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AN OMITTING TYPES THEOREM WITH AN APPLICATION TO THE CONSTRUCTION OF GENERIC STRUCTURES

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Abstract.

We give a forcing-free construction of f-generic structures. The construction uses an omitting types theorem of independent interest.

Introduction.

Throughout we consider theories T (i.e. deductively closed sets of sentences) formalized in some countable first order language L. (The countability of L is an essential restriction.) There is associated with each such theory T a certain class of structures $\mathscr{F}_T$, the class of $T-f$-generic structures, see [1]. This class is usually constructed using f-forcing; we will construct $\mathscr{F}_T$ by omitting types.

It is no surpise that $\mathscr{F}_T$ can be constructed in this way (by omitting types). The members of $\mathscr{F}_T$ are the completing models of T^f ($=\mathrm{Th}(\mathscr{F}_T)$) and hence are those structures which omit certain types Γ_φ (see lemma 1); equivalently (as mentioned in [8, theorem 1.2]) the members of $\mathscr{F}_T$ are those structures which omit certain other types p_φ. The catch is that to define Γ_φ or p_φ we must refer to the forcing relation, whereas the method used here makes no use at all of forcing.

I am grateful to G. Cherlin for several comments on [10] which proved relevant to the problem discussed here; to R. Cusin for showing me a preprint containing theorem 1; and to A. Macintyre for several specific and general points.

Omitting types theorems.

Let $v_0, v_1, v_2, \ldots$ be the variables of the underlying language L.

By a type we will here mean a set of formulas Γ such that the set $fv(\Gamma)$ of free variables occurring in Γ is a subset of $\{v_0, \ldots, v_k\}$ for some

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integer $k \geqq 0$. We will sometimes indicate $fv(\Gamma)$ by writing $\Gamma(v_0, \ldots, v_k)$. We use the standard notions of realizing and omitting a type.

Let T be some theory. A type Γ is T-np (non-principal over T) if there is no formula ψ consistent with T such that $T \vdash \psi \to \gamma$, for each $\gamma \in \Gamma$. (Clearly it is sufficient to consider only those ψ with $fv(\psi) \subseteqq fv(\Gamma)$.)

The following theorem is well-known.

THEOREM A. *Let T be some fixed theory and Γ some countable collection of T-np types. For each sentence σ consistent with T there is a countable structure $\mathfrak{A}$ such that*

$$\begin{aligned} &\text{(Ai)} \quad \mathfrak{A} \models T\,, \\ &\text{(Aii)} \quad \mathfrak{A} \models \sigma\,, \\ &\text{(Aiii)} \quad \mathfrak{A} \textit{ omits each type in } \Gamma\,. \end{aligned}$$

For each integer $n \geqq 0$ let $\forall_n (\exists_n)$ be the set of formulas logically equivalent to formulas in prenex normal form whose prenex consists of n blocks of quantifiers, the first block being universal (existential), the second block being existential (universal), the third block being universal (existential), etc. For each two structures $\mathfrak{A}, \mathfrak{B}$ let $\mathfrak{A} \prec_n \mathfrak{B}$ mean that $\mathfrak{A} \subseteqq \mathfrak{B}$ and

$$\mathfrak{A} \models \varphi[x] \Rightarrow \mathfrak{B} \models \varphi[x]$$

holds for all formulas $\varphi \in \forall_n$ and all $\mathfrak{A}$-assignments x. We note that for each theory T, $\mathfrak{A} \models T_n \cap \forall_{n+1}$ if and only if $\mathfrak{A} \prec_n \mathfrak{B}$ for some $\mathfrak{B} \models T$.

A type Γ is $T-(n)$-np if $\Gamma \subseteqq \forall_{n+1}$ and there is no formula $\psi \in \exists_{n+1}$ consistent with T such that $T \vdash \psi \to \gamma$, for each $\gamma \in \Gamma$.

We will need theorem A as well as the following refinement.

THEOREM B. *Let T be some fixed theory, $n \geqq 0$ some integer, and Γ some countable collection of $T-(n)$-np types. For each $\sigma \in \exists_{n+1}$ consistent with T there is a countable structure $\mathfrak{A}$ such that*

$$\begin{aligned} &\text{(Bi)} \quad \mathfrak{A} \models T \cap \forall_{n+1}\,, \\ &\text{(Bii)} \quad \mathfrak{A} \models \sigma\,, \\ &\text{(Biii)} \quad \mathfrak{A} \textit{ omits each type in } \Gamma\,. \end{aligned}$$

Clearly theorems A, B are of the same family. Theorem B is also related to a theorem of Chang, [2].

