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AN EXACT SEQUENCE OF BRAID GROUPS

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1. Definitions and statement of the main theorem.

Throughout this paper, M will be a surface (2-manifold). Let $P_n=\{p_1,p_2,\ldots p_n\}$ be a fixed n-point set for each n, and let F_nM be the set of all embeddings of P_n in M. Thus $F_nM \subseteqq M^n$, the n-fold cartesian product of M with itself, which may be considered to be the set of all maps from P_n to M. Let $i: F_nM \to M^n$ be the inclusion map. Choose a fixed embedding x^0 of P_n in M, and denote $x^0(p_i)$ by ${x^0}_i$ for $1 \leqq i \leqq n$. Then $\pi_1(F_nM,x^0)$ is called the group of unpermuted (pure) n-strand braids on the surface M, and the induced map

$$i_* : \pi_1(F_nM,x^0) \to \pi_1(M^n,x^0)$$

can be described simply as follows: Since M^n is a cartesian product,

$$\pi_1(M^n,x^0) = \textstyle\prod_{i=1}^n \pi_1(M,{x^0}_i) .$$

A path $\{x(t): t \in [0,1]\}$ which represents an element of $\pi_1(F_nM,x^0)$, may be restricted to $\{p_i\} \subseteqq P_n$, to yield a loop

$$\{x(t)(p_i) : t \in [0,1]\}$$

which represents an element α_i of $\pi_1(M,{x^0}_i)$ for each i. Then for any path class $[x]$ of $\pi_1(F_nM,x^0)$, the element $i_*([x])$ is given by the n-tuple of path classes

$$(\alpha_1,\alpha_2,\ldots,\alpha_n) \in \textstyle\prod_{i=1}^n \pi_1(M,{x^0}_i) = \pi_1(M^n,x^0) .$$

Let D^2 be an open disk in M which contains the n points ${x^0}_1,{x^0}_2,\ldots,{x^0}_n$. Then F_nD^2, which is the set of all embeddings of P_n in D^2, may be identified with a subset of F_nM by composing any map of F_nD^2 with the inclusion $D^2 \subseteqq M$. Let $j: F_nD^2 \to F_nM$ be the resulting identification. Then the induced map

$$j_* : \pi_1(F_nD^2,x^0) \to \pi_1(F_nM,x^0)$$

takes any n-strand unpermuted braid on D^2 and considers it as a braid on M. Since D^2 is homeomorphic to R^2, the group $\pi_1(F_nD^2,x^0)$ is isomorphic to the classical (Artin) unpermuted braid group (see [5]).

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We can readily discern two distinct types of phenomena which are exhibited by the representatives (geometric braids) of elements of the group $\pi_1(F_nM, x^0)$ of n-strand unpermuted braids on M:

(1) There is "classical braiding" of the strands which may be thought of as taking place in an open disk $D^2 \subseteq M$.

(2) There is wandering of the individual strands about on the surface M.

The principal result of this paper, Theorem 1, may be thought of as saying intuitively that for a closed surface M of genus $g \geqq 1$, nothing else happens in $\pi_1(F_nM, x^0)$.

THEOREM 1. *If M is a closed surface other than S^2 or P^2, then in the following sequence of (not necessarily Abelian) groups*

$$1 \to \pi_1(F_nD^2, x^0) \xrightarrow{j_*} \pi_1(F_nM, x^0) \xrightarrow{i_*} \prod_{i=1}^n \pi_1(M, x^0_i) \to 1$$

the kernel of each homomorphism is equal to the normal closure (consequence) of the image of the previous homomorphism in the sequence.

This theorem was conjectured for closed, orientable surfaces by Joan Birman in [2], where she proves Lemmas 1 and 3, and asserts without proof that Lemma 2 is true and Lemma 5 false when the genus $g \geqq 1$. Her proofs are based on the fibrations of Fadell and Neuwirth [4]. It is, however, possible to give proofs of all but the simplest parts of Theorem 1 based on the single geometric device of associating to each unpermuted braid in M, something which resembles a braid in a suitable covering space $\tilde{M}$ of M. Accordingly, geometric proofs (some new) have been given for all lemmas to provide greater unity of approach. The proof of Theorem 1 occupies the remainder of this paper. A more complete exposition of the device of "periodic braids in a covering space" than is necessary in the present context will appear separately [6].

