to K; moreover, this is the case if and only if for each x in $A(K)^*$ the set $K\cap(x+K)$ is either empty, or a single point, or it contains $y+\lambda K$ for some y in $A(K)^*$ and some real $\lambda>0$.

Theorem 5. *Suppose that $K=\mathrm{co}(F\cup G)$ where $F=u^{-1}(0)$ and $G=u^{-1}(1)$ for some u in $A(K)$. Then the following statements are equivalent.*

(i) *$A(K)^*$ has the unique minimal decomposition property.*

(ii) *$A(F)^*$ and $A(G)^*$ both have the unique minimal decomposition property, and F and G are complementary split faces of K.*

Proof. (i) $\Rightarrow$ (ii). If F and G are not split faces of K then there exist x_1,x_2 in F and y_1,y_2 in G with $x_1\neq x_2$, $y_1\neq y_2$, and $\frac{1}{2}x_1+\frac{1}{2}y_1=\frac{1}{2}x_2+\frac{1}{2}y_2$.

It follows that x_1, x_2 belong to $K \cap (K + x_1 - y_2)$ and therefore there exists some z in $A(K)^*$ and $\lambda > 0$ such that $z + \lambda K$ is contained in $K \cap (K + x_1 - y_2)$. However, for y in $K \cap (K + x_1 - y_2)$ we have $y = x + x_1 - y_2$ for some x in K, so that $0 \leqq u(y) \leqq 1$ and $u(y) = u(x) + u(x_1) - u(y_2) = u(x) - 1 \leqq 0$. Hence u is constant on $K \cap (K + x_1 - y_2)$, but is evidently not constant on $z + \lambda K$. This contradiction proves that F and G are complementary split faces of K.

The closed unit ball of $A(K)^*$ is $\operatorname{co}(K \cup -K)$, which is the convex hull of $\operatorname{co}(F \cup -F)$ and $\operatorname{co}(G \cup -G)$, the closed unit balls of $A(F)^*$ and $A(G)^*$ respectively. It is straightforward now to verify that $A(F)^*$ and $A(G)^*$ have the unique minimal decomposition property.

(ii) $\Rightarrow$ (i). Since F and G are complementary split faces of K every z in $A(K)^*$ has a unique decomposition $z = x + y$, where x belongs to $\operatorname{lin} F$ and y belongs to $\operatorname{lin} G$, and moreover $\|z\| = \|x\| + \|y\|$. Let $\|z\| = 1$ and let

$$z = \lambda v_1 - (1-\lambda) w_1 = \lambda v_2 - (1-\lambda) w_2$$

for $0 \leqq \lambda \leqq 1$ and v_i, w_i belonging to K. Then

$$\begin{aligned} z &= \lambda(\lambda_1 x_1 + (1-\lambda_1) y_1) - (1-\lambda)(\mu_1 x_1' + (1-\mu_1) y_1') \\ &= \lambda(\lambda_2 x_2 + (1-\lambda_2) y_2) - (1-\lambda)(\mu_2 x_2' + (1-\mu_2) y_2') \end{aligned}$$

where $0 \leqq \lambda_i, \mu_i \leqq 1$, x_i, x_i' belong to F and y_i, y_i' belong to G. We therefore have

$$\lambda\lambda_1 x_1 - (1-\lambda)\mu_1 x_1' = \lambda\lambda_2 x_2 - (1-\lambda)\mu_2 x_2' ,$$

and also

$$\|\lambda\lambda_1 x_1 - (1-\lambda)\mu_1 x_1'\| + \|\lambda(1-\lambda_1) y_1 - (1-\lambda)(1-\mu_1) y_1'\| = 1 ,$$

so that

$$\|\lambda\lambda_1 x_1 - (1-\lambda)\mu_1 x_1'\| = \lambda\lambda_1 + (1-\lambda)\mu_1 .$$

Since $A(F)^*$ has the unique minimal decomposition property we must have $\lambda\lambda_1 x_1 = \lambda\lambda_2 x_2$ and $(1-\lambda)\mu_1 x_1' = (1-\lambda)\mu_2 x_2'$. Using the fact that $A(G)^*$ has the unique minimal decomposition property, in a similar manner we obtain $v_1 = v_2$ and $w_1 = w_2$, so that $A(K)^*$ has the unique minimal decomposition property.

If A is a unital B^*-algebra with state space S then $\operatorname{lin} S$ is the set of hermitian linear functionals in A^* and it is well-known (cf. [8]) that $A(S)^* = \operatorname{lin} S$ has the unique minimal decomposition property. This property of $\operatorname{lin} S$ is not however sufficient to distinguish B^*-algebras amongst complex unital B-algebras A with state space S. In fact, the state space of any Dirichlet algebra is a Bauer simplex, in which case $\operatorname{lin} S$ is a vector lattice, and so certainly possesses the unique minimal decomposition property. We can however obtain the following result in this context.