Following [2] we say a type Γ is $T-(n+2)$-existential if $\Gamma \subseteqq \exists_{n+2}$ and there is no type $\Delta \subseteqq \exists_{n+1}$ consistent with T such that for each $\gamma \in \Gamma$ there is some $\delta \in \Delta$ with $T \vdash \delta \to \gamma$.

THEOREM C. *Let T be some fixed theory, $n \geqq 0$ some integer. For each model $\mathfrak{M}$ of T is some structure $\mathfrak{A}$, of the same cardinality as $\mathfrak{M}$, such that*

(Ci) $\mathfrak{A} \models T \cap \forall_{n+1}$,
(Cii) $\mathfrak{M} \prec_n \mathfrak{A}$,
(Ciii) $\mathfrak{A}$ *omits each $T-(n+2)$-existential type* .

This theorem occurs in [2, § 3], however, the following remarks should be noted.

(a) Chang's n is our $n+1$.
(b) Chang assumes that T is $\forall_{n+2}$-axiomatizable (our n) and proves $\mathfrak{A} \models T$. This makes no essential difference. We see from lemma 0 (below) that in the presence of (Ciii) we can strengthen (Ci) to $\mathfrak{A} \models T \cap \forall_{n+2}$.
(c) (Cii) is stronger than Chang's (ii), but Chang does verify (Cii), see [2, (3) on p. 67].

(Ciii) can be given in an equivalent, more understandable form. To do this we use the type $\Gamma(\varphi, n+1)$ that is

$$\{\varphi\} \cup \{\neg\psi : \psi \in \exists_{n+1}, fv(\psi) \subseteq fv(\varphi), T \vdash \psi \rightarrow \varphi\}$$

for formulas $\varphi \in \forall_{n+1}$. Such a type is easily seen to be $T-(n+2)$-existential, (see [2, p. 65, E.g. B]), and so is also $T-(n)$-np.

LEMMA 0. *Suppose $\mathfrak{A} \models T \cap \forall_{n+1}$. The following are equivalent.*

(i) $\mathfrak{A}$ *omits each $T-(n+2)$-existential type.*
(ii) $\mathfrak{A}$ *omits $\Gamma(\varphi, n+1)$, for each $\varphi \in \forall_{n+1}$.*
(iii) *For each model $\mathfrak{B}$ of T, if $\mathfrak{A} \prec_n \mathfrak{B}$ then $\mathfrak{A} \prec_{n+1} \mathfrak{B}$.*

We can now make a direct comparison between B and C. Clearly

$$\text{(Bi)} = \text{(Ci)}$$

and

$$\text{(Bii)} \Leftarrow \text{(Cii)}$$

Also, provided we have $\Gamma(\varphi, n+1) \in \boldsymbol{\Gamma}$ for each $\varphi \in \forall_{n+1}$,

$$\text{(Biii)} \Rightarrow \text{(Ciii)} .$$

The stronger version of B obtained by replacing (Bii) by (Cii) is false.

Proof of B.

Anyone familiar with the proof of A will be able to provide a proof of B himself. We will not give all the details of the proof, but just outline the main points.

Let T, n, Γ, σ be given. We form a new language M from L by adjoining a sequence $(a_i : i < \omega)$ of new constant symbols. We refer to these as parameters. We construct an M-structure $\langle \mathfrak{A}, (a_i : i < \omega)\rangle$ such that $\mathfrak{A}$ is the required L-structure and each element of $\mathfrak{A}$ is some a_i. To do this we construct a set of M-sentences $X \subseteqq \exists_{n+1}$ such that the following hold.

(1) $T \cup X$ is consistent.

(2) $\sigma \in X$.

(3) For each M-sentence $\tau \in \exists_{n+1}$, if $T \cup X \cup \{\tau\}$ is consistent then $\tau \in X$.

(4) For each $\Gamma(v_0, \ldots, v_k) \in \boldsymbol{\Gamma}$ and parameters $a_{i_0}, \ldots, a_{i_k}$ there is some $\gamma(v_0, \ldots, v_k) \in \Gamma$ such that $\neg\gamma(a_{i_0}, \ldots, a_{i_k}) \in X$.