2. Start of the proof of Theorem 1.

The assertion of Theorem 1, which may be called "exactness" in the category of non-Abelian groups (noting the necessary modification from the more familiar definition of a short exact sequence of Abelian groups), reduces to several simpler assertions. At $\pi_1(F_nD^2, x^0)$, all we must check is that $\ker(j_*) = \{1\}$. At $\pi_1(F_nM, x^0)$, it is sufficient to check that $\operatorname{im}(j_*) \subseteq \ker(i_*)$ and that $\ker(i_*) \subseteq \langle \operatorname{im}(j_*) \rangle$, where $\langle \operatorname{im}(j_*) \rangle$ is the normal closure (consequence) of $\operatorname{im}(j_*)$ in $\pi_1(F_nM, x^0)$, that is

$$\{\gamma = \prod_k \alpha_k \beta_k \alpha_k^{-1} : \ \alpha_k \in \pi_1(F_nM, x^0), \beta_k \in \operatorname{im}(j_*)\}.$$

Some simplification has been possible since $\langle \mathrm{im}(j_*)\rangle$ is the smallest normal subgroup containing $\mathrm{im}(j_*)$, and $\ker(i_*)$ is a normal subgroup of $\pi_1(F_nM, x^0)$. At $\prod_{i=1}^n \pi_1(M, x^0{}_i)$, it is sufficient to check that $\mathrm{im}(i_*) = \prod_{i=1}^n \pi_1(M, x^0{}_i)$, i.e. that i_* is onto.

LEMMA 1. *For any surface M we have*

$$\mathrm{im}(i_*) = \prod_{i=1}^n \pi_1(M, x^0{}_i) .$$

PROOF. Given any loop class $\alpha_i \in \pi_1(M, x^0{}_i)$, there is an $x^0{}_i$-based loop a_i in $M - \{x^0{}_1, x^0{}_2, \ldots, x^0{}_{i-1}, x^0{}_{i+1}, \ldots, x^0{}_n\}$ which represents α_i, since any loop representing x_i which passes through a point $x^0{}_j$ for $j \neq i$ can be modified by a small homotopy to avoid $x^0{}_j$. The homotopy $\{b_i(t): t \in [0,1]\}$ of maps of $P_n \to M$ defined by

$$\begin{aligned} b_i(t)(p_j) &= a_i(t) \quad \text{if } j = i , \\ &= x^0{}_j \quad \text{if } j \neq i , \end{aligned}$$

is a representative of a braid $[b_i] \in \pi_1(F_nM, x^0)$, where brackets denote path class. Then

$$\begin{aligned} &i_*([b_1]\cdot[b_2]\cdot\ldots\cdot[b_n]) \\ &\quad = ([a_1], 1, 1, \ldots, 1)\cdot(1, [a_2], 1, 1, \ldots, 1)\cdot\ldots\cdot(1, 1, \ldots, 1, [a_n]) \\ &\quad = ([a_1], [a_2], \ldots, [a_n]) = (\alpha_1, \alpha_2, \ldots, \alpha_n) . \end{aligned}$$

Thus i_* is onto.

LEMMA 2. *For any surface M we have*

$$\mathrm{im}(j_*) \subseteq \ker(i_*) .$$

PROOF. If the path $\{x(t): t \in [0,1]\}$ represents an element $[x] \in \mathrm{im}(j_*)$, then for every $t \in [0,1]$ and every i, $1 \leq i \leq n$, we have $x(t)(p_i) \in D^2$. Thus the loop

$$\{x(t)(p_i) : \ t \in [0,1]\}$$

is homotopic to the trivial loop $x^0{}_i$ in D^2 since D^2 is simply connected, and thus

$$\{x(t)(p_i) : \ t \in [0,1]\}$$

is homotopic to the trivial loop $x^0{}_i$ in M for every i. We see immediately from the description of i_* in section 1, that $i_*([x]) = (1, 1, \ldots, 1)$, the identity of $\prod_{i=1}^n \pi_1(M, x^0{}_i)$, and the lemma follows.

LEMMA 3. *If M is any compact surface except S^2 or P^2, then* $\ker(j_*) = \{1\}$.

PROOF. First we need that M has a covering space $\tilde{M}$ which is homeomorphic to a subset of R^2. It is sufficient to consider orientable surfaces M, since every non-orientable surface has an orientable double covering, and any covering space of a covering space is itself a covering space. If M is closed and not S^2, then its universal covering space $\tilde{M}$ is homeomorphic to R^2. If M has boundary, then M is a subset of a closed orientable surface M_1. Let $e_1\colon \tilde{M}_1 \to M_1$ be the universal covering space of M_1, let $\tilde{M}=e_1^{-1}(M)$, and let $e=e_1|\tilde{M}$. Then $\tilde{M}_1$ is homeomorphic to R^2 if genus $(M_1)>0$. If genus $(M_1)=0$, then

