MATH. SCAND. 33 (1973), 55–68

ARITHMETIC NORMALITY FOR PROJECTIVE EMBEDDINGS OF FLAG MANIFOLDS

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1. Introduction.

Let X be a variety over a field k, and let $\mathscr{L}$ be a very ample invertible sheaf on X. Then $\mathscr{L}$ defines an embedding in $\boldsymbol{P}_k{}^m = \boldsymbol{P}(H^0(X,\mathscr{L}))$ where $m = \operatorname{rank}(H^0(X,\mathscr{L}))$ as follows: set

$$R_\nu = \operatorname{Im}[(H^0(X,\mathscr{L}))^{\otimes\nu} \to H^0(X,\mathscr{L}^{\otimes\nu})]$$

and $R = \oplus_{\nu=0}^{\infty} R_\nu$. Then the surjection $\boldsymbol{S}(H^0(X,\mathscr{L})) \to R$, where $\boldsymbol{S}(H^0(X,\mathscr{L}))$ is the symmetric algebra of $H^0(X,\mathscr{L})$, defines a closed immersion

$$X = \operatorname{Proj}(R) \to \boldsymbol{P}_k{}^m .$$

X is said to be arithmetically normal (respective arithmetically Cohen-Macauly) for the given embedding if the homogeneous coordinate ring R is normal (respective Cohen-Macauly).

This paper concerns the question of arithmetic normality for projective embeddings of flag schemes $\boldsymbol{D}_\pi(\mathscr{E})$ of type $\pi = (p_1,\ldots,p_r)$ over a field k, and, more generally, of subschemes of $\boldsymbol{D}_\pi(\mathscr{E})$ of the type $X_{\boldsymbol{a}} = X_{a_1,\ldots,a_r}$ described in section 2 ($\mathscr{E}$ is a vector space over k; see section 1 for notation). Previously, these questions have been studied by purely algebraic methods by J. I. Igusa who proved (in [3]) that Grassmannians, that is the case $\pi = (p,q)$, are arithmetically normal for the Plücker embedding. Igusa's method was extended by T. Nishimura [7], who proved that flag schemes of the type $\boldsymbol{D}_{(p_1,p_2,p_3)}(\mathscr{E})$ are arithmetically normal for the standard embedding. More recently, it has been proved, also by purely algebraic methods, by M. Hochster [2] and D. Laksov [6], that Grassmannians and their Schubert subvarieties are arithmetically Cohen-Macauly for the Plücker embedding.

Our method is to prove that certain maps of cohomology groups are surjective (a more detailed description follows below), and then apply a well-known theorem of Serre (see section 4). Our results in terms of normality of coordinate rings are the following:

Recieved December 4, 1972.

(1) For arbitrary $\boldsymbol{a}=(a_1,\ldots,a_r)$ as in section 2 (and with $S=\operatorname{Spec}(k)$), the scheme $X_{\boldsymbol{a}}$ is arithmetically normal for any embedding into a projective m-space $\boldsymbol{P}_k^m$ (theorem (4.3)).

In particular, all flag schemes over k are arithmetically normal for any embedding into a projective m-space. This, together with a previous result [9, theorem (3.8.1)], yield the following:

(2) An arbitrary flag scheme $\boldsymbol{D}_\pi(\mathscr{E})$ over k is arithmetically normal and Cohen-Macauly for any embedding of $\boldsymbol{D}_\pi(\mathscr{E})$ into a projective m-space $\boldsymbol{P}_k^m$ (theorem (4.5)).

To show that $X_{\boldsymbol{a}}$ is arithmetically normal for any projective embedding, it is enough, by the theorem of Serre mentioned above, to show that the canonical map (notation as in section 4)

$$H^0(X_{\boldsymbol{a}},\mathscr{M}_{\boldsymbol{a}}^{\delta})\otimes H^0(X_{\boldsymbol{a}},\mathscr{M}_{\boldsymbol{a}}^{\gamma})\to H^0(X_{\boldsymbol{a}},\mathscr{M}_{\boldsymbol{a}}^{\delta+\gamma})$$

is surjective for all $\delta=(\delta_1,\ldots,\delta_r)$ and $\gamma=(\gamma_1,\ldots,\gamma_r)$ satisfying $\delta_i\geqq 0$ and $\gamma_i\geqq 0$ for $1\leqq i\leqq r-1$.

In section 3 we consider the more general situation where $X_{\boldsymbol{a}}$ is defined over a locally noetherian scheme S. The main result of section 3 is:

(3) Assume δ and γ satisfy the conditions above. Then the canonical homomorphism

$$g_{\boldsymbol{a},*}(\mathscr{M}_{\boldsymbol{a}}^{\delta})\otimes g_{\boldsymbol{a},*}(\mathscr{M}_{\boldsymbol{a}}^{\gamma})\to g_{\boldsymbol{a},*}(\mathscr{M}_{\boldsymbol{a}}^{\delta+\gamma})$$

is an epimorphism of $\mathscr{O}_S$-modules (theorem (3.7)). Here $g_{\boldsymbol{a}}\colon X_{\boldsymbol{a}}\to S$ is the structure morphism.

In particular, if $D=\boldsymbol{D}_{(1,\ldots,1)}(\mathscr{E})$ is the full flag scheme over S and $f\colon D\to S$ is the structure morphism, then for any two non increasing sequences $\alpha=(\alpha_1,\ldots,\alpha_e)$ and $\beta=(\beta_1,\ldots,\beta_e)$ (that is $\alpha_i\geqq\alpha_{i+1}$ and $\beta_i\geqq\beta_{i+1}$ for all i), the canonical homomorphism

$$f_*(\mathscr{L}^{\alpha})\otimes f_*(\mathscr{L}^{\beta})\to f_*(\mathscr{L}^{\alpha+\beta})$$

is an epimorphism of $\mathscr{O}_S$-modules.

In sections 1 and 2 we fix the notation and recall some basic facts which we need.

1. Notation.

Let S be a scheme. For any morphism of schemes $T\to S$ and any $\mathscr{O}_S$-module $\mathscr{E}$, denote by $\mathscr{E}_T$ the pullback to T of $\mathscr{E}$.

An $\mathcal{O}_T$-module $\mathcal{Q}$ is called a *q-quotient* of $\mathcal{E}_T$ if $\mathcal{Q}$ is locally free of constant rank q and is a quotient module of $\mathcal{E}_T$ (that is, there is a surjection of $\mathcal{O}_T$-modules $\mathcal{E}_T \to \mathcal{Q}$).