(5) For each formulas $\varphi(v_0, \ldots, v_k)$, $\psi(v_0, \ldots, v_k)$ if

$$(\exists v_0, \ldots, v_k)\varphi \in X, \qquad \neg(\forall v_0, \ldots, v_k)\psi \in X$$

then there are parameters $a_{i_0}, \ldots, a_{i_k}$ such that

$$\varphi(a_{i_0}, \ldots, a_{i_k}) \in X, \qquad \neg\psi(a_{i_0}, \ldots, a_{i_k}) \in X .$$

We say $\varphi(a_{i_0}, \ldots, a_{i_k})$, $\neg\psi(a_{i_0}, \ldots, a_{i_k})$ are instances of $(\exists v_0, \ldots, v_k)\varphi$, $\neg(\forall v_0, \ldots, v_k)\psi$.

We construct X as the union of a chain

$$X_0 \subseteqq X_1 \subseteqq \ldots \subseteqq X_m \subseteqq \ldots, \qquad m < \omega$$

of finite M-sets $X_m \subseteqq \exists_{m+1}$. We put $X_0 = \{\sigma\}$ and at each step $X_m \mapsto X_{m+1}$ we consider some triple

$$(\tau, \Gamma(v_0, \ldots, v_k),\ (a_{i_0}, \ldots, a_{i_k}))$$

where τ is an M-sentence in $\exists_{n+1}$, $\Gamma \in \boldsymbol{\Gamma}$, and $a_{i_0}, \ldots, a_{i_k}$ are parameters. We arrange the construction in such a way that every such triple is considered at some stage.

Given X_m we construct X_{m+1} so that the following hold.

(6) If $T \cup X_m \cup \{\tau\}$ is consistent then $\tau \in X_{m+1}$.

(7) There is some $\gamma \in \Gamma$ such that $\neg\gamma(a_{i_0}, \ldots, a_{i_k}) \in X_{m+1}$.

(8) If we have put into X_{m+1} some sentence of the form $(\exists v_0, \ldots, v_k)\varphi$ or $\neg(\forall v_0, \ldots, v_k)\psi$ then we have also put in instances.

We note that (7) is possible since Γ is $T-(n)$-np, so we do not have

$$T \cup X_m \vdash \gamma(a_{i_0} \dots a_{i_k})$$

for all $\gamma \in \Gamma$. Also (8) is possible since only finitely many parameters occur in X_m, so there are unused parameters available as witnesses.

The construction.

For each theory T let $\mathscr{S}_T$ be the class of submodels of T, that is the class of structures $\mathfrak{A}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ for some model $\mathfrak{B}$ of T. Two theories T, T' are co-theories (mutually model consistent) if $\mathscr{S}_T = \mathscr{S}_{T'}$, equivalently if $T \cap \forall_1 = T' \cap \forall_1$. A structure $\mathfrak{A}$ is a completing model of T if $\mathfrak{A} \in \mathscr{S}_T$ and

$$\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B}$$

holds for all models $\mathfrak{B}$ of T.

The following lemma is (well-known and) easily proved.

LEMMA 1. *For each theory T and structure $A \in \mathscr{S}_T$, the following are equivalent.*

(i) *$\mathfrak{A}$ is a completing model of T.*
(ii) *For each formula φ, $\mathfrak{A}$ omits Γ_φ.*

Here Γ_φ is the type $\Gamma(\varphi, 0)$ that is

$$\{\varphi\} \cup \{\neg\theta : \ \theta \in \exists_1, \ fv(\theta) \subseteq fv(\varphi), \ T \vdash \theta \rightarrow \varphi\}.$$

The following lemma is due to Cusin [4, theorem 1′]; it is proved using lemma 1 and theorem A.

THEOREM 2. *For each theory T the following are equivalent.*

(i) *T is the theory of its completing models.*
(ii) *For each formula φ consistent with T there is some formula $\theta \in \exists_1$ consistent with T such that $T \vdash \theta \rightarrow \varphi$.*

From now on let T be some fixed (but arbitrary) theory.

In [9] we showed that there is exactly one class of structures $\mathscr{F}$ such that

(ωi) $T, T^* = \mathrm{Th}(\mathscr{F})$ are co-theories,
(ωii) $\mathscr{F}$ is the class of completing models of T^*.

Uniqueness followed by standard model theoretic arguments (compactness, method of diagrams, etc.), but existence used f-forcing. Here we construct $\mathscr{F}$ using theorems A, B.

From [10] it also follows that for each integer $n \geqq 0$ there is at most one class $\mathscr{F}_n$ such that

(ni) $T, T_n = \mathrm{Th}(\mathscr{F}_n)$ are co-theories,
($n$ii) $\mathscr{F}_n$ is the class of submodels $\mathfrak{A}$ of T such that

$$\mathfrak{A} \subseteqq \mathfrak{B} \Rightarrow \mathfrak{A} \prec_n \mathfrak{B}$$

holds for all models $\mathfrak{B}$ of T_n.