$$\tilde{M}_1-\text{disk} = S^2-\text{disk} \cong \mathsf{R}^2 .$$

Suppose that $\{x(t)\colon t\in[0,1]\}$ represents a braid $[x]\in\pi_1(F_nD^2,x^0)$ and that $\{x(t)\colon t\in[0,1]\}$ is homotopic in F_nM to the identity braid representative $\{x^0\colon t\in[0,1]\}$. We must show that $\{x(t)\colon t\in[0,1]\}$ is homotopic in F_nD^2 to the identity braid representative. Since $D^2\subseteqq M$ is contractible, $e^{-1}(D^2)$ is the union of disjoint subsets of $\tilde{M}$, each mapped by e homeomorphically onto D^2. Choose one such set $\tilde{D}^2$, and let $\tilde{e}=e|\tilde{D}^2$. Denote $\tilde{e}^{-1}(x^0{}_i)$ by $\tilde{x}^0{}_i$ for each i. Now for each $t\in[0,1]$ and each i, the point $x(t)(p_i)\in D^2$. Thus the map

$$\bar{x}\colon [0,1]\times P_n \to \tilde{M}$$

defined by

$$\bar{x}(t,p_i) = \tilde{e}^{-1}x(t)(p_i)$$

covers the map $x\colon[0,1]\times P_n\to M$. The homotopy in F_nM of $\{x(t)\colon t\in[0,1]\}$ to the identity braid representative is given by some map

$$w\colon [0,1]\times[0,1]\times P_n \to M$$

such that

$$w(0,t,p_i) = x(t)(p_i), \quad w(1,t,p_i) = x^0{}_i ,$$
$$w(s,0,p_i) = x^0{}_i = w(s,1,p_i)$$

for every s, t and i; and in addition for each s and t,

$$w(s,t,p_i) \neq w(s,t,p_j) \quad \text{if } i\neq j .$$

Treating w as a homotopy on s of $x\colon[0,1]\times P_n\to M$, there is a covering homotopy $\bar{w}$ of $\bar{x}\colon[0,1]\times P_n\to\tilde{M}$. Then

$$\bar{w}(0,t,p_i) = \bar{x}(t,p_i), \quad \bar{w}(s,0,p_i) = \tilde{x}^0{}_i = \bar{w}(s,1,p_i) ,$$

and $e\bar{w}(1,t,p_i)=x^0{}_i$ for every s, t and i, and in addition for each s and t,

$$\bar{w}(s,t,p_i) \neq \bar{w}(s,t,p_j) \quad \text{if } i\neq j .$$

Since $e^{-1}(x^0{}_i)$ is discrete and $\overline{w}(1,0,p_i)=\tilde{x}^0{}_i$, then $\overline{w}(1,t,p_i)=\tilde{x}^0{}_i$ for each t and i.

We recall that $\tilde{M}$ may be considered to be a subset of R^2. Now the image of $\overline{x}$ is compact, and $\operatorname{im}(\overline{x}) \subseteqq \tilde{D}^2 \subseteqq \tilde{M} \subseteqq \mathsf{R}^2$. The image of $\overline{w}$ is also compact, and $\operatorname{im}(\overline{w}) \subseteqq \tilde{M} \subseteqq \mathsf{R}^2$. Thus there is a homeomorphism $h\colon \mathsf{R}^2 \to \mathsf{R}^2$ which leaves $\operatorname{im}(\overline{x})$ pointwise fixed and which maps $\operatorname{im}(\overline{w})$ into $\tilde{D}^2$. The proof of Lemma 3 is complete when we have shown that $\tilde{e}h\overline{w}$ is a homotopy in F_nD^2 of $\{x(t)\colon t \in [0,1]\}$ to the identity braid representative. First

$$\tilde{e}h\overline{w}(0,t,p_i) = \tilde{e}h\overline{x}(t,p_i) = \tilde{e}\overline{x}(t,p_i) = x(t)(p_i)$$

since h is pointwise fixed on $\operatorname{im}(\overline{x})$. Second

$$\tilde{e}h\overline{w}(s,0,p_i) = \tilde{e}h(\tilde{x}^0{}_i) = \tilde{e}(\tilde{x}^0{}_i) = x^0{}_i\,,$$

and

$$\tilde{e}h\overline{w}(s,1,p_i) = \tilde{e}h(\tilde{x}^0{}_i) = \tilde{e}(\tilde{x}^0{}_i) = x^0{}_i$$

since $\tilde{x}^0{}_i \in \operatorname{im}(\overline{x})$ for each i. Third

$$\tilde{e}h\overline{w}(1,t,p_i) = \tilde{e}h(\tilde{x}^0{}_i) = \tilde{e}(\tilde{x}^0{}_i) = x^0{}_i$$

for the same reason. Last, we check that for each s and t, the map $u(s,t)\colon P_n \to D^2$ given by

$$u(s,t)(p_i) = \tilde{e}h\overline{w}(s,t,p_i)$$

is in F_nD^2. Since $h(\operatorname{im}(\overline{w})) \subseteqq \tilde{D}^2$, and $\tilde{e}\colon \tilde{D}^2 \to D^2$, we have $u(s,t)(p_i) \in D^2$ for each i. If $i \neq j$, then since h and $\tilde{e}$ are homeomorphisms, and $\overline{w}(s,t,p_i) \neq \overline{w}(s,t,p_j)$, we have

$$u(s,t)(p_i) = \tilde{e}h\overline{w}(s,t,p_i) \neq \tilde{e}h\overline{w}(s,t,p_j) = u(s,t)(p_j)\,.$$

Lemma 3 is now proved.