In this paper let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-module of constant rank e. The schemes we consider will be S-schemes.

Let $r \geqq 1$ and let $\pi = (p_1, \ldots, p_r)$ be a sequence of positive integers satisfying $\sum_{i=1}^{r} p_i = e$. Then set $q_j = \sum_{i=1}^{j} p_i$ for $1 \leqq j \leqq r$, and for convenience set $p_0 = q_0 = 0$. Denote by $\boldsymbol{D}_\pi(\mathcal{E})$ or $\boldsymbol{D}_{(p_1, \ldots, p_r)}(\mathcal{E})$ the *flag scheme of type* π. That is, $\boldsymbol{D}_\pi(\mathcal{E})$ represents the functor whose values in an S-scheme T is the set of sequences of $\mathcal{O}_T$-modules

$$\mathcal{E}_T \to \mathcal{R}_{r-1} \to \mathcal{R}_{r-2} \to \ldots \to \mathcal{R}_1$$

where $\mathcal{R}_j$ is a q_j-quotient of $\mathcal{E}_T$ for all j. For convenience we will set $\mathcal{R}_r = \mathcal{E}_T$ and $\mathcal{R}_0 = 0$. We will call such a sequence a *π-sequence of quotients of $\mathcal{E}_T$*.

Set $X = \boldsymbol{D}_\pi(\mathcal{E})$. Then X comes equipped with a *universal π-sequence of quotients of E_X*

$$\mathcal{E}_X \to \mathcal{Q}_{r-1} \to \mathcal{Q}_{r-2} \to \ldots \to \mathcal{Q}_1 .$$

That is, if T is an S-scheme, then for every π-sequence of quotients of $\mathcal{E}_T$

$$\mathcal{E}_T \to \mathcal{R}_{r-1} \to \mathcal{R}_{r-2} \to \ldots \to \mathcal{R}_1$$

there is a unique morphism $\varphi\colon T \to X$ such that this sequence is isomorphic to the pullback to T of the universal sequence on X in the sense that we have a commutative diagram

$$\begin{array}{ccccccccc}
\mathcal{E}_T & \to & \mathcal{R}_{r-1} & \to & \mathcal{R}_{r-2} & \to & \ldots & \to & \mathcal{R}_1 \\
\| & & \downarrow\cong & & \downarrow\cong & & & & \downarrow\cong \\
\mathcal{E}_T & \to & \varphi^*\mathcal{Q}_{r-1} & \to & \varphi^*\mathcal{Q}_{r-2} & \to & \ldots & \to & \varphi^*\mathcal{Q}_1
\end{array}$$

We call $\mathcal{Q}_j$ *the universal q_j-quotient of* $\mathcal{E}_X$. Throughout this paper, X and $\mathcal{Q}_j$, $1 \leqq j \leqq r$, will be as above.

In the case $r = 2$, setting $p = p_1$ and $q = p_2$, the scheme $\boldsymbol{D}_{(p,q)}(\mathcal{E})$ is called the *Grassmannian of p-quotients of* $\mathcal{E}$. We will use the notation $\boldsymbol{G}_p(\mathcal{E})$ as well as $\boldsymbol{D}_{(p,q)}(\mathcal{E})$. For $p = 1$, this is the fibred projective space $\boldsymbol{P}(\mathcal{E})$.

For the case $\pi = (1, \ldots, 1)$, we fix the following special notation:

Set $D = \boldsymbol{D}(\mathcal{E}) = \boldsymbol{D}_\pi(\mathcal{E})$ and let

$$\mathcal{E}_D \to \mathcal{P}_{e-1} \to \mathcal{P}_{e-2} \to \ldots \to \mathcal{P}_1$$

be the universal sequence on D. Then set

$$\mathcal{L}_j = \ker(\mathcal{P}_j \to \mathcal{P}_{j-1})$$

for $1 \leqq j \leqq e$. If $\alpha = (\alpha_1, \ldots, \alpha_e)$ is a sequence of integers, set

$$\mathscr{L}^\alpha = \mathscr{L}_1^{\alpha_1}\mathscr{L}_2^{\alpha_2} \ldots \mathscr{L}_e^{\alpha_e} = \mathscr{L}_1^{\otimes\alpha_1} \otimes \mathscr{L}_2^{\otimes\alpha_2} \otimes \ldots \otimes \mathscr{L}_e^{\otimes\alpha_e} .$$

Denote by $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_e$ the natural basis for $\mathsf{Z}^{\oplus e}$.

REMARK (1.1). Notice that for an arbitrary $\pi = (p_1, \ldots, p_r)$, the π-sequence

$$\mathscr{E}_D \to \mathscr{P}_{q_{r-1}} \to \mathscr{P}_{q_{r-2}} \to \ldots \to \mathscr{P}_{q_1}$$

on $D = \boldsymbol{D}(\mathscr{E})$ defines a canonical morhism $h: \boldsymbol{D}(\mathscr{E}) \to \boldsymbol{D}_\pi(\mathscr{E})$ such that $\mathscr{P}_{q_j} \cong h^*(\mathscr{Q}_j)$ for all j.

2. Special subschemes of $\boldsymbol{D}_\pi(\mathscr{E})$.

In this section we recall the definitions and some basic properties of some particular subschemes of $X = \boldsymbol{D}_\pi(\mathscr{E})$. For proofs and details, we refer to [9, (1.4)].

With the notation of section 1. let $\boldsymbol{a} = (a_1, \ldots, a_r)$ be an r-sequence of integers satisfying

$$e \geqq a_1 \geqq a_2 \geqq \ldots \geqq a_r \geqq 0 , \tag{2.1}$$

$$a_i \leqq e - q_i \quad \text{for } 1 \leqq i \leqq r , \tag{2.2}$$

and assume there is a sequence of locally free submodules of $\mathscr{E}$

$$\mathscr{A}_r \subset \mathscr{A}_{r-1} \subset \ldots \subset \mathscr{A}_1 \subset \mathscr{E}$$

such that $\mathscr{A}_i$ has constant rank a_i and is locally a direct summand in $\mathscr{A}_{i-1}$ for all i. Notice that (2.2) implies $a_r = 0$.

We will call such a sequence an *$\boldsymbol{a}$-sequence of subbundles of $\mathscr{E}$* (where we use the term subbundle to mean a locally free submodule of $\mathscr{E}$ which locally is a direct summand in $\mathscr{E}$).

