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TOPOLOGIES ON THE EXTREME POINTS OF COMPACT CONVEX SETS II

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Abstract.

We continue the study of topologies on the extreme points of compact convex sets begun in [6]. Here, we analyze the auxiliary condition (C2) introduced there. We find sufficient conditions for the topologies of L-ideals and of split faces to satisfy (C2). We also apply these results to the generalized peak set topology of function algebras.

Introduction.

In [6] we studied certain topologies on the extreme points of compact convex sets. In particular, let K be a compact convex subset of a locally convex topological (real or complex) vector space E. Let $\mathscr{T}$ be any collection of closed convex subsets of K. Let

$$R = \bigcap \{T_\alpha :\ \emptyset \neq T_\alpha \in \mathscr{T}\}\,.$$

Suppose $\mathscr{T}$ satisfies the following conditions:

1. $\emptyset, K \in \mathscr{T}$.
2. $T_1, T_2 \in \mathscr{T} \ \Rightarrow\ \mathrm{co}(T_1, T_2) \in \mathscr{T}$.
3. $T_\alpha \in \mathscr{T}$ each $\alpha \in A \ \Rightarrow\ \bigcap T_\alpha \in \mathscr{T}$.
4. $\mathrm{Ext}(T) = \mathrm{Ext}(K) \cap T$ for each $T \in \mathscr{T}$, $T \neq R$.
5. $\mathrm{Ext}(R) = \mathrm{Ext}(K) \cap R$ or $\emptyset = \mathrm{Ext}(K) \cap R$.

We use $\mathrm{Ext}(\cdot)$ for the extreme points of the given set. Given the collection $\mathscr{T}$ as above, we define the τ-topology on $\mathrm{Ext}(K)$ by the following scheme:

$$F \subseteq \mathrm{Ext}(K) \text{ is } \tau\text{-closed} \iff F = \mathrm{Ext}(K) \cap T \text{ for some } T \in \mathscr{T}\,.$$

Hence, all τ-closed sets are of the form $F = \mathrm{Ext}(T)$ for some $T \in \mathscr{T}$, but the converse need not hold. We seek conditions under which the following statement is true.

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STATEMENT 0.1. *Suppose K is metrizable. Then the following are equivalent for a fixed $p \in \mathrm{Ext}(K)$:*

1. *The τ-topology is first countable at p.*
2. *The τ-topology is locally compact at p.*
3. *The τ-topology is locally sequentially compact at p.*

Further, if the τ-topology is first countable for each $p \in \mathrm{Ext}(K)$, it is second countable.

For $q \in K$, we take $T(q)$ to be the minimal element of $\mathscr{T}$ which contains q. (It exists by properties 1 and 3 of $\mathscr{T}$.) We let

$$\Phi(q) = \mathrm{Ext}(K) \cap T(q) \quad \text{for each } q \in K .$$

We note that $\Phi(q)$ is τ-closed for each $q \in K$ and that $\Phi(q) \neq \emptyset$ for $q \notin R$.

We say that the τ-topology satisfies condition (C2) if the following holds:

If the sequence $\{p_n\} \subseteq \mathrm{Ext}(K)$ converges to q, then all the τ-cluster points of $\{p_n\}$ are in $\Phi(q)$.

We then have the following result.

THEOREM 0.2. [6, Theorem 2.5]. *Suppose K is metrizable satisfying* (C2). *Then Statement* 0.1 *holds.*

We outlined in [6] several examples of collections $\mathscr{T}$ which satisfy (1)–(5) above. Of these, the topology generated by the split faces need not satisfy condition (C2). Also, we do not know whether the topology generated by the weak* closed L-ideals always satisfies (C2). The purpose of this paper is to give sufficient conditions for either of these topologies to satisfy (C2) and hence for Statement 0.1 to hold for them.

In section 1 we consider necessary and sufficient conditions for $\mathscr{T}$ to satisfy condition (C2). To more easily verify this condition, in section 2 we consider a group G of transformations acting on a compact metric space Z. We introduce the concept of G acting smoothly on Z. We apply this to a group of one-to-one $\mathscr{T}$-affine maps of $\mathrm{Ext}(K)$ onto $\mathrm{Ext}(K)$. The notions of sections 1 and 2 are then applied in section 3 to the two topologies in question. Finally in section 4 we shall show that the generalized peak sets for a function algebra are the closed sets for a τ-topology for which Statement 0.1 holds.

1. Discussion of condition (C2).

We let K, E, and $\mathscr{T}$ be as in the introduction. $\mathscr{T}$ is said to be *smooth* if the following condition holds:

Whenever the sequence $\{p_n\} \subseteq \operatorname{Ext}(K)$ *converges to* q, *we have*

$$\bigcup_{n=1}^{\infty} \Phi(p_n)^- \cup T(q)$$

is closed in K.