3. The factorization theorem.

We interrupt the proof of Theorem 1, to prove the existence of a factorization of any unpermuted n-strand braid on a surface into a product of braids, each of which has all but one strand fixed (Theorem 2). In the classical case of the surface R^2, Artin [1] derives a stronger result, a unique factorization. Although our factorization could be obtained from the fibrations of Fadell and Neuwirth [4], the direct geometric proof given here, based on the braid coordinates of Artin is more consistent with the approach of this paper. These methods were also used by Dahm [3] to obtain a similar factorization theorem in a more general setting.

LEMMA 4. *Let M be a surface, and let P be a fixed subset of $P_n = \{p_1, p_2, \ldots, p_n\}$. Let $\{x(t): t \in [0,1]\}$ be a path in F_nM such that $x(t)(p_i)$ is in the interior of M for all $t \in [0,1]$ and $1 \leqq i \leqq n$, and such that whenever $p_i \in P$ and $p_j \notin P$, then $x(t)(p_j) \neq x(0)(p_i)$ for every $t \in [0,1]$. Let $\{y(t): t \in [0,1]\}$ be the path defined by*

$$\begin{aligned} y(t)(p_i) &= x(0)(p_i) \quad \text{if } p_i \in P\,, \\ &= x(t)(p_i) \quad \text{if } p_i \notin P\,. \end{aligned}$$

Then $\{y(t): t \in [0,1]\}$ is a path in F_nM, and there exists a path $\{z(t): t \in [1,2]\}$ in F_nM such that $\{x(t): t \in [0,1]\}$ is homotopic to the product of $\{y(t): t \in [0,1]\}$ and $\{z(t): t \in [1,2]\}$, and for all $t \in [1,2]$, whenever $p_i \notin P$, then $z(t)(p_i) = x(1)(p_i)$.

PROOF. The hypothesis, $x(t)(p_j) \neq x(0)(p_i)$ whenever $p_i \in P$ and $p_j \notin P$ guarantees that $\{y(t): t \in [0,1]\}$ is a path in F_nM. Since $y(t)(p_i)$ is always in the interior of M, we may use an extension of isotopy theorem or Artin's braid coordinates, to extend the isotopy $y(t)(y(0))^{-1}$ defined on $y(0)(P_n)$ to get a continuous family $\{h_t: t \in [0,1]\}$ of homeomorphisms $h_t: M \to M$ such that

$$h_t y(0)(p_i) = y(t)(p_i)$$

for $1 \leqq i \leqq n$ and all $t \in [0,1]$, and h_0 is the identity map on M. Let $h_t = h_0$ and $x(t) = x(0)$ for $t \leqq 0$, and let $h_t = h_1$ and $x(t) = x(1)$ for $t \geqq 1$. Then

$$\{h_t : t \in (-\infty, \infty)\} \quad \text{and} \quad \{x(t) : t \in (-\infty, \infty)\}$$

are continuous families. Define $\{z(t): t \in [1,2]\}$ by

$$\begin{aligned} z(t)\,(p_i) &= h_1 h_{t-1}^{-1} x(t-1)(p_i) \quad \text{if } p_i \in P\,, \\ &= x(1)(p_i) \quad \text{if } p_i \notin P\,. \end{aligned}$$

We define a family $\{u(t,s): t \in [0,2], s \in [0,1]\}$ of mappings $P_n \to M$ by

$$u(t,s)(p_i) = h_t h_{t-s}^{-1} x(t-s)(p_i)\,.$$

Noting that $\{x(t): t \in [0,1]\}$ is homotopic to (and represents the same braid as) $\{x(t): t \in [0,2]\}$, we complete the proof of Lemma 4 by checking $\big((1)$–(5) below$\big)$ that

$$\{u(t,s) : t \in [0,2], s \in [0,1]\}$$

is a homotopy in F_nM of $\{x(t): t \in [0,2]\}$ to the product of $\{y(t): t \in [0,1]\}$ and $\{z(t): t \in [1,2]\}$.