Denote by $X_{\mathscr{A}_1, \ldots, \mathscr{A}_r}$ the scheme of zeros of the maps $\mathscr{A}_{i,X} \to \mathscr{Q}_i$ for $1 \leqq i \leqq r-1$, where $\mathscr{A}_{i,X}$ denotes the pullback to X of $\mathscr{A}_i$. That is, $X_{\mathscr{A}_1, \ldots, \mathscr{A}_r}$ represents the functor whose values in an S-scheme T are the morphisms $t: T \to X$ such that the composite map $\mathscr{A}_{i,T} \to \mathscr{E}_T \to t^*\mathscr{Q}_i$ is the zero homomorphism for every i. Recall that the schemes $X_{\mathscr{A}_1, \ldots, \mathscr{A}_r}$ are closed subschemes of X which are smooth over S (see [9, (1.4)]).

Since it will always be clear which sequence $\mathscr{A}_1, \ldots, \mathscr{A}_r$ we refer to, we will for convenience use the notation $X_{\boldsymbol{a}} = X_{a_1, \ldots, a_r}$ as well as $X_{\mathscr{A}_1, \ldots, \mathscr{A}_r}$.

REMARK (2.3). The notation $X_{\boldsymbol{a}}$ will be particularly convenient when $\mathscr{E}$ is free. When that is the case, we will fix a global basis for $\mathscr{E}$ and let a sequence $(a_1,\ldots,a_r)$ as above correspond to the unique sequence $\mathscr{A}_1,\ldots,\mathscr{A}_r$ we get by, for every i taking $\mathscr{A}_i$ to be the submodule of $\mathscr{E}$ generated by the first a_i elements of the basis. This gives a unique scheme $X_{\boldsymbol{a}}$ for every sequence $\boldsymbol{a}=(a_1,\ldots,a_r)$.

For convenience, we fix a special notation for the case $\pi=(1,\ldots,1)$. In that case we have an $(a_1,\ldots,a_e)$-sequence of subbundles of $\mathscr{E}$

$$\mathscr{A}_e \subset \mathscr{A}_{e-1} \subset \ldots \subset \mathscr{A}_1 \subset \mathscr{E}$$

where $\boldsymbol{a}=(a_1,\ldots,a_e)$ satisfies

$$(2.1') \qquad e \geqq a_1 \geqq a_2 \geqq \ldots \geqq a_e \geqq 0\,,$$

$$(2.2') \qquad a_i \leqq e-i \quad \text{for } 1 \leqq i \leqq e\,.$$

We will denote by $D_{\mathscr{A}_1,\ldots,\mathscr{A}_r}$ or $D_{\boldsymbol{a}}$ (instead of $X_{\mathscr{A}_1,\ldots,\mathscr{A}_e}$ and $X_{\boldsymbol{a}}$) the subscheme of $D=\boldsymbol{D}(\mathscr{E})$ corresponding to the above sequence. So $D_{\mathscr{A}_1,\ldots,\mathscr{A}_e}$ (or $D_{\boldsymbol{a}}$) is the scheme of zeros of the maps $\mathscr{A}_{i,D}\to\mathscr{P}_i$ for $1\leqq i\leqq e-1$.

We will use the following proposition, which is proved in [9, (1.4)]. The version we give here is valid for the case that $\mathscr{E}$ is free, which is the case we will need in our applications. So let $\mathscr{E}$ be free, fix a global basis for $\mathscr{E}$, and let the correspondence between sequences $\boldsymbol{a}=(a_1,\ldots,a_e)$ satisfying (2.1′) and (2.2′) and subschemes $D_{\boldsymbol{a}}$ of D be as described in remark (2.3).

PROPOSITION (2.4). *Let $\boldsymbol{a}=(a_1,\ldots,a_e)$ be as above. Fix j, $1\leqq j\leqq e$, and assume $\boldsymbol{b}=\boldsymbol{a}+\varepsilon_j$ satisfies* (2.1′) *and* (2.2′). *Then $D_{\boldsymbol{b}}$ is a divisor in $D_{\boldsymbol{a}}$ and we have an exact sequence*

$$0 \to \mathscr{L}_{j,\boldsymbol{a}}^{-1} \to \mathscr{O}_{D_{\boldsymbol{a}}} \to \mathscr{O}_{D_{\boldsymbol{b}}} \to 0$$

where $\mathscr{L}_{j,\boldsymbol{a}}^{-1}$ is the restriction to $D_{\boldsymbol{a}}$ of $\mathscr{L}_j^{-1}$.

3. The main theorem.

Let the notation be as in 1 and 2. Assume in addition that S is locally noetherian. $\alpha,\beta,\ldots$ will denote sequences of integers which will be exponents to the $\mathscr{L}_i$'s.

Let $\boldsymbol{a}=(a_1,\ldots,a_e)$ satisfy (2.1′) and (2.2′) and assume, as in section 2, that $a_1,\ldots,a_e$ are the ranks of the modules in a sequence

$$\mathscr{A}_e \subset \mathscr{A}_{e-1} \subset \ldots \subset \mathscr{A}_1 \subset \mathscr{E}$$

of subbundles of $\mathscr{E}$. Recall (se remark (2.3)) that when $\mathscr{E}$ is free, every sequence $(a_1,\ldots,a_e)$ which satisfies (2.1′) and (2.2′) corresponds to an $(a_1,\ldots,a_e)$-sequence of subbundles of $\mathscr{E}$.

Let $f\colon \mathbf{D}(\mathscr{E}) \to S$ be the structure morphism, and let $f_{\boldsymbol{a}}\colon D_{\boldsymbol{a}} \to S$ be the restriction of f to $D_{\boldsymbol{a}} = D_{\mathscr{A}_1,\ldots,\mathscr{A}_e}$. So we have the commutative diagram

$$\begin{array}{ccc} D_{\boldsymbol{a}} & \xrightarrow{\;\subset\;} & \mathbf{D}(\mathscr{E}) \\ & {\scriptstyle f_{\boldsymbol{a}}} \searrow \qquad \swarrow {\scriptstyle f} & \\ & S & \end{array}$$

Let $\mathscr{L}_{j,\boldsymbol{a}}$ be the restriction to $D_{\boldsymbol{a}}$ of $\mathscr{L}_j$, and for any sequence $\alpha = (\alpha_1,\ldots,\alpha_e)$ of integers, let $\mathscr{L}_{\boldsymbol{a}}^{\alpha}$ be the restriction to $D_{\boldsymbol{a}}$ of $\mathscr{L}^{\alpha}$. Finally denote by $R^l f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\alpha})$ the l'th higher direct image of $\mathscr{L}_{\boldsymbol{a}}$ under $f_{\boldsymbol{a}}$.