COROLLARY 6. *Let A be a complex unital Banach algebra with state space S and with $Z = \mathrm{co}(S \cup -iS)$. Then A is a B^*-algebra if and only if $A(Z)^*$ has the unique minimal decomposition property.*

PROOF. If A is a B^*-algebra then, as noted above, $\mathrm{lin}\, S$, and similarly $\mathrm{lin}(-iS)$, has the unique minimal decomposition property. Theorem 5 shows that $A(Z)^*$ has the same property, since S and $-iS$ are complementary split faces of Z.

Conversely, if $A(Z)^*$ has the unique minimal decomposition property then, by Theorem 5, S is a split face of Z. The Vidav–Palmer theorem (cf. [3, Theorem 4]) now shows that A is a B^*-algebra.

It should be noted that Corollary 6, shows that A is a B^*-algebra if and only if Z has the intersection property referred to above, whereas A is a $C(X)$ if and only if Z has the intersection property which characterizes simplexes.

4.

Again in this section K will denote a compact convex set, and ∂K will denote its set of extreme points. The following result is probably known.

THEOREM 7. *Let F be a closed face of K and let $\{F_n\}$ be a sequence of closed split faces of K such that $\partial F = \bigcup_{n=1}^{\infty} \partial F_n$. Then F is a split face of K. If, in addition, each F_n is a simplex then F is a simplex.*

PROOF. Let μ be a boundary measure in the annihilator $A(K)^{\perp}$ of $A(K)$. In order to show that F is a split face of K we need to show that the restriction measure μ_F belongs to $A(K)^{\perp}$ (cf. [1, II.6.12]). Since each F_n is a closed split face of K, each μ_{F_n} belongs to $A(K)^{\perp}$. By taking convex hulls of finite collections of the F_n if necessary we may assume that $\{F_n\}$ is an increasing sequence and put $G = \bigcup_{n=1}^{\infty} F_n$. Therefore, if f is in $A(K)$ then $f\chi_{F_n}$ converges to $f\chi_G$ pointwise on K, and so the dominated convergence theorem gives

$$0 = \int_K f\, d\mu_{F_n} = \int_G f\, d\mu_F\,.$$

Now μ_F is a boundary measure on F (cf. [2, Lemma 1]) and hence vanishes on the G_δ-set $\bigcap_{n=1}^{\infty}(F \setminus F_n)$ which is disjoint from ∂F. Therefore, we obtain $\int_K f\, d\mu_F = 0$, so that μ_F belongs to $A(K)^{\perp}$, and F is a split face.

Now suppose also that each F_n is a simplex. In this case each μ_{F_n} is a boundary measure belonging to $A(F_n)^{\perp}$, and hence is zero, by the Choquet–Meyer uniqueness theorem. Since μ_F is supported by $\bigcup_{n=1}^{\infty} F_n$ it follows that $\mu_F = 0$, and hence F is also a simplex.

There are many non-simplexes K with the property that every extreme point is a split face; for example, if K is the set Z, described in previous sections, for any non-trivial function algebra. The following result shows that any such K must necessarily have uncountably many extreme points.

COROLLARY 8. *If K has at most countably many extreme points, each of which is a split face of K, then K is a simplex.*

Theorem 7 and Corollary 8 have an application to function algebras, as the next result shows.

COROLLARY 9. *Let A be a function algebra on X with Choquet boundary ∂A. If ∂A is covered by a sequence $\{E_n\}$ of generalized peak sets for A in X, such that $A \mid E_n = C(E_n)$ for each n, then $A = C(X)$. In particular, if ∂A is countable then $A = C(X)$.*

PROOF. Let S be the state space of A, and let $Z = \mathrm{co}(S \cup -iS)$. Then the sets $\mathrm{co}(E_n \cup -iE_n)$ are Bauer simplexes and split faces of Z (cf. [6]). Since the faces $\{\mathrm{co}(E_n \cup -iE_n)\}$ cover ∂Z, Theorem 7 shows that Z is a simplex. Hence S is a split face of Z so that $A(S)$, which can be identified with $\mathrm{re}\, A$, is uniformly closed. The Hoffman–Wermer theorem shows now that $A = C(X)$. Every point in ∂A is a generalized peak point for A, and so the last statement follows directly.

If A is a function algebra on X such that $A \mid E_n = C(E_n)$ for each n, where $\{E_n\}$ is a sequence of closed subsets of X covering X, then it is known [10] that $A = C(X)$. Using Corollary 9, together with a theorem of Varopoulos we obtain an associated result.

COROLLARY 10. *Let A be a function algebra on X and let $\{E_n\}$ be a sequence of subsets of ∂A which are closed in X and such that $\partial A = \bigcup_{n=1}^{\infty} E_n$ and $A \mid E_n = C(E_n)$ for each n. Then $A = C(X)$.*

PROOF. The conditions on each E_n imply that E_n is generalized peak set for A [12]. The result now follows from Corollary 9.

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