(1) $u(t,0)(p_i) = h_t h_t^{-1} x(t)(p_i) = x(t)(p_i) \qquad 0 \leqq t \leqq 2, \ p_i \in P_n\,.$

(2) $u(t,1)(p_i)$

$$= h_t h_{t-1}^{-1} x(t-1)(p_i) = \begin{cases} h_t h_0^{-1} x(0)(p_i) & 0 \leqq t \leqq 1\,, \\ h_1 h_{t-1}^{-1} x(t-1)(p_i) & 1 \leqq t \leqq 2\,, \end{cases}$$

$$= \begin{cases} h_t h_0^{-1} y(0)(p_i) & 0 \leqq t \leqq 1\,, \\ h_1 h_{t-1}^{-1} x(t-1)(p_i) & 1 \leqq t \leqq 2,\ p_i \in P\,, \\ h_1 h_{t-1}^{-1} y(t-1)(p_i) & 1 \leqq t \leqq 2,\ p_i \notin P\,, \end{cases}$$

$$= \begin{cases} h_t y(0)(p_i) & 0 \leqq t \leqq 1\,, \\ h_1 h_{t-1}^{-1} x(t-1)(p_i) & 1 \leqq t \leqq 2,\ p_i \in P\,, \\ h_1 y(0)(p_i) & 1 \leqq t \leqq 2,\ p_i \notin P\,, \end{cases}$$

$$= \begin{cases} y(t)(p_i) & 0 \leqq t \leqq 1\,, \\ h_1 h_{t-1}^{-1} x(t-1)(p_i) & 1 \leqq t \leqq 2,\ p_i \in P\,, \\ y(1)(p_i) = x(1)(p_i) & 1 \leqq t \leqq 2,\ p_i \notin P\,, \end{cases}$$

$$= \begin{cases} y(t)(p_i) & 0 \leqq t \leqq 1\,, \\ z(t)(p_i) & 1 \leqq t \leqq 2\,, \end{cases}$$

(3)
$$\begin{aligned} u(0,s)(p_i) &= h_0 h_{-s}^{-1} x(-s)(p_i) \\ &= h_0 h_0^{-1} x(0)(p_i) \\ &= x(0)(p_i) \qquad \text{for } s \in [0,1],\ p_i \in P_n\,. \end{aligned}$$

(4)
$$\begin{aligned} u(2,s)(p_i) &= h_2 h_{2-s}^{-1} x(2-s)(p_i) \\ &= h_1 h_1^{-1} x(1)(p_i) \\ &= x(1)(p_i) \\ &= x(2)(p_i) \qquad \text{for } s \in [0,1],\ p_i \in P_n\,. \end{aligned}$$

(5) For each $s \in [0,1]$, the map $u(t,s)\colon P_n \to M$ is 1–1 for each $t \in [0,2]$, since $x(t-s)$, h_{t-s}^{-1}, and h_t are always 1–1 maps.

THEOREM 2. *Any n-strand unpermuted braid $[x] \in \pi_1(F_nM, x^0)$ on a surface M may be expressed as a product*

$$[x] = [y_1]\cdot[y_2]\cdot \ldots \cdot[y_n]\,,$$

such that each factor $[y_j] \in \pi_1(F_nM, x^0)$ has a representative $\{y_j(t)\colon t \in [0,1]\}$ with $y_j(t)(p_i) = x^0{}_i$ whenever $i \neq j$, (that is only the j-th strand of y_j moves). Furthermore if for some subset $J \subseteq \{1,2,\ldots,n\}$, a representative $\{x(t)\colon t \in [0,1]\}$ of $[x]$ has $x(t)(p_j) = x^0{}_j$ for all $t \in [0,1]$ whenever $j \in J$, then for each $j \in J$, the representative $\{y_j(t)\colon t \in [0,1]\}$ of $[y_j]$ can be selected to be the identity braid, $y_j(t) = x^0$ for all $t \in [0,1]$.

PROOF. Let $Q = \{p_i\colon i \in J\}$. For convenience, we parametrize all paths in F_nM with $t \in [0,1]$, and drop the homotopy notation used above for such paths. The product of a path y and a path z will be denoted by $y \cdot z$. Let $z_0 = x$. Then A_j and B_j below are true for $j = 0$.

(A_j) The path z_j is a closed x^0-based loop in $F_n M$.
(B_j) Whenever $i \leqq j$ or $i \in J$, then $z_j(t)(p_i) = x^0{}_i$ for all $t \in [0,1]$.

We shall show recursively, for $1 \leqq j \leqq n-1$, that if we have a path z_{j-1} such that A_{j-1} and B_{j-1} are satisfied, we can find paths y_j and z_j such that A_j, B_j, and C_j through F_j below are all satisfied.

(C_j) The path y_j is a closed x^0-based loop in $F_n M$.
(D_j) Whenever $i \neq j$, then $y_j(t)(p_i) = x^0{}_i$ for all $t \in [0,1]$.
(E_j) Whenever $j \in J$, then $y_j(t) = x^0$ for all $t \in [0,1]$.
(F_j) $[z_{j-1}] = [y_j] \cdot [z_j]$.