We recall the following results (for proofs, see [9, (3.4.1) and (3.6.1)], or [4]).

Proposition (3.1). *With $\boldsymbol{a} = (a_1,\ldots,a_e)$ and $\alpha = (\alpha_1,\ldots,\alpha_e)$ as above, assume there is an integer j, $1 \leqq j \leqq e$, such that $a_j = a_{j+1}$ and $\alpha_j = \alpha_{j+1} - 1$. Then we have $R^l f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\alpha}) = 0$ for all l.*

Theorem (3.2). *Let $\boldsymbol{a} = (a_1,\ldots,a_e)$ be arbitrary. Assume $\alpha = (\alpha_1,\ldots,\alpha_e)$ satisfies $\alpha_i \geqq \alpha_{i+1} - 1$ for $1 \leqq i \leqq e-1$. Then we have $R^l f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\alpha}) = 0$ for all $l \neq 0$.*

We are now in a position to prove the main theorem of this paper.

Theorem (3.3). *Let the notation be as above. Assume the sequences $\alpha = (\alpha_1,\ldots,\alpha_e)$ and $\beta = (\beta_1,\ldots,\beta_e)$ are non increasing (that is, $\alpha_i \geqq \alpha_{i+1}$ and $\beta_i \geqq \beta_{i+1}$ for $1 \leqq i \leqq e-1$). Then the canonical map*

$$f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\alpha}) \otimes f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\beta}) \to f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\alpha+\beta})$$

is an epimorphism of $\mathscr{O}_S$-modules.

Proof. The statement is local on S, so we may assume that $\mathscr{E}$ is free. Then we have the situation described in (2.3), and (2.4) applies.

Let $\alpha = (\alpha_1,\ldots,\alpha_e)$ and $\beta = (\beta_1,\ldots,\beta_e)$ be non increasing. The theorem will be proved by induction on $\sum_{i=1}^{e}(e - a_i - i)$. For $\sum_{i=1}^{e}(e - a_i - i) = 0$, condition (2.2′) implies $a_i = e - i$ for all i, hence we have $D_{\boldsymbol{a}} = S$ by the definition of $D_{\boldsymbol{a}}$. So in this case the theorem is trivial because $\mathscr{L}_{\boldsymbol{a}}^{\alpha}$ and $\mathscr{L}_{\boldsymbol{a}}^{\beta}$ are both equal to the structure sheaf $\mathscr{O}_S$.

Now let $\sum_{i=1}^{e}(e-a_i-i)=m>0$, and assume by induction that the theorem is true for sequences $\boldsymbol{a}'=(a_1',\ldots,a_e')$ which satisfy $\sum_{i=1}^{e}(e-a_i'-i)<m$. Choose k such that $e-a_k-k>0$ and $a_k=a_{k+1}$ (for example the largest k such tnat $e-a_k-k>0$; note that $k<e$ since we have $e-a_e-e=0$), and let j be the smallest number such that $a_j=a_k$. Then we have $e-a_j-j\geqq e-a_k-k>0$ and $a_{j-1}>a_j$. Hence the sequence $\boldsymbol{b}=\boldsymbol{a}+\varepsilon_j$ satisfies (2.1′) and (2.2′). Therefore, by (2.4), $D_{\boldsymbol{b}}$ is a divisor in $D_{\boldsymbol{a}}$ and we have an exact sequence

$$0\to\mathscr{L}_{j,\boldsymbol{a}}^{-1}\to\mathcal{O}_{D_{\boldsymbol{a}}}\to\mathcal{O}_{D_{\boldsymbol{b}}}\to 0$$

which yields the exact sequences

$$C_1\colon 0\to\mathscr{L}_{\boldsymbol{a}}^{\alpha-\varepsilon_j}\to\mathscr{L}_{\boldsymbol{a}}^{\alpha}\to\mathscr{L}_{\boldsymbol{b}}^{\alpha}\to 0$$

and

$$C_2\colon 0\to\mathscr{L}_{\boldsymbol{a}}^{\alpha+\beta-\varepsilon_j}\to\mathscr{L}_{\boldsymbol{a}}^{\alpha+\beta}\to\mathscr{L}_{\boldsymbol{b}}^{\alpha+\beta}\to 0\,.$$

Since $\alpha=(\alpha_1,\ldots,\alpha_e)$ is non increasing, the sequence $\alpha'=\alpha-\varepsilon_j$ will satisfy the condition $\alpha_i'\geqq\alpha_{i+1}'-1$ for $1\leqq i\leqq e-1$. Therefore, by (3.2), the long exact sequence of higher direct images under $f_{\boldsymbol{a}}$ which corresponds to the sequence C_1 above, will reduce to the exact sequence

$$C_3\colon 0\to f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\alpha-\varepsilon_j})\to f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\alpha})\to f_{\boldsymbol{b},*}(\mathscr{L}_{\boldsymbol{b}}^{\alpha})\to 0\,.$$

Similarly, since $\alpha+\beta$ clearly is non increasing, the sequence $\alpha''=\alpha+\beta-\varepsilon_j$ will satisfy the condition $\alpha_i''\geqq\alpha_{i+1}''-1$ for $1\leqq i\leqq e-1$, and we get as above, by (3.2), the exact sequence

$$0\to f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\alpha+\beta-\varepsilon_j})\to f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\alpha+\beta})\to f_{\boldsymbol{b},*}(\mathscr{L}_{\boldsymbol{b}}^{\alpha+\beta})\to 0\,.$$

Tensoring the sequence C_3 with $f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\beta})$, we get the following diagram of canonical maps with exact columns

$$(*)\qquad\begin{array}{ccc}
 & & 0\\
 & & \downarrow\\
f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\alpha-\varepsilon_j})\otimes f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\beta}) & \xrightarrow{\sigma'} & f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\alpha+\beta-\varepsilon_j})\\
\downarrow & & \downarrow{\scriptstyle\lambda}\\
f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\alpha})\otimes f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\beta}) & \xrightarrow{\sigma} & f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\alpha+\beta})\\
{\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\lambda'}\\
f_{\boldsymbol{b},*}(\mathscr{L}_{\boldsymbol{b}}^{\alpha})\otimes f_{\boldsymbol{a},*}(\mathscr{L}_{\boldsymbol{a}}^{\beta}) & \xrightarrow{\sigma''} & f_{\boldsymbol{b},*}(\mathscr{L}_{\boldsymbol{b}}^{\alpha+\beta})\\
\downarrow & & \downarrow\\
0 & & 0
\end{array}$$

Actually, it follows from (3.2) and the theory for cohomology and base-change that $f_{a,*}(\mathscr{L}_a^{\beta})$ is locally free (see EGA III (7.7) and (7.8)), hence the upper left map in the diagram is injective. But we will not need this.