LEMMA 1.1. *Suppose* $\mathscr{T}$ *is smooth and the sequence* $\{p_n\} \subseteq \operatorname{Ext}(K)$ *converges to* q. *Then*

$$\begin{aligned} \bigcup_n \Phi(p_n) \cup \Phi(q) &= \operatorname{Ext}(K) \cap \operatorname{co}^-(T(q), \textstyle\bigcup_n T(p_n)) \\ &= \operatorname{Ext} \operatorname{co}^-(T(q), \textstyle\bigcup_n T(p_n)) . \end{aligned}$$

PROOF. Let $S = \operatorname{co}^-(T(q), \bigcup_n T(p_n))$. Then

$$S = \operatorname{co}^-(\textstyle\bigcup_n \operatorname{co}^- \Phi(p_n), \operatorname{co}^- \operatorname{Ext} T(q))$$

by the Krein–Milman Theorem and therefore

$$S = \operatorname{co}^-(\textstyle\bigcup_n \Phi(p_n), \operatorname{Ext} T(q)) .$$

So

$$\operatorname{Ext} S \subseteq [\textstyle\bigcup_n \Phi(p_n) \cup \operatorname{Ext} T(q)]^-$$

by the Milman Theorem and hence

$$\operatorname{Ext} S \subseteq \textstyle\bigcup_n \Phi(p_n)^- \cup T(q)$$

since $\mathscr{T}$ is smooth. Since $\Phi(p_n)^- \subseteq T(p_n) \subseteq S$ we know that all points of $\Phi(p_n)^-$ which are not extreme in $T(p_n)$ cannot be extreme in S. Similar statements are true about $T(q)$. Hence

$$\operatorname{Ext}(S) \subseteq \textstyle\bigcup_n \Phi(p_n) \cup \operatorname{Ext} T(q) .$$

If $T(q) \neq R$, then $\operatorname{Ext} T(q) = \Phi(q)$. If $T(q) = R$, then $T(q) \subseteq \Phi(p_n)$ for each n, and so, as above

$$\operatorname{Ext} T(q) \cap \operatorname{Ext}(S) \subseteq \textstyle\bigcup_n \Phi(p_n) .$$

Hence

$$\operatorname{Ext}(S) \subseteq \textstyle\bigcup_n \Phi(p_n) \cup \Phi(q) .$$

On the other hand, clearly

$$\operatorname{Ext}(S) \supseteq \operatorname{Ext}(K) \cap S \supseteq \textstyle\bigcup_n \Phi(p_n) \cup \Phi(q)$$

and the lemma is proven.

We may now give necessary and sufficient conditions for $\mathscr{T}$ to satisfy condition (C2).

THEOREM 1.2. *$\mathscr{T}$ satisfies* (C2) *if and only if the following conditions hold for any sequence $\{p_n\}$ such that $p_n \to q$ and $\{p_n\} \subseteq \mathrm{Ext}(K)$*:

1. $\mathrm{co}^-(T(q), \bigcup_n T(p_n)) \in \mathscr{T}$,
2. $\bigcap_k \bigcup_{n \geq k} \Phi(p_n) \subseteq \Phi(q)$,
3. $\bigcup_n \Phi(p_n)^- \cup T(q)$ *is closed, i.e. $\mathscr{T}$ is smooth.*

PROOF. We first assume (1), (2), and (3). Let $\{p_n\} \subseteq \mathrm{Ext}(K)$ and $p_n \to q$. Let

$$D_k = \mathrm{Ext}(K) \cap \mathrm{co}^-(T(q), \textstyle\bigcup_{n \geq k} T(p_n)) .$$

From (1), D_k is τ-closed so clearly each τ-cluster point of $\{p_n\}$ is in D_k for each k. Hence, it suffices to show that $\bigcap_k D_k = \Phi(q)$. But from Lemma 1.1,

$$D_k = \textstyle\bigcup_{n \geq k} \Phi(p_n) \cup \Phi(q)$$

and from (2), $\bigcap_k D_k = \Phi(q)$.

Now suppose $\mathscr{T}$ satisfies condition (C2). Let $\{p_n\} \subseteq \mathrm{Ext}(K)$ and $p_n \to q$. We show first that $\mathscr{T}$ is smooth. Let

$$p \in (\textstyle\bigcup_n \Phi(p_n))^- - \bigcup_n \Phi(p_n)^- .$$

It suffices to show that $p \in T(q)$. If $p \in R$, there is nothing to prove. So we may assume there is a $z \in \Phi(p)$. Let $k_n \in \Phi(p_n)$ and $k_n \to p$. If $\{p_n\}$ does not τ-cluster to z, there is a τ-closed set F and a subnet $\{p_\alpha\}$ such that $p \notin F$ while $p_\alpha \in F$. Find $T \in \mathscr{T}$ so that $\mathrm{Ext}(T) = F$. Then $p_\alpha \in T$ and so $k_\alpha \in \Phi(p_\alpha) \subseteq T(p_\alpha) \subseteq T$. As $\{k_\alpha\}$ is cofinal in $\{k_n\}$, $k_n \to p$ and so $p \in T$. Hence $z \in \Phi(p) = \mathrm{Ext}\, T(p) \subseteq \mathrm{Ext}\, T = F$, a contradiction. Hence $\{p_n\}$ τ-clusters at z and so by (C2) $z \in \Phi(q)$. As this is true for each $z \in \Phi(p)$, we have $\Phi(p) \subseteq \Phi(q)$ and so $p \in T(p) \subseteq T(q)$. A very similar argument shows that

$$\textstyle\bigcap_k \bigcup_{n \geq k} \Phi(p_n) \subseteq \{\tau\text{-cluster points of } \{p_n\}\}$$

and so by (C2) is a subset of $\Phi(q)$. Finally, we show that

$$\mathrm{co}^-(T(q), \textstyle\bigcup_n T(p_n)) \in \mathscr{T} .$$

As $\mathscr{T}$ is smooth, using Lemma 1.1 this is equivalent to showing that $S = \bigcup_n \Phi(p_n) \cup \Phi(q)$ is τ-closed. If not, let p belong to the τ-closure of S minus S. As $\Phi(q)$ is τ-closed, p belongs to the τ-closure of $\bigcup_n \Phi(p_n)$. It is therefore a τ-cluster point of $\{p_n\}$ and so by (C2) is in $\Phi(q)$. Hence S is τ-closed and the proof is complete.

We now wish to give easily verifiable conditions which will imply the three conditions of Theorem 1.2. Recall first that a topological space is R_0 if closures of points are either disjoint or equal.

PROPOSITION 1.3. *Suppose the τ-topology is R_0 and that $\Phi(p)$ is compact for each $p \in \mathrm{Ext}(K)$. Suppose $\{p_n\} \subseteq \mathrm{Ext}(K)$ and $p_n \to q$. Then*

$$\bigcap_k \bigcup_{n \geqq k} \Phi(p_n) \subseteq \Phi(q) .$$

PROOF. If $\bigcap_k \bigcup_{n \geqq k} \Phi(p_n) = \emptyset$, we are done. So suppose

$$z \in \bigcap_k \bigcup_{n \geqq k} \Phi(p_n) . \tag{1.1}$$

Hence $z \in \mathrm{Ext}(K)$ and for each k, there is $n_k \geqq k$ with $z \in \Phi(p_{n_k})$. As the τ-topology is R_0, $\Phi(z) = \Phi(p_{n_k})$ for each k. Since $\Phi(z)$ is compact, $q \in \Phi(z)$ and so $z \in \Phi(z) = \Phi(q)$ since the topology is R_0. As this holds for each z satisfying (1.1) we are again done.

So much for the second hypothesis. As to the first, we have the following characterization of $\mathrm{co}^-(T(q), \bigcup_n T(p_n))$ which will be of use in the applications in section 3. We shall return to the hypothesis that $\mathscr{T}$ is smooth in the next section after we discuss transformation groups.

PROPOSITION 1.4. *Let K be metrizable and let $\mathscr{T}$ be smooth. Let $\{p_n\} \subseteq \mathrm{Ext}(K)$ converge to q. Suppose $p_n \notin T(q)$ for each n, and $p_n \notin T(p_m)$ for each n, m. Let*

$$F = \{\beta f + \textstyle\sum_n \alpha_n f_n : f \in T(q), f_n \in T(p_n) , \\ \beta \geqq 0,\ \alpha_n \geqq 0,\ \textstyle\sum_n \alpha_n + \beta = 1\} .$$

Then $F = \mathrm{co}^-(T(q), \bigcup_n T(p_n))$.

PROOF. Let $S = \mathrm{co}^-(T(q), \bigcup_n T(p_n))$. From Lemma 1.1,

$$\mathrm{Ext}(S) = \bigcup_n \Phi(p_n) \cup \Phi(q) .$$

We give the proof for the case $\Phi(q) \neq \emptyset$ and leave the case $\Phi(q) = \emptyset$ to the reader. By hypothesis, the right hand side is a disjoint union of a countable number of Borel sets. Let $x \in S$ and find μ, a probability measure supported by $\mathrm{Ext}(S)$, such that $x = r(\mu) =$ resultant of μ. [For the existence of μ see 1, Theorem I.4.8.] Then

$$\mu = \textstyle\sum_n \alpha_n \nu_n + \beta\nu ,$$

where ν_n is a probability measure supported by $\Phi(p_n)$, ν a probability measure supported by $\Phi(q)$, $\alpha_n \geqq 0$, $\beta \geqq 0$, and $\sum_n \alpha_n + \beta = 1$. Thus

$$x = r(\mu) \\ = \textstyle\sum_n \alpha_n r(\nu_n) + \beta r(\nu) ,$$

and since ν_n, ν are supported by disjoints sets

$$x = \sum_n \alpha_n f_n + \beta f,$$

where $f_n \in \text{co}^- \Phi(p_n) = T(p_n)$ and $f \in \text{co}^- \Phi(q) = T(q)$. Thus $S \subseteqq F$. The other inclusion is trivial.

2. The group of $\mathcal{T}$-affine maps.

Before discussing the applications we first must consider a somewhat different setting. Let G be a group of transformations on a compact Hausdorff space Z, i.e., there is a homomorphism π taking G to the $1-1$ maps $\pi(g)$ of Z onto Z. We note that $\pi(g)$ for $g \in G$ need not be continuous. We will denote the action of $\pi(g)$ on z by $g \cdot z$ where $g \in G$ and $z \in Z$. If $p \in Z$ we let $G(p)$ be the orbit of p under this action of G, i.e.

$$G(p) = \{g \cdot p : g \in G\}.$$

We say that G *acts smoothly* on Z if the following condition holds:

Whenever the sequence $\{p_n\}$ *converges to* q *in* Z *the union*

$$\bigcup_{n=1}^{\infty} G(p_n)^- \cup G(q)^-$$

is a closed subset of Z.