This is done as follows: Let

$$P = \{p_{j+1}, p_{j+2}, \ldots, p_n\} - Q .$$

Altering z_{j-1} by a small homotopy which does not move the endpoints or the ith strand for $i \leqq j$ or $i \in J$ if necessary, we may obtain a closed x^0-based loop z'_{j-1} such that $z'_{j-1}(t)(p_i)$ is in the interior of M for all i and t, and such that whenever $p_i \in P$ and $p_k \notin P$, then

$$z'_{j-1}(t)(p_k) \neq z'_{j-1}(0)(p_i) \quad \text{for every } t \in [0,1] .$$

Thus we apply Lemma 4 to factor $z'_{j-1} = y_j \cdot z_j$, with

$$\begin{aligned} y_j(t)(p_i) &= z'_{j-1}(0)(p_i) \quad \text{if } p_i \in P , \\ &= z'_{j-1}(t)(p_i) \quad \text{if } p_i \notin P , \end{aligned}$$

and $z_j(t)(p_i) = z'_{j-1}(1)(p_i)$ whenever $p_i \notin P$. Since z'_{j-1} is a closed x^0-based loop,

$$z'_{j-1}(0)(p_i) = x^0{}_i = z'_{j-1}(1)(p_i)$$

for every i. Thus $y_j(1)(p_i) = x^0{}_i$ for every i. Since $z'_{j-1} = y_j \cdot z_j$, we see that

$$\begin{gathered} y_j(0) = z'_{j-1}(0) = x^0, \quad z_j(0) = y_j(1) = x^0 , \\ z_j(1) = z'_{j-1}(1) = x^0 . \end{gathered}$$

Thus y_j and z_j are closed x^0-based loops and A_j, C_j and F_j are satisfied. To show D_j, suppose that $i \neq j$. If $i \leqq j-1$ or if $i \in J$, then $p_i \notin P$, and B_{j-1} implies that

$$y_j(t)(p_i) = z'_{j-1}(t)(p_i) = z_{j-1}(t)(p_i) = x^0{}_i$$

for all $t \in [0,1]$. If $i \geqq j+1$ and $i \notin J$, then $p_i \in P$, and

$$y_j(t)(p_i) = z'_{j-1}(0)(p_i) = x^0{}_i \quad \text{for all } t \in [0,1] .$$

To show E_j, suppose $j \in J$. By D_j, it suffices to show

$$y_j(t)(p_j) = x^0{}_j \quad \text{for all } t \in [0,1] .$$

But $j \in J$ implies that $p_j \notin P$, and thus

$$y_j(t)(p_j) = z'{}_{j-1}(t)(p_j) = z_{j-1}(t)(p_j) = x^0{}_j$$

for all $t \in [0,1]$ by B_{j-1}. Thus A_j through F_j have been established for $1 \leqq j \leqq n-1$. Let $y_n = z_{n-1}$. The assertions F_j show that

$$[x] = [z_0] = [y_1]\cdot[y_2]\cdot \ldots \cdot[y_n] .$$

The assertions C_j, D_j and E_j in the conclusion of Theorem 2 have been shown for $1 \leqq j \leqq n-1$, and C_n, D_n, and E_n follow immediately from A_{n-1} and B_{n-1}.

4. Completion of the proof of Theorem 1.

The proof of Theorem 1 will be complete when we have shown:

LEMMA 5. *If M is any closed surface except S^2 and P^2, then* $\ker(i_*) \subseteqq \langle \operatorname{im}(j_*)\rangle$, *where* $\langle \operatorname{im}(j_*)\rangle$ *is the normal closure (consequence) of* $\operatorname{im}(j_*)$ *in* $\pi_1(F_nM, x^0)$.

PROOF. We must show that if $[x] \in \pi_1(F_nM, x^0)$ is such that

$$i_*([x]) = (1,1,\ldots,1) \in \textstyle\prod_{i=1}^n \pi_1(M, x^0{}_i) ,$$

then for some $\alpha_1,\alpha_2,\ldots,\alpha_r \in \pi_1(F_nM,x^0)$ and some $\beta_1,\beta_2,\ldots,\beta_r \in \operatorname{im}(j_*)$ we may factor

$$[x] = \textstyle\prod_{k=1}^r \alpha_k\beta_k\alpha_k{}^{-1} .$$

As a consequence of Theorem 2, it is sufficient to prove this assertion whenever $[x]=[y_j]$ has a representative y_j such that for all $i \neq j$, then $y_j(t)(p_i) = x^0{}_i$ for all t, that is when only one strand moves. In this case

$$i_*([y_j]) = (1,1,\ldots,1)$$

implies that

$$\{y_j(t)(p_j) : t \in [0,1]\} = 1 \in \pi_1(M, x^0{}_j) .$$

Let $e\colon \tilde{M} \to M$ be the universal covering space of M. Then $\tilde{M}$ is homeomorphic to R^2. We fix one such homeomorphism and identify $\tilde{M}$ with R^2 by means of it. Thus all of the geometric properties of R^2 in its usual coordinate system may be applied to $\tilde{M}$ below. As in the proof of Lemma 3, we choose a component $\tilde{D}^2$ of $e^{-1}(D^2)$, and let $\tilde{e} = e\,|\,\tilde{D}^2$. Again, $\tilde{e}$ is a homeomorphism and we let $\tilde{x}^0{}_i = \tilde{e}^{-1}(x^0{}_i)$ for each i. Let

$$E^0{}_j = \{x^0{}_1, x^0{}_2, \ldots, x^0{}_{j-1}, x^0{}_{j+1}, x^0{}_{j+2}, \ldots, x^0{}_n\} .$$