We have $\sum_{i=1}^{e}(e-b_i-i)=m-1$. So by our induction hypothesis, the theorem is true for D_b (that is, for the sequence $b=(b_1,\ldots,b_e)$). Therefore the map

$$f_{b,*}(\mathscr{L}_b^{\alpha})\otimes f_{b,*}(\mathscr{L}_b^{\beta})\to f_{b,*}(\mathscr{L}_b^{\alpha+\beta})$$

is an epimorphism. Also, since the sequence C_1 is exact for arbitrary non increasing α, in particular for $\alpha=\beta$, we see that the map $f_{a,*}(\mathscr{L}_a^{\beta})\to f_{b,*}(\mathscr{L}_b^{\beta})$ is an epimorphism. Hence σ'' in the diagram (*) above is the composite of the two epimorphisms

$$f_{b,*}(\mathscr{L}_b^{\alpha})\otimes f_{a,*}(\mathscr{L}_a^{\beta})\to f_{b,*}(\mathscr{L}_b^{\alpha})\otimes f_{b,*}(\mathscr{L}_b^{\beta})\to f_{b,*}(\mathscr{L}_a^{\alpha+\beta})$$

and is therefore itself an epimorphism.

Now, using the diagram (*) together with the surjectivity of σ'', we will prove the theorem for D_a in two steps. First we show that the map

$$f_{a,*}(\mathscr{L}_a^{\alpha})\otimes f_{a,*}(\mathscr{L}_a^{\beta})\to f_{a,*}(\mathscr{L}_a^{\alpha+\beta})$$

is surjective when β satisfies the condition $\beta_j=\beta_{j+1}$. Then we use that result to prove the surjectivity for arbitrary (non increasing) α and β.

(1) Assume $\beta_j=\beta_{j+1}$. The proof goes by induction on $\alpha_j-\alpha_{j+1}$.

For $\alpha_j-\alpha_{j+1}=0$, the sequence $\alpha'=\alpha-\varepsilon_j$ satisfies $\alpha_j'=\alpha_{j+1}'-1$. Therefore, since $a_j=a_{j+1}$ by our choice of j, proposition (3.1) implies $f_{a,*}(\mathscr{L}_a^{\alpha-\varepsilon_j})=0$. Furthermore, since we by assumption have $\beta_j=\beta_{j+1}$, the sequence $\alpha''=\alpha+\beta-\varepsilon_j$ also satisfies the condition $\alpha_j''=\alpha_{j+1}''-1$, and (3.1) yields $f_{a,*}(\mathscr{L}_a^{\alpha+\beta-\varepsilon_j})=0$. So the maps λ and λ' in diagram (*) are both isomorphisms, and σ is an epimorphism since σ'' is.

Now let $\alpha_j-\alpha_{j+1}=n>0$ and assume by induction that the map

$$f_{a,*}(\mathscr{L}_a^{\alpha'})\otimes f_{a,*}(\mathscr{L}_a^{\beta})\to f_{a,*}(\mathscr{L}_a^{\alpha'+\beta})$$

is surjective for every non increasing $\alpha'=(\alpha_1',\ldots,\alpha_e')$ which satisfies $\alpha_j'-\alpha_{j+1}'<n$. Since α is non increasing and we have $\alpha_j>\alpha_{j+1}$, the sequence $\alpha'=\alpha-\varepsilon_j$ is clearly non increasing and satisfies $\alpha_j'-\alpha_{j+1}'=n-1$. Hence the map σ' in the diagram (*) is an epimorphism by the induction hypothesis above. A diagram chasing shows that the surjectivity of σ' and σ'' implies that σ is surjective. This concludes the case $\beta_j=\beta_{j+1}$.

(2) Now let $\alpha=(\alpha_1,\ldots,\alpha_e)$ and $\beta=(\beta_1,\ldots,\beta_e)$ be arbitrary (non increasing). To show that the map σ in the diagram (*) is an epimorphism, we will again use induction on $\alpha_j-\alpha_{j+1}$.

For $\alpha_j - \alpha_{j+1} = 0$, we have the situation of case (1) above, but with the roles of α and β switched. Hence σ is surjective by the case (1) result.

The induction now proceeds exactly as in case (1). This concludes the proof of theorem (3.3).

(3.4) There is a natural extension of theorem (3.3) to a corresponding theorem on a subscheme $X_{\boldsymbol{a}} = X_{a_1,\ldots,a_r}$ of a flag scheme $X = \boldsymbol{D}_\pi(\mathscr{E})$ where $\pi = (p_1,\ldots,p_r)$ is arbitrary and where $a_1,\ldots,a_r$ are the ranks of an r-sequence of subbundles of $\mathscr{E}$ as in section 2.

For the rest of this section, fix the following notation. Denote by $\mathscr{Q}_{i,\boldsymbol{a}}$ the restriction to $X_{\boldsymbol{a}}$ of the universal q_i-quotient $\mathscr{Q}_i$ on X. Set

$$\mathscr{M}_i = \Lambda^{q_i} \mathscr{Q}_i, \quad 1 \leqq i \leqq r,$$

and denote by $\mathscr{M}_{i,\boldsymbol{a}}$ the restriction to $X_{\boldsymbol{a}}$ of $\mathscr{M}_i$ for all i. For any r-sequence $\delta = (\delta_1,\ldots,\delta_r)$ of integers, set

$$\mathscr{M}^\delta = \mathscr{M}_1^{\delta_1} \ldots \mathscr{M}_r^{\delta_r} = \mathscr{M}_1^{\otimes\delta_1} \otimes \ldots \otimes \mathscr{M}_r^{\otimes\delta_r}$$

and let $\mathscr{M}_{\boldsymbol{a}}^\delta$ be the restriction to $X_{\boldsymbol{a}}$ of $\mathscr{M}^\delta$.