These classes (when they exist) form a chain

$$(h) \qquad \mathscr{F}_0 \supseteqq \mathscr{F}_1 \supseteqq \ldots \supseteqq \mathscr{F}_n \supseteqq \ldots \supseteqq \mathscr{F}$$

with $\mathscr{F} = \bigcap \{\mathscr{F}_n : n < \omega\}$. We will construct (h) step by step.

First note that $\mathscr{F}_0$ must be $\mathscr{S}_T$, and so there is no existence problem here. (In fact there is no existence problem for $\mathscr{F}_1$ since $\mathscr{F}_1 = \mathscr{E}_T$, a class constructed by means of theorem C. See [9] for details.) We must provide a construction of $\mathscr{F}_{n+1}$ from $\mathscr{F}_n$. This we do using [10, theorem 4].

Suppose we have $\mathscr{F}_n$ (for some integer $n \geqq 0$), and consider the class $\mathscr{K}$ of submodels $\mathfrak{A}$ of T such that

$$\mathfrak{A} \subseteqq \mathfrak{B} \Rightarrow \mathfrak{A} \prec_{n+1} \mathfrak{B}$$

holds for all models $\mathfrak{B}$ of T_n. Our problem is, of course, to show that $\mathscr{K}$ is non-empty.

THEOREM 3. *For each sentence* $\sigma \in \exists_{n+1}$ *consistent with* T_n, *there is some* $\mathfrak{A} \in \mathscr{K}$ *with* $\mathfrak{A} \models \sigma$.

PROOF. For each formula φ let Γ_φ be the type

$$\{\varphi\} \cup \{\neg\theta : \ \theta \in \exists_1, \ fv(\theta) \subseteqq fv(\varphi), \ T_n \vdash \theta \to \varphi\}\,.$$

We first show that for $\varphi \in \forall_{n+1}$, Γ_φ is a $T_n - (n)$-np type.

Suppose $\psi \in \exists_{n+1}$ is such that $T_n \vdash \psi \to \gamma$, for each $\gamma \in \Gamma_\varphi$. (We will show that $T_n \cup \{\psi\}$ is inconsistent.)

If ψ is consistent with T_n then (ni) gives $\mathfrak{A} \models \psi[x]$ for some $\mathfrak{A} \in \mathscr{F}_n$ and $\mathfrak{A}$-assignment x. But then ($n$ii) gives $\mathfrak{A} \models \theta[x]$ for some $\theta \in \exists_1$ where $T_n \vdash \theta \to \psi$. Now we have $T_n \vdash \psi \to \varphi$, and we can suppose that $fv(\theta) \subseteqq fv(\psi) \subseteqq fv(\varphi)$, so that $\neg\theta \in \Gamma_\varphi$. Thus we also have $T_n \vdash \psi \to \neg\theta$, so that $T_n \vdash \neg\theta$. This contradicts $\mathfrak{A} \models \theta[x]$, and so Γ_φ is $T_n - (n) -$np.

Now let $\Gamma = \{\Gamma_\varphi : \varphi \in \forall_{n+1}\}$, $\sigma \in \exists_{n+1}$ be consistent with T_n. Theorem B gives us some structure $\mathfrak{A}$ such that

(i) $\mathfrak{A} \models T_n \cap \forall_{n+1}$,
(ii) $\mathfrak{A} \models \sigma$,
(iii) $\mathfrak{A}$ omits each Γ_φ, for $\varphi \in \forall_{n+1}$.

We show that $\mathfrak{A} \in \mathscr{K}$.

First (i) gives $\mathfrak{A} \prec_n \mathfrak{B}$ for some $\mathfrak{B} \models T_n$. In particular (using (ni)) $\mathfrak{A}$ is a submodel of T.

Second, suppose that $\mathfrak{A} \subseteq \mathfrak{B}$ for some $\mathfrak{B} \models T_n$, $\mathfrak{A} \models \varphi[x]$ for some $\varphi \in \forall_{n+1}$ and $\mathfrak{A}$-assignment x. From (iii) we get $\mathfrak{A} \models \theta[x]$ for some $\theta \in \exists_1$ where $T_n \vdash \theta \to \varphi$. Thus $\mathfrak{B} \models \theta[x]$ (since $\theta \in \exists_1$) and so $\mathfrak{B} \models \varphi[x]$ (since $\mathfrak{B} \models T_n$). Hence we get $\mathfrak{A} \prec_{n+1} \mathfrak{B}$.