Let $\tilde{X}^0{}_i = e^{-1}(\{x^0{}_i\})$ and let

$$\tilde{E}^0{}_j = \bigcup_{i \neq j} \tilde{X}^0{}_i = e^{-1}(E^0{}_j) .$$

Since

$$\{y_j(t)(p_j) : t \in [0,1]\} = 1 \in \pi_1(M, x^0{}_j) ,$$

the covering homotopy $\{\bar{y}(t) : t \in [0,1]\}$ in $\tilde{M}$ which starts at $\tilde{x}^0{}_j$ is a closed loop. Since $y_j \in F_n M$, we have $y_j(t)$ is a 1–1 map for each $t \in [0,1]$, and thus $\bar{y}_j(t) \notin \tilde{E}^0{}_j$ for any $t \in [0,1]$. So $\{\bar{y}_j(t) : t \in [0,1]\}$ is a closed $\tilde{x}^0{}_j$-based loop in $\tilde{M} - \tilde{E}^0{}_j$.

Let S_j be the set of all paths

$$\bar{u} = \{\bar{u}(t) : t \in [0,1]\}$$

in $\tilde{M} - \tilde{E}^0{}_j$ such that $\bar{u}(0)$ and $\bar{u}(1) \in \tilde{X}^0{}_j = e^{-1}(\{x^0{}_j\})$. Then the path $\{e\bar{u}(t) : t \in [0,1]\}$ is a closed $x^0{}_j$-based loop in $M - E^0{}_j$. If $\bar{u} \in S_j$, we may define a braid representative $\{u(t) : t \in [0,1]\}$ by

$$\begin{aligned} u(t)(p_i) &= e\bar{u}(t) \quad \text{if } i = j , \\ &= x^0{}_i \quad \text{if } i \neq j . \end{aligned}$$

We define $\varphi_j(\bar{u})$ to be the geometric braid u. If $\bar{u}$ and $\bar{v}$ are composable paths in S_j, then

$$\varphi_j(\bar{u} \cdot \bar{v}) = \varphi_j(\bar{u}) \cdot \varphi_j(\bar{v}) .$$

If $\bar{u} \in S_j$, then $\bar{u}^{-1} \in S_j$, and $\varphi_j(\bar{u}^{-1}) = (\varphi_j(\bar{u}))^{-1}$. If $\bar{u} \in S_j$ is homotopic in $\tilde{M} - \tilde{E}^0{}_j$ to a path $\bar{u}_1 \in S_j$ by a homotopy $\{\bar{u}_s : s \in [0,1]\}$ which fixes end points, then $\{\varphi_j(\bar{u}_s) : s \in [0,1]\}$ is a homotopy of x^0-based loops in $F_n M$ between $\varphi_j(\bar{u})$ and $\varphi_j(\bar{u}_1)$. In particular, for the $\bar{y}_j$ and y_j with which we are working, $\varphi_j(\bar{y}_j) = y_j$, and if $\bar{z}$ is any $\tilde{x}^0{}_j$-based loop in $\tilde{M} - \tilde{E}^0{}_j$ which is homotopic in $\tilde{M} - \tilde{E}^0{}_j$ to $\bar{y}_j$, then $\varphi_j(\bar{z})$ represents the same element

$$[\varphi_j(\bar{z})] = [\varphi_j(\bar{y}_j)] = [y_j] \in \pi_1(F_n M, x^0)$$

as does $\varphi_j(\bar{y}_j) = y_j$.

Let A_j be the set of all paths

$$\bar{a} = \{\bar{a}(t) : t \in [0,1]\}$$

in $\tilde{M} - \tilde{E}^0{}_j$ such that $\bar{a}(0) = \tilde{x}^0{}_j$ and $\bar{a}(1) \in \tilde{X}^0{}_j = e^{-1}(\{x^0{}_j\})$. Thus $A_j \subseteq S_j$, and for each $\bar{a} \in A_j$, the geometric braid $\varphi_j(\bar{a})$ represents an element $\alpha \in \pi_1(F_n M, x^0)$. Let B_j be the set of all paths

$$\bar{b} = \{\bar{b}(t) : t \in [0,1]\}$$

in $\tilde{M} - \tilde{E}^0{}_j$ satisfying the following two conditions

(1) For some $\tilde{x}^r{}_j \in \tilde{X}^0{}_j$, the end points $\bar{b}(0) = \tilde{x}^r{}_j = \bar{b}(1)$.
(2) $\bar{b}(t) \in \tilde{D}^r$ for all $t \in [0,1]$, where $\tilde{D}^r$ is the component of $e^{-1}(D^2)$ which contains $\tilde{x}^r{}_j$.