Corresponding to the r-sequence $(a_1,\ldots,a_r)$, define the e-sequence $\boldsymbol{b} = (b_1,\ldots,b_e)$ by setting $b_i = a_j$ for $q_{j-1} < i \leqq q_j$ and $1 \leqq j \leqq r$. Let $D_{\boldsymbol{b}}$ be the corresponding subscheme of $D = \boldsymbol{D}(\mathscr{E})$.

Corresponding to the r-sequence $(\delta_1,\ldots,\delta_r)$, define the e-sequence $\alpha = (\alpha_1,\ldots,\alpha_e)$ by setting $\alpha_i = \sum_{l=j}^r \delta_l$ for $q_{j-1} < i \leqq q_j$ and $1 \leqq j \leqq r$.

Remark (3.5). Notice that the sequence $(\alpha_1,\ldots,\alpha_e)$ is non increasing if and only if the δ_i's are non negative for $1 \leqq i \leqq r-1$.

Let $h\colon D \to X$ be the canonical morphism (see remark (1.1)). From the definition of h, we have the isomorphisms $h^*\mathscr{Q}_i \simeq \mathscr{P}_{q_i}$ for $1 \leqq i \leqq r$. Hence we also get $h^*(\mathscr{M}_i) \simeq \Lambda^{q_i}\mathscr{P}_{q_i}$ for all i. Now the exact sequences (see definition of $\mathscr{L}_k$ in section 1)

$$0 \to \mathscr{L}_k \to \mathscr{P}_k \to \mathscr{P}_{k-1} \to 0$$

yield the isomorphisms

$$\Lambda^k \mathscr{P}_k \cong \mathscr{L}_k \otimes \Lambda^{k-1} \mathscr{P}_{k-1} \quad \text{for } 1 \leqq k \leqq e .$$

By induction on k we get the isomorphisms

$$\Lambda^k \mathscr{P}_k \cong \mathscr{L}_1 \mathscr{L}_2 \ldots \mathscr{L}_k, \quad 1 \leqq k \leqq e .$$

This together with the isomorphism $h^*(\mathscr{M}_i) \simeq \Lambda^{q_i}\mathscr{P}_{q_i}$ above, yield

$$h^*(\mathscr{M}_i) \cong \mathscr{L}_1 \mathscr{L}_2 \ldots \mathscr{L}_{q_i} \quad \text{for all } i .$$

From the definition of $\alpha=(\alpha_1,\ldots,\alpha_e)$ above, it is then easy to see that we get an isomorphism

$$h^*(\mathscr{M}^\delta) \cong \mathscr{L}^\alpha .$$

The way we defined $\boldsymbol{b}=(b_1,\ldots,b_e)$, we will have a cartesian diagram (see [9, (1.4)])

$$\begin{array}{ccc} D_{\boldsymbol{b}} & \longrightarrow & D \\ {\scriptstyle h_{\boldsymbol{a}}}\downarrow & \square & \downarrow{\scriptstyle h} \\ X_{\boldsymbol{a}} & \longrightarrow & X \end{array}$$

We therefore get the isomorphism

$$h_{\boldsymbol{a}}^*(\mathscr{M}_{\boldsymbol{a}}^{\delta}) \cong \mathscr{L}_{\boldsymbol{b}}^{\alpha}$$

where $h_{\boldsymbol{a}}$ is as in the diagram above. Let $g: X \to S$ be the structure morphism of X, and denote by $g_{\boldsymbol{a}}$ the restriction of g to $X_{\boldsymbol{a}}$. We have a commutative diagram

$$\begin{array}{ccc} D_{\boldsymbol{b}} & \longrightarrow & X_{\boldsymbol{a}} \\ {\scriptstyle f_{\boldsymbol{b}}}\big\downarrow & & \big\downarrow{\scriptstyle g_{\boldsymbol{a}}} \\ & S & \end{array}$$

It is proved in [9, (3.3.1)], that we in the situation above have an isomorphism

$$R^l g_{\boldsymbol{a},*}(\mathscr{M}_{\boldsymbol{a}}^{\delta}) \cong R^l f_{\boldsymbol{b},*}(\mathscr{L}_{\boldsymbol{b}}^{\alpha}) \tag{3.6}$$

for every l.

THEOREM (3.7). *Let* $\boldsymbol{a}=(a_1,\ldots,a_r)$ *be as above. Assume the sequences* $\delta=(\delta_1,\ldots,\delta_r)$ *and* $\gamma=(\gamma_1,\ldots,\gamma_r)$ *satisfy* $\delta_i \geqq 0$ *and* $\gamma_i \geqq 0$ *for* $1 \leqq i \leqq r-1$. *Then the canonical map*

$$g_{\boldsymbol{a},*}(\mathscr{M}_{\boldsymbol{a}}^{\delta}) \otimes g_{\boldsymbol{a},*}(\mathscr{M}_{\boldsymbol{a}}^{\gamma}) \to g_{\boldsymbol{a},*}(\mathscr{M}_{\boldsymbol{a}}^{\delta+\gamma})$$

is an epimorphism of $\mathcal{O}_S$*-modules.*

PROOF. By the preceding discussion, we have a commutative diagram

$$\begin{array}{ccc} g_{\boldsymbol{a},*}(\mathscr{M}_{\boldsymbol{a}}^{\delta}) \otimes g_{\boldsymbol{a},*}(\mathscr{M}_{\boldsymbol{a}}^{\gamma}) & \xrightarrow{\ \sigma\ } & g_{\boldsymbol{a},*}(\mathscr{M}_{\boldsymbol{a}}^{\delta+\gamma}) \\ {\scriptstyle \lambda}\big\downarrow \cong & & {\scriptstyle \lambda'}\big\downarrow \cong \\ f_{\boldsymbol{b},*}(\mathscr{L}_{\boldsymbol{b}}^{\alpha}) \otimes f_{\boldsymbol{b},*}(\mathscr{L}_{\boldsymbol{b}}^{\beta}) & \xrightarrow{\ \sigma'\ } & f_{\boldsymbol{b},*}(\mathscr{L}_{\boldsymbol{b}}^{\alpha+\beta}) \end{array}$$

where λ and λ' are isomorphisms (see (3.6)). Here $\beta=(\beta_1,\ldots,\beta_e)$ is defined by setting $\beta_i=\sum_{l=j}^{r}\gamma_l$ for $q_{j-1}<i\leqq q_j$ and $1\leqq j\leqq r$. The diagram shows that σ is an epimorphism if and only if σ' is. The conditions on δ and γ imply that α and β are non increasing (see (3.5)). Hence σ' is an epimorphism by theorem (3.3), and theorem (3.7) follows.