We say that G acts *equi-continuously* on the compact metric space Z if for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that for each $p, q \in Z$ and $g \in G$ we have

$$\text{dist}(p,q) < \delta \Rightarrow \text{dist}(g \cdot p, g \cdot q) < \varepsilon .$$

PROPOSITION 1.1. *If* G *acts equi-continuously on the compact metric space* Z *then* G *acts smoothly.*

PROOF. It clearly suffices to show the following: if $z_n \in G(p_n)^-$ and $z_n \to z$ then $z \in G(q)^-$. Suppose $\{z_n\}$ and z satisfy these conditions. Clearly we may find $g_n \in G$ satisfying $g_n \cdot p_n \to z$. By going to a subsequence and re-indexing, we may assume that there is a $z' \in G(q)^-$ such that $g_n \cdot q \to z'$. Let $\varepsilon > 0$ be given and find $\delta = \delta(\varepsilon/3)$ in the definition of equi-continuity. Then there is an N such that for all $n \geqq N$ we have:

$$\text{dist}(p_n, q) < \delta, \quad \text{dist}(g_n \cdot p_n, z) < \varepsilon/3,$$

$$\text{dist}(g_n \cdot q, z') < \varepsilon/3 .$$

Take $g = g_N$ and apply equi-continuity to get that $\text{dist}(z, z') < \varepsilon$. Hence $z = z' \in G(q)^-$.

COROLLARY 1.2. *Let G be compact and Z be a compact metric space. Suppose $G \times Z \to Z$ by $(g,z) \mapsto g \cdot z$ is jointly continuous. Then G acts equi-continuously and, so, smoothly.*

COROLLARY 1.3. *Let G be finite and suppose $\pi(g) \in C(Z)$ for each $g \in Z$. Then G acts equi-continuously and, so, smoothly.*

We now return to the setting in the introduction. We take for Z the set $\operatorname{Ext}(K)^-$. A map $g: Z \to Z$ is *$\mathscr{T}$-affine* if

$$\{g(p) : p \in T \cap Z\} = T \cap Z \quad \text{for each } T \in \mathscr{T} .$$

Let H be the set of all 1–1 maps of Z onto Z which are $\mathscr{T}$-affine. H is clearly a group under composition; let G be a subgroup. G is a group of transformations of the compact set Z. If $p \in Z$, we let $G(p)$ be the orbit of p under G. Note that

$$G(p) \subseteqq T(p) \cap Z$$

for each $p \in Z$. We say that G is *large* [compare with 2, page 429] if for each $p \in \operatorname{Ext}(K)$ we have $G(p)^- = \Phi(p)^-$. Note that we have not assumed that any $g \in G$ is necessarily continuous or is affine on Z. Neither have we assumed that the map $g \mapsto g(p)$ is continuous for any $p \in Z$.

PROPOSITION 2.4. *Suppose G is large.*

1. *If G acts smoothly on Z, then $\mathscr{T}$ is smooth.*
2. *Suppose G is compact and the map $g \mapsto g(p)$ is continuous for each $p \in \operatorname{Ext}(K)$. Then the τ-topology is R_0. Further, it is T_0 (and hence T_1) iff $G = \{I\}$.*
3. *If, in addition to the hypotheses of 2, G maps $\operatorname{Ext}(K)$ into $\operatorname{Ext}(K)$, then $\Phi(p)$ is compact for each $p \in \operatorname{Ext}(K)$.*

PROOF. As $G(q)^- \subseteqq T(q)$ for each $q \in Z$, (1) is immediate. The hypotheses of (2) easily yields $G(p) = \Phi(p)^-$ for each $p \in \operatorname{Ext}(K)$ and so (2) and (3) are clear.

3. Two applications.

The first application we shall make of the previous results is to the topology generated by the weak* closed L-ideals [see 3]. We let V be any separable real Banach space and let K be the unit ball of V^* with the weak* topology. A map e from V^* to V^* is an L-projection if it satisfies:

1. $e^2 = e$,
2. $\|p\| = \|ep\| + \|p - ep\|$ for all $p \in V^*$.

A subspace of V^* is a L-ideal if it is the range of an L-projection. We take $\mathscr{T}$ to be the collection of intersections of weak* closed L-ideals with K. It is not known whether (C2) holds for $\mathscr{T}$ in general. We give sufficient conditions below, however.

LEMMA 3.1. *Suppose $\{L_n\}$ is a collection of mutually disjoint L-ideals. Let $f_n \in L_n$. Let α_n be real numbers satisfying $\sum_n \|\alpha_n f_n\| < \infty$. Then*

$$\|\textstyle\sum_n \alpha_n f_n\| = \sum_n \|\alpha_n f_n\| .$$

PROOF. As a finite sum of L-ideals is again an L-ideal [3, Proposition 3.14], for each N we have

$$\|\textstyle\sum_{n=1}^N \alpha_n f_n\| = \sum_{n=1}^N \|\alpha_n f_n\| .$$

Obvious estimates using the triangle inequality complete the proof.

THEOREM 3.2. *Let K be the unit ball of V^* for a separable Banach space V. Let $\mathscr{T}$ be the collection of intersections of weak* closed L-ideals with K. Suppose the following hold:*

1. *$\mathscr{T}$ is smooth.*
2. *Whenever $\{p_n\} \subseteqq \operatorname{Ext}(K)$ and $p_n \to q$, we have*

$$\textstyle\bigcap_k \bigcup_{n \geqq k} T(p_n) \subseteqq \Phi(q) .$$

Then $\mathscr{T}$ satisfies (C2) *and so Statement* 0.1 *holds.*

PROOF. From Theorem 1.2 we need only show the following: Suppose $\{p_n\} \subseteqq \operatorname{Ext}(K)$ converges to q. Then

$$\operatorname{co}^-(T(q), \textstyle\bigcup_n T(p_n)) \in \mathscr{T} .$$

We let F be the closed convex hull of $T(q)$ and $\bigcup_n T(p_n)$. Without loss of generality, we assume that $p_n \notin T(q)$ and $p_m \notin T(p_n)$ for each n,m. By Proposition 1.4,

$$F = \{\beta f + \textstyle\sum_n \alpha_n f_n \mid f \in T(q), f_n \in T(p_n), \beta \geqq 0, \alpha_n \geqq 0, \beta + \sum_n \alpha_n = 1\} .$$

Let L be the linear span of F and

$$L' = \{\beta f + \textstyle\sum_n \alpha_n f_n \mid f \in T(q), f_n \in T(p_n), \sum_n \|\alpha_n f_n\| + \|\beta f\| < \infty\} .$$

We first claim that $L' \cap K = F$. Indeed, let

$$g = \beta f + \textstyle\sum_n \alpha_n f_n \in L' \cap K .$$

Noting that each set in $\mathscr{T}$ is symmetric, by replacing f_n $[f]$ by a multiple of modulus one wherever necessary, we may assume $\beta \geqq 0$, $\alpha_n \geqq 0$ for each n. If $g=0$, then $g \in F$. So we may assume $g \neq 0$. By Lemma 3.1, we have

$$\|g\| = \beta\|f\| + \sum_n \alpha_n \|f_n\| .$$

Let

$$\begin{aligned} f' &= f\|g\|/\|f\| && \text{if } f \neq 0 , \\ f' &= 0 && \text{if } f = 0 , \\ f_n' &= f_n\|g\|/\|f_n\| && \text{if } f_n \neq 0 , \\ f_n' &= 0 && \text{if } f_n = 0 , \\ \beta' &= \beta\|f\|/\|g\|; \quad \alpha_n' = \alpha_n\|f_n\|/\|g\| . \end{aligned}$$

Then $f' \in T(q)$ and $f_n' \in T(p_n)$. Also

$$\begin{aligned} g &= \beta' f' + \textstyle\sum_n \alpha_n' f_n', \\ 1 &= \beta' + \textstyle\sum_n \alpha_n' . \end{aligned}$$

Hence $g \in F$. Using Lemma 3.1, the other inclusion is clear.

We next claim that $L \subseteqq L'$, Indeed, let $g^1, \ldots, g^M \in F$ and consider $\gamma^1 g^1 + \ldots + \gamma^M g^M$. Since the infinite sums involved in the definition of $g^1, \ldots, g^M$ converge absolutely, we may re-arrange the summands at will. Hence

$$\begin{aligned} \gamma^1 g^1 + \ldots + \gamma^M g^M &= \gamma^1 \beta^1 f^1 + \textstyle\sum_n \gamma^1 \alpha_n^1 f_n^1 + \ldots \\ &= (\gamma^1 \beta^1 f^1 + \ldots + \gamma^M \beta^M f^M) + \ldots . \end{aligned}$$

Note that $\gamma^1 \beta^1 f^1 + \ldots + \gamma^M \beta^M f^M \in T(q)$ and similar for $T(p_n)$. As

$$\begin{aligned} \|\gamma^1 \beta^1 f^1 + \ldots + \gamma^M \beta^M f^M\| &+ \textstyle\sum_n \|\gamma^1 \alpha_n^1 f_n^1 + \ldots + \gamma^M \alpha_n^M f_n^M\| \\ &\leqq \|\gamma^1 g^1\| + \ldots + \|\gamma^M g^M\| < \infty , \end{aligned}$$

$\gamma^1 g^1 + \ldots + \gamma^M g^M \in L'$.

Since $F \subseteqq L$, $L \cap K = F$. By the Krein–Smulian Theorem, L is weak* closed and, so, norm closed. Clearly L includes $(T(q) + \sum_n T(p_n))^-$ where the bar indicates norm closure and where we consider sums of finitely many elements. On the other hand, Lemma 3.1 implies that

$$(T(q) + \textstyle\sum_n T(p_n))^- \supseteqq L'$$

and so

$$L = L' = (T(q) + \textstyle\sum_n T(p_n))^- .$$

As the latter is an L-ideal [3, Proposition 3.14], we have $F = L \cap K \in \mathscr{T}$.

Combining the results of section 2 with Theorem 3.2 yields

COROLLARY 3.3. *Let $\mathscr{T}$ be as in Theorem 3.2. Let G be a subgroup of the group of 1–1 $\mathscr{T}$-affine maps of* $\mathrm{Ext}(K)^-$ *onto* $\mathrm{Ext}(K)^-$. *Suppose the following hold:*

1. *G acts equi-continuously [or only smoothly] on* $\mathrm{Ext}(K)^-$.
2. *G is large.*
3. *Whenever $\{p_n\} \subseteqq \mathrm{Ext}(K)$ converges to q, we have*

$$\bigcap_k \bigcup_{n \geqq k} \Phi(p_n) \subseteqq \Phi(q) .$$

Then Statement 0.1 *holds.*

Finally using the results of sections 1 and 2 we get

COROLLARY 3.4. *Let $\mathscr{T}$ and G be as in Corollary* 3.3. *Suppose the following hold:*

1. *The map $G \times \mathrm{Ext}(K)^- \to \mathrm{Ext}(K)^-$ is jointly continuous [or just continuous in G and equi-continuous in* $\mathrm{Ext}(K)^-$].
2. *G is large.*
3. *G is compact.*
4. *G maps* $\mathrm{Ext}(K)$ *into* $\mathrm{Ext}(K)$.

Then Statement 0.1 *holds.*

We note that for V a separable Lindenstrauss space, we have $G = \mathbf{Z}_2$ and so corollary 3.4 applies.

The second application we shall make is to the topology generated by the closed split faces [see 1; 2]. Let K be a compact convex set in an lctvs E. Let F be a closed face of K. We let F' be the union of all faces disjoint from F. It is always true that $K = \mathrm{co}(F \cup F')$ [2, Cor. 1.2]. Thus for each $x \in K$, there are points $f \in F$, $f' \in F'$ and $0 \leqq \alpha \leqq 1$ satisfying

$$x = \alpha f + (1-\alpha) f' .$$

The face F is said to be *split* if F' is a face and if for each $x \in K - (F \cup F')$ the elements f, f', and α in the above decomposition are all unique. We take $\mathscr{T}$ to be the set of all closed split faces of K. There are examples in which $\mathscr{T}$ does not satisfy condition (C2). We present sufficient conditions below.

THEOREM 3.5. *Let K be a compact convex set in a locally convex topological vector space E. Let $\mathscr{T}$ be the collection of closed split faces of K. Suppose the following hold:*

1. *$\mathscr{T}$ is smooth.*
2. *Whenever $\{p_n\} \subseteqq \operatorname{Ext}(K)$ and $p_n \to q$, we have*

$$\bigcap_k \bigcup_{n \geqq k} T(p_n) \subseteqq T(q) .$$

3. *The τ-topology is R_0.*

Then $\mathscr{T}$ satisfies (C2) *and so Statement* 0.1 *holds.*

PROOF. We arrange, by translation if need be, that $0 \in \operatorname{Ext}(K)$. Again, from Theorem 1.2 we need only show the following: Suppose $\{p_n\} \subseteqq \operatorname{Ext}(K)$ converges to q. Then $\text{co}^-(T(q), \bigcup_n T(p_n)) \in \mathscr{T}$. We let F be the closed convex hull of $T(q)$ and $\bigcup_n T(p_n)$. We assume, without loss of generality, that $p_n \notin T(q)$ and $p_n \notin T(p_m)$ for each n,m. By Proposition 1.4,

$$F = \{\beta f + \textstyle\sum_n \alpha_n f_n : f \in T(q), f_n \in T(p_n), \beta \geqq 0 , \\ \alpha_n \geqq 0, \beta + \textstyle\sum_n \alpha_n = 1\} .$$

CLAIM (1). *F is a closed face.*

JUSTIFICATION. Let $x,y \in K$ and suppose $g = \lambda x + (1-\lambda)y \in F$ for $0 < \lambda < 1$. We must show that x and y belong to F. Since $g \in F$ we may write

$$g = \lambda x + (1-\lambda)y = \textstyle\sum_n \alpha_n f_n + \beta f . \tag{3.1}$$

As $\Phi(p_i) \cap \Phi(p_j) = \emptyset$ for $i \neq j$ we have that $T(p_i) \cap T(p_j) = \emptyset$ for $i \neq j$. A trivial induction argument can be used to extend [2, Lemma 2.1] to get for each N a unique decomposition:

$$\begin{aligned} x &= \textstyle\sum_{n=1}^N a_n x_n + a_{N,r} x_{N,r} \\ y &= \textstyle\sum_{n=1}^N b_n y_n + b_{N,r} y_{N,r} \\ g &= \textstyle\sum_{n=1}^N \alpha_n f_n + \sum_{n=1}^N d_n g_n + d_{N,r} g_{N,r} \end{aligned} \tag{3.2}$$

where $x_n \in T(p_n)$, $x_{N,r} \in T(p_1)' \cap \ldots \cap T(p_N)'$ and $\sum_{n=1}^N a_n + a_{N,r} = 1$ [and similar statements for the other decompositions in (3.2)]. Neither a_n nor x_n depend on N since $T(p_n)$ is split [similar for the others]. As $T(p_n)$ is split, we get from (3.1)

$$\lambda a_n x_n + (1-\lambda) b_n y_n = \alpha_n f_n + d_n g_n \quad \text{for } 1 \leqq n \leqq N . \tag{3.3}$$

Choose convergent subsequences of $\{a_{N,r}\}$, $\{x_{N,r}\}$, and the corresponding sequences in (3.2). In the limit we have (with obvious notation)

$$(3.4)\qquad \begin{aligned} x &= \textstyle\sum_{n=1}^{\infty} a_n x_n + a_r x_r \\ y &= \textstyle\sum_{n=1}^{\infty} b_n y_n + b_r y_r \\ g &= \textstyle\sum_{n=1}^{\infty} \alpha_n f_n + \sum_{n=1}^{\infty} d_n g_n + d_r g_r + \beta f . \end{aligned}$$

Using (3.1) and (3.3) we easily get

$$\lambda a_r x_r + (1-\lambda) b_r y_r + \textstyle\sum_{n=1}^{\infty} d_{n,n} g_{n,n} = \beta f .$$

If $\beta = 0$, then $a_r = b_r = 0$ and the representation (3.4) shows F is a face. If $\beta \neq 0$, by dividing through by β and using $T(q)$ is a face, one easily gets that $[a_r = 0$ or $x_r \in T(q)]$ and $[b_r = 0$ or $y_r \in T(q)]$. Hence again $x, y \in F$ and so F is indeed a face.

Claim (2). $F' = T(q)' \cap \bigcap_n T(p_n)'$.

Justification. Clearly, $F' \subseteq T(q)' \cap \bigcap_n T(p_n)'$. Conversely suppose

$$x \in T(q)' \cap \textstyle\bigcap_n T(p_n)' \cap F .$$

Then $x = \beta f + \sum_n \alpha_n f_n$ since $x \in F$. Since $x \in T(q)'$ which is a face, $\beta = 0$. Similarly, $\alpha_n = 0$ each n. Since $\beta + \sum_n \alpha_n = 1$, we have

$$T(q)' \cap \textstyle\bigcap_n T(p_n)' \cap F = \emptyset .$$

Since $T(q)' \cap \bigcap_n T(p_n)'$ is a face, it is a subset of F'.

Claim (3). *F is a split face.*

Justification. From Claim 2, F' is a face. It follows from Claim 1 that each point $x \in K$ has a representation as a linear combination of elements of F and F'. So we need only show uniqueness of such decompositions. Hence, let $x \in K$ and suppose

$$x = \gamma_1 g_1 + (1-\gamma_1) g_1' = \gamma_2 g_2 + (1-\gamma_2) g_2'$$

for $0 \leqq \gamma_i \leqq 1$, $g_i \in F$, and $g_i' \in F'$. Let

$$g_i = \beta_i f_i + \textstyle\sum_n \alpha_{n,i} f_{n,i} \qquad i = 1, 2$$

as above. Fix N. Then

$$x = \gamma_i \alpha_{N,i} f_{N,i} + [\textstyle\sum_{n \neq N} \alpha_{n,i} f_{n,i} + \gamma_i \beta_i f_i + (1-\gamma_i) g_i] \qquad i = 1, 2 .$$

From Claim (1) $\mathrm{co}^-(T(q), \bigcup_{n \neq N} T(p_n))$ is a face. As

$$\mathrm{Ext}\ \mathrm{co}^-(T(q), \textstyle\bigcup_{n \neq N} T(p_n)) \cap \Phi(p_N) = \emptyset$$

using Lemma 1.1, we have that it is disjoint from $T(p_N)$ and so is a subset of $T(p_N)'$. Since $T(p_N)$ is split, we have

$$\gamma_1\alpha_{N,1} = \gamma_2\alpha_{N,2}, \quad f_{N,1} = f_{N,2} \quad \text{for each } N .$$

Hence

$$\gamma_1\beta_1 f_1 + (1-\gamma_1)g_1' = \gamma_2\beta_2 f_2 + (1-\gamma_2)g_2' . \tag{3.5}$$

Adding the requisite coefficient times zero to both sides of (3.5) we would have two representations in terms of $T(q)$ and $F' \subseteq T(q)'$. Since $T(q)$ is split, $\gamma_1\beta_1 = \gamma_2\beta_2$, $f_1 = f_2$, $(1-\gamma_1) = (1-\gamma_2)$, and $g_1' = g_2'$. Clearly the two representations of x are the same and we are done.

Combining the results of section 1 and 2 with Theorem 3.5 we get

COROLLARY 3.6. *Let $\mathscr{T}$ be as in Theorem* 3.5. *Let G be a subgroup of the group of* 1–1 *$\mathscr{T}$-affine maps of* $\mathrm{Ext}(K)^-$ *onto* $\mathrm{Ext}(K)^-$. *Suppose the following hold*:

1. *The map* $G \times \mathrm{Ext}(K)^- \to \mathrm{Ext}(K)^-$ *is jointly continuous [or just continuous in G and equi-continuous in* $\mathrm{Ext}(K)^-$].
2. *G is large.*
3. *G is compact.*
4. *G maps* $\mathrm{Ext}(K)$ *into* $\mathrm{Ext}(K)$.

Then Statement 0.1 *holds.*

Further specializing we get

COROLLARY 3.7. *Let $\mathscr{T}$ be as in Theorem* 3.5. *Suppose the τ-topology is* T_1. *Then Statement* 0.1 *holds.*

4. Function algebras.

We shall here apply Corollary 3.4 to the generalized peak set topology of function algebras [see, for example, 4, p. 113]. Let X be a compact Hausdorff space. We denote by $C(X)$ the complex valued continuous functions on X and equip it with the supremum norm. A closed subalgebra A of $C(X)$ is a *function algebra* if it contains the constants and separates the points of X. A subset E of X is a *peak set* if there is an $f \in A$ such that $f|E = 1$ and $|f(y)| < 1$ if $y \in X - E$. We call a closed subset E of X a *generalized peak set* if it is the intersection of peak sets. Clearly a generalized peak set is a peak set iff it is a G_δ-set. If E is a generalized peak set, then we let

$$I_E = \{a \in A : a \equiv 0 \text{ on } E\}.$$

We let S be the state space of A, i.e.

$$S = \{F \in A^* : \|F\| = 1 = F(\mathbf{1})\}.$$

We note that $\{\delta(x): x \in X\} \subseteq S$ and we identify X with this subset of S. We let K be the unit ball of A^*.

THEOREM 4.1. [7, Theorem 3.1 and Theorem 1.2]. *If E is a generalized peak set, then $I_E{}^\perp$ is a weak* closed L-ideal in A^*.*

Conversely, if J is a weak closed L-ideal in A^*, then $J \cap X$ is a generalized peak set. Further, each weak* closed L-ideal in $A^*{}_r$, i.e. A^* considered as a real Banach space, is a complex subspace of A^*.*

THEOREM 4.2. *The maps given in Theorem 4.1 are inverse of each other. Hence there is a one-to-one correspondence between the intersections of weak* closed L-ideals with K and generalized peak sets.*

PROOF. Let E be a generalized peak set. We claim that $I_E{}^\perp \cap X = E$. Indeed,

$$I_E{}^\perp \cap X = \{x \in X : (\forall a \in A) a \equiv 0 \text{ on } E \Rightarrow a(x) = 0\}.$$

Let $x \notin E$. Then there is a peak set P containing E with $x \notin P$. Thus, there is $f \in A$ with $f|P \equiv 1$ and $|f(y)| < 1$ for $y \in X - P$. Then $\mathbf{1} - f|E = 0$ and $(\mathbf{1} - f)(x) \neq 0$ so $x \notin I_E{}^\perp \cap X$ and the other conclusion is trivial. Now let J be a weak* closed L-ideal. We claim that $I^\perp_{J\cap X} = J$. Indeed, let $J' = I^\perp_{J\cap X}$. Then, by the first part of the proof and Theorem 4.1, $J' \cap X = J \cap X$. Since $X \supseteq \text{Ext}(S)$, $J' \cap S = J \cap S$. However, the circled closed convex hull of S is K. Since the L-ideals are complex subspaces, we have $J' \cap K = J \cap K$. But then $J = J'$.

THEOREM 4.3. *Let A be a function algebra on a compact metric space X. Then the following are equivalent for each peak point $p \in X$:*

1. *The peak set topology is first countable at p.*
2. *The peak set topology is locally compact at p.*
3. *The peak set topology is sequentially locally compact at p.*

Further, if the peak set topology is first countable for each peak point $p \in X$, it is second countable.

PROOF. Let $\mathscr{T}$ be the collection of intersections of weak* closed L-ideals in A^* with K. Let $s \in \text{Ext}(S)$ and let $Z(s) = \{zs: |z| = 1\}$. We recall that the collection of peak points of X is precisely $\mathrm{Ext}(S)$ [4, Theorem 2.3.4]. Using Theorems 4.1 and 4.2, we see that $Z(s)$ is τ-closed and is the minimal τ-closed set containing s. Hence $\Phi(s)=Z(s)$. Further, if $p \in \mathrm{Ext}(K)$, then $p=zs$ for $|z|=1$ and $s \in \mathrm{Ext}(S)$. Hence $G=$ circle group is a subgroup of all one-to-one $\mathscr{T}$-affine maps of $\mathrm{Ext}(K)^-$ onto $\mathrm{Ext}(K)^-$. Clearly G is compact, large, and maps $\mathrm{Ext}(K)$ into $\mathrm{Ext}(K)$. By taking the standard metric for K [see 5 p. 426, for instance], one easily checks that the map $G \times \mathrm{Ext}(K)^- \to \mathrm{Ext}(K)^-$ is jointly continuous. Thus by Corollary 3.4, Statement 0.1 applies to the τ-topology. But Theorem 4.2 implies that the τ-topology is the same as the peak set topology and the theorem follows immediately.

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