Thus $B_j \subseteq S_j$, and for each $\bar{b} \in B_j$, not only does the geometric braid $\varphi_j(\bar{b})$ represent an element $\beta \in \pi_1(F_n M, x^0)$, but the element $\beta \in \mathrm{im}(j_*)$.

Thus if we can show that $\bar{y}_j$ is homotopic in $\tilde{M} - \tilde{E}^0{}_j$ to a product

$$\prod_{k=1}^r \bar{a}_k \bar{b}_k \bar{a}_k^{-1}$$

with $\bar{a}_1, \bar{a}_2, \ldots \bar{a}_r \in A_j$, and $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_r \in B_j$, then

$$\begin{aligned}[y_j] &= [\varphi_j(\bar{y}_j)] = [\varphi_j(\textstyle\prod_{k=1}^r \bar{a}_k \bar{b}_k \bar{a}_k^{-1})] = [\textstyle\prod_{k=1}^r \varphi_j(\bar{a})\varphi_j(\bar{b})\varphi_j(\bar{a})^{-1}] \\ &= \textstyle\prod_{k=1}^r \alpha_k \beta_k \alpha_k^{-1},\end{aligned}$$

where $\alpha_k = [\varphi_j(\bar{a}_k)] \in \pi_1(F_n M, x^0)$ and $\beta_k = [\varphi_j(\bar{b}_k)] \in \mathrm{im}(j_*)$ for $k = 1, 2, \ldots, r$, and the lemma would be proved.

Let $\tilde{C}$ be an open Euclidean disk in $\tilde{M}$ centered at $\tilde{x}^0{}_j$ with radius large enough so that $\mathrm{im}(\bar{y}_j) \subseteq \tilde{C}$ (see Figure 1). Let $\tilde{E}_j = \tilde{E}^0{}_j \cap \tilde{C}$. Choose a coordinate system in $\tilde{M} = \mathrm{R}^2$ such that no two points of $\tilde{E}_j \cup \{\tilde{x}^0{}_j\}$ lie on the same vertical line. This is always possible since $\tilde{E}_j \cup \{\tilde{x}^0{}_j\}$ is a finite set. Let $L(\tilde{E}_j)$ be the union of all vertical lines through points of $\tilde{E}_j$. For any $\tilde{x}^0{}_j$-based loop $\bar{u}$, let $N(\bar{u})$ be the number of values of t for which $\bar{u}(t) \in L(\tilde{E}_j)$. Consider the set W of pairs of paths $(\bar{u}, \bar{v})$ such that

$$\bar{u} = \prod_{k=1}^r \bar{a}_k \bar{b}_k \bar{a}_k^{-1}$$

for some $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_r \in A_j$ and some $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_r \in B_j$, and such that $\bar{u} \cdot \bar{v}$ is homotopic to $\bar{y}_j$ in $\tilde{C} - \tilde{E}_j$. By the definition of A_j, we see that $\bar{v}$ must be a closed $\tilde{x}^0{}_j$-based loop. Since the trivial path $\bar{u} = \tilde{x}^0{}_j$ satisfies the condition for $\bar{u}$, and since $L(\tilde{E}_j)$ consists of a finite number of lines allowing us to alter $\bar{y}_j$ by a small homotopy in $\tilde{C} - \tilde{E}_j$ to a path $\bar{v}$ for which $N(\bar{v})$ is finite, we see that the minimum of $N(\bar{v})$ for $(\bar{u}, \bar{v}) \in W$ is finite. We choose a pair $(\bar{u}, \bar{v}) \in W$ for which $N(\bar{v})$ has the smallest possible value.

If $N(\bar{v}) = 0$, then $\mathrm{im}(\bar{v}) \cap L(\tilde{E}_j) = \emptyset$, and since each component of $\tilde{C} - L(\tilde{E}_j)$ is simply connected, $\bar{v}$ is homotopic in $\tilde{C} - L(\tilde{E}_j)$ to the trivial path $\tilde{x}^0{}_j$. Thus $\bar{y}_j$ is homotopic in $\tilde{C} - \tilde{E}_j \subseteq \tilde{M} - \tilde{E}^0{}_j$ to $\bar{u}$, and the lemma is proved.

We now show that $N(\bar{v})$ cannot be greater than zero, and thereby complete the proof of Lemma 3. Suppose $N(\bar{v}) > 0$ were true. Then at every point of $\mathrm{im}(\bar{v}) \cap L(\tilde{E}_j)$, the path $\bar{v}$ crosses a line $L \subseteq L(\tilde{E}_j)$, since otherwise a small homotopy of $\bar{v}$ would reduce $N(