4. Applications.

In this section we let the base scheme S be equal to Spec(k) for some field k. Except for that, let the notation be as in (3.4).

PROPOSITION (4.1). *Assume* $\delta=(\delta_1,\ldots,\delta_r)$ *satisfies* $\delta_i\geqq 1$ *for* $1\leqq i\leqq r-1$. *Then* $\mathscr{M}_a^{\delta}$ *is very ample.*

PROOF. X_a is a closed subscheme of $X=\boldsymbol{D}_\pi(\mathscr{E})$, and $\mathscr{M}_a^{\delta}$ is the restriction to X_a of $\mathscr{M}^{\delta}=\mathscr{M}_1^{\delta_1}\ldots\mathscr{M}_r^{\delta_r}$ on X, where $\mathscr{M}_i=\Lambda^{q_i}\mathscr{Q}_i$. The epimorphisms $\mathscr{E}_X\to\mathscr{Q}_i$ define morphisms $\boldsymbol{D}_\pi(\mathscr{E})\to\boldsymbol{G}_{q_i}(\mathscr{E})$ for all i. Hence we get a morphism

$$\boldsymbol{D}_\pi(\mathscr{E})\to Y=\boldsymbol{G}_{q_1}(\mathscr{E})\times_k\ldots\times_k\boldsymbol{G}_{q_{r-1}}(\mathscr{E})$$

which is easily seen to be a closed immersion (see [9, (2.1)]). Now, if $\bar{\mathscr{Q}}_i$ is the universal q_i-quotient on $\boldsymbol{G}_{q_i}(\mathscr{E})$, $1\leqq i\leqq r-1$, then the invertible sheaf $\Lambda^{q_i}\bar{\mathscr{Q}}_i$ is very ample (and defines the Plücker embedding of $\boldsymbol{G}_{q_i}(\mathscr{E})$). Hence $(\Lambda^{q_i}\bar{\mathscr{Q}}_i)^{\otimes\delta_i}$ is very ample for all $\delta_i\geqq 1$. It defines an embedding

$$\varphi_i:\boldsymbol{G}_{q_i}(\mathscr{E})\subsetneq\boldsymbol{P}((\Lambda^{q_i}\mathscr{E})^{\otimes\delta_i}).$$

The product of the φ_i's yields an embedding of Y into the product space

$$\times_{i=1}^{r-1}\boldsymbol{P}((\Lambda^{q_i}\mathscr{E})^{\otimes\delta_i}).$$

Composing this with the Segre embedding yields an embedding

$$Y\subsetneq\boldsymbol{P}(\otimes_{i=1}^{r-1}(\Lambda^{q_i}\mathscr{E})^{\otimes\delta_i})=\boldsymbol{P}_k^{m}$$

where we have put

$$m=\prod_{i=1}^{r-1}\binom{e}{q^i}^{\delta_i}-1.$$

The composite map

$$X_a\subsetneq X\subsetneq Y\subsetneq\boldsymbol{P}_k^{m}$$

yields the desired embedding of X_a into $\boldsymbol{P}_k^{m}$. It is clear from the construction of this map that the pullback to X_a of the canonical invertible sheaf on $\boldsymbol{P}_k^{m}$ is $\mathscr{M}_a^{\delta}$. This concludes the proof of the proposition.

REMARK (4.2). It can be shown that the invertible sheaves of the type $\mathscr{M}_a^\delta$, where $\delta_i \geqq 1$ for $1 \leqq i \leqq r-1$, are the only very ample sheaves on X_a, so that every embedding of X_a into a projective m-space $\boldsymbol{P}_k^m$ is as described above.

Now with $\delta = (\delta_1, \ldots, \delta_r)$ satisfying $\delta_i \geqq 1$, $1 \leqq i \leqq r-1$, as above, set

$$R_\nu^\delta = \operatorname{Im}[(H^0(X_a, \mathscr{M}_a^\delta))^{\otimes \nu} \to H^0(X_a, (\mathscr{M}_a^\delta)^{\otimes \nu})] \quad \text{for } \nu \geqq 0 ,$$

and set

$$R^\delta = \oplus_{\nu=0}^\infty R_\nu^\delta .$$

The graded ring R^δ is the homogeneous coordinate ring for X_a relative to the projective embedding given above, and we have $X_a \cong \operatorname{Proj}(R^\delta)$ and $\mathscr{M}_a^\delta = R^\delta(1)^\sim$. Let $\mathfrak{m}^\delta = \oplus_{\nu=1}^\infty R_\nu^\delta$ be the irrelevant ideal in R^δ.

THEOREM (4.3). *For any δ as above, the corresponding homogeneous coordinate ring R^δ of X_a is normal. Equivalently* (*by remark* (4.2)), *X_a is arithmetically normal for any embedding into a projective space $\boldsymbol{P}_k^m$.*

PROOF. For convenience, set $R = R^\delta$ and $\mathfrak{m} = \mathfrak{m}^\delta$. By Serre's criterion (EGA IV (5.8.6)), R is normal if it is regular in codimension 1, and if we have $\operatorname{depth}_{\mathfrak{p}}(R) \geqq 2$ for every prime ideal $\mathfrak{p}$ in R of codimension $\geqq 2$. Now X_a is smooth over k (see [9, (1.4)]), hence $\operatorname{Spec}(R)$ is smooth, and therefore regular, outside the vertex. So to check that R is normal, it is enough to check that $\operatorname{depth}_{\mathfrak{m}}(R) \geqq 2$. To do this, consider the canonical map

$$\Phi : R \to \oplus_{\nu=-\infty}^\infty H^0(X_a, \mathcal{O}_{X_a}(\nu)) .$$

By a theorem of Serre ([8, 77, proposition 2]; for the version we use here, see [9, (6.4.1)]) we have $\operatorname{depth}_{\mathfrak{m}}(R) \geqq 2$ if Φ is bijective.

Now, since $H^0(X_a, \mathcal{O}_{X_a}(\nu))$ is equal to zero for $\nu < 0$, Φ is bijective if and only if the map

$$H^0(X_a, (\mathscr{M}_a^\delta))^{\otimes \nu} \to H^0(X_a, (\mathscr{M}_a^\delta)^{\otimes \nu})$$

is surjective for every $\nu \geqq 0$. Clearly, it is enough to show that the map

$$H^0(X_a, (\mathscr{M}_a^\delta)^{\otimes \nu}) \otimes H^0(X_a, \mathscr{M}_a^\delta) \to H^0(X_a, (\mathscr{M}_a^\delta)^{\otimes(\nu+1)})$$

is surjective for every $\nu \geqq 0$. But this follows immediately from theorem (3.7) since $\delta = (\delta_1, \ldots, \delta_r)$ and $\nu\delta = (\nu\delta_1, \nu\delta_2, \ldots, \nu\delta_r)$ satisfy the condition $\delta_i \geqq 0$ and $\nu\delta_i \geqq 0$ for $1 \leqq i \leqq r-1$. This concludes the proof of theorem (4.3).

In the special case $a_1 = a_2 = \ldots = a_r = 0$, we have the following stronger theorem.

THEOREM (4.5). *Let $\pi = (p_1, \ldots, p_r)$ be arbitrary. Then the flag scheme $\boldsymbol{D}_\pi(\mathscr{E})$ is arithmetically normal and Cohen-Macauly for any embedding of $\boldsymbol{D}_\pi(\mathscr{E})$ into a projective space $\boldsymbol{P}_k{}^m$.*

PROOF. With the same notation as in the proof of theorem (4.4) (now with $X_a = \boldsymbol{D}_\pi(\mathscr{E})$), Serre's theorem (see the proof of (4.4) above) says that R is Cohen-Macauly if Φ is bijective and $H^i(\boldsymbol{D}_\pi(\mathscr{E}), (\mathscr{M}^\delta)^{\otimes \nu})$ vanishes for all $\nu \in \mathsf{Z}$ and $1 \leqq i \leqq \dim_k(\boldsymbol{D}_\pi(\mathscr{E})) - 1$.

As in the proof of (4.4), Φ is bijective. It remains to see that $H^i(\boldsymbol{D}_\pi(\mathscr{E}), \mathscr{M}^{\nu\delta}) = 0$ for all ν and $1 \leqq i \leqq \dim_k(\boldsymbol{D}_\pi(\mathscr{E})) - 1$, where $\nu\delta = (\nu\delta_1, \ldots, \nu\delta_r)$. Now, with $\alpha = (\alpha_1, \ldots, \alpha_e)$ as in (3.4), we have the isomorphisms

$$H^i(\boldsymbol{D}_\pi(\mathscr{E}), \mathscr{M}^{\nu\delta}) \cong H^i(\boldsymbol{D}(\mathscr{E}), \mathscr{L}^{\nu\alpha})$$

for all i (see (3.6) and [9, (3.3.1)]). If $\nu \geqq 0$, we have $H^i(\boldsymbol{D}(\mathscr{E}), \mathscr{L}^{\nu\alpha}) = 0$ for $i \neq 0$ by theorem (3.2). If $\nu < 0$, we see that we have

$$\nu\alpha_1 = \ldots = \nu\alpha_{q_1} < \nu\alpha_{q_1+1} = \ldots = \nu\alpha_{q_2} < \ldots,$$

that is, $\nu\alpha_i$ is constant on the intervals $(q_{j-1}, q_j]$, $1 \leqq j \leqq r$, and we have $\nu\alpha_{q_j} < \nu\alpha_{q_j+1}$ for all j. Hence we have $H^i(\boldsymbol{D}(\mathscr{E}), \mathscr{L}^{\nu\alpha}) = 0$ for $i \neq \dim_k(\boldsymbol{D}_\pi(\mathscr{E}))$ by [9, theorem (3.8.1)]. This concludes the proof of theorem (4.5).

We include the following consequence of theorem (4.3). Let $a^i = (a_1{}^i, \ldots, a_{r_i}{}^i)$, $1 \leqq i \leqq n$, be sequences of integers satisfying (2.1) and (2.2), and let $X_i = X_{a^i}$ be the corresponding varieties. Set $Y = X_1 \times \ldots \times X_n$, the product taken over k. Let $\mathscr{N}^i$ be a very ample sheaf for X_i, that is $\mathscr{N}^i$ is of the type described in proposition (4.1) (see remark (4.2)). Then $\mathscr{N} = \otimes_{i=1}^n \mathscr{N}^i$ is a very ample sheaf, and the corresponding homogeneous coordinate ring for Y is

$$R = \oplus_{\nu=0}^\infty \operatorname{Im}[(H^0(Y, \mathscr{N}))^{\otimes \nu} \to H^0(Y, \mathscr{N}^{\otimes \nu})].$$

THEOREM (4.6). *The homogeneous coordinate ring R for the product $Y = X_1 \times \ldots \times X_n$ is normal. That is, Y is arithmetically normal for any embedding into a projective m-space $\boldsymbol{P}_k{}^m$.*

PROOF. Y is smooth over k since it is the product of smooth varieties. Hence, by the same argument as in the proof of theorem (4.3), Y is arithmetieally normal if the maps

$$H^0(Y,\mathcal{N}^{\otimes\nu}) \otimes H^0(Y,\mathcal{N}) \xrightarrow{\sigma} H^0(Y,\mathcal{N}^{\otimes(\nu+1)})$$

are surjective for all $\nu \geqq 0$. We have

$$\mathcal{N}^{\otimes\nu} = (\otimes_{i=1}^{n}\mathcal{N}_i)^{\otimes\nu} = \otimes_{i=1}^{n}(\mathcal{N}_i^{\otimes\nu}).$$

Hence, by the Künneth formula, we have $H^0(Y,\mathcal{N}^{\otimes\nu}) \cong \otimes_{i=1}^{n} H^0(Y,\mathcal{N}_i^{\otimes\nu})$ for all ν. So we get a commutative diagram

$$\begin{array}{ccc} H^0(Y,\mathcal{N}^{\otimes\nu}) \otimes H^0(Y,\mathcal{N}) & \xrightarrow{\sigma} & H^0(Y,\mathcal{N}^{\otimes(\nu+1)}) \\ \cong\downarrow & & \downarrow\cong \\ \otimes_{i=1}^{n}(H^0(Y,\mathcal{N}_i^{\otimes\nu}) \otimes H^0(Y,\mathcal{N}_i)) & \xrightarrow{\sigma'} & \otimes_{i=1}^{n}(H^0(Y,\mathcal{N}_i^{\otimes(\nu+1)})) \end{array}$$

where the vertical maps are isomorphisms.

By theorem (3.7), σ' is the tensor product of surjective maps, and is therefore surjective. We conclude that σ is surjective.

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