COROLLARY 4. *$\emptyset \neq \mathscr{K} \subseteq \mathscr{F}_n$, $T_n \subseteq \mathrm{Th}(\mathscr{K})$, $\mathrm{Th}(\mathscr{K}) \cap \forall_{n+1} \subseteq T_n$, in particular T, $\mathrm{Th}(\mathscr{K})$ are co-theories.*

To show that $\mathscr{K} = \mathscr{F}_{n+1}$ we must verify that

(K) $\mathscr{K}$ is the class of submodels $\mathfrak{A}$ of T such that

$$\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec_{n+1} \mathfrak{B}$$

holds for all models $\mathfrak{B}$ of $\mathrm{Th}(\mathscr{K})$.

First consider $\mathfrak{A} \in \mathscr{K}$ (so that $\mathfrak{A}$ is a submodel of T) and suppose $\mathfrak{A} \subseteq \mathfrak{B}$ for some $\mathfrak{B} \models \mathrm{Th}(\mathscr{K})$. In particular $\mathfrak{B} \models T_n$ so that (by definition of $\mathscr{K}$) $\mathfrak{A} \prec_{n+1} \mathfrak{B}$.

Secondly, consider any $\mathfrak{A}$ satisfying the property of (K), in particular $\mathfrak{A}$ is a submodel of T. Now suppose that $\mathfrak{A} \subseteq \mathfrak{B}$ for some $\mathfrak{B} \models T_n$. Then (since $\mathrm{Th}(\mathscr{K}) \cap \forall_{n+1} \subseteq T_n$) we have $\mathfrak{B} \prec_n \mathfrak{C}$ for some $\mathfrak{C} \models \mathrm{Th}(\mathscr{K})$. Thus (K) gives $\mathfrak{A} \prec_{n+1} \mathfrak{C}$, and so $\mathfrak{A} \prec_{n+1} \mathfrak{B}$. Hence $\mathfrak{A} \in \mathscr{K}$, as required.

We have now constructed the chain (h) except for $\mathscr{F}$. We also have a chain

$$T_0 \subseteq T_1 \subseteq \ldots \subseteq T_n \subseteq \ldots, \quad n < \omega$$

of co-theories of T. Let $T^* = \bigcup \{T_n : n < \omega\}$ so that T, T^* are co-theories. To show that $\mathscr{F}$ exists it is sufficient to show that T^* is the theory of its completing models. We do this using theorem 2.

Consider any formula φ consistent with T^*. We have $\varphi \in \forall_{n+1}$ for some n, and φ is consistent with T_n. Conditions (ni,ii) now give $T_n \vdash \theta \to \varphi$ for some $\theta \in \exists_1$ consistent with T_n. But T_n, T^* are co-theories, so θ is consistent with T^*. Also $T^* \vdash \theta \to \varphi$, so we may apply theorem 2.

Other remarks.

(1) Theorems A, B depend heavily on the countability of L and Γ, however, we can replace "Γ countable" by "Γ meager in the appropriate stone space". Does this lead to interesting results about f-generic structures?

(2) The construction of $\mathscr{F}_T$ given here is analogous to the construction of $\mathscr{G}_T$ (the $T-F$-generic structures) using a certain chain

$$\mathscr{G}_0 \supseteqq \mathscr{G}_1 \supseteqq \dots \supseteqq \mathscr{G}_n \supseteqq \dots \supseteqq \mathscr{G}.$$

This chain is built up step by step using a quantifier count as a measure of complexity (as we have done here). In particular the constructions at each step are similar but different. Cherlin noted that there was a certain construction $\mathscr{K} \mapsto \mathscr{K}'$ such that when iterated gave a chain

$$\mathscr{G}_0 \supseteqq \mathscr{G}_0' \supseteqq \mathscr{G}_0'' \supseteqq \dots \supseteqq \mathscr{G}_0^{(n)} \supseteqq \dots \supseteqq \mathscr{G}$$

with $\mathscr{G} = \bigcap \{\mathscr{G}_0^{(n)} : n < \omega\}$. Details can be found in [3].

Is there a corresponding construction which gives $\mathscr{F}_T$? The following may be relevant.

(3) Theorems A, B are clearly related. Indeed if we put "$n = \omega$" in B we get A. Presumably there is a common generalization of A, B which is concerned with an unspecified set of formulas F. This set F would have to satisfy certain restrictions. The general theorem would be such that $F = \{\text{all formulas}\}$ gives A and $F = \forall_n$ gives B. (See [5] and [6].)

(4) Can B be deduced from A (or A from B)? Can A, B be proved using f-forcing?

(5) The whole of the method used here can be lifted to suitable countable fragments of $L_{\omega_1\omega}$ in the manner of [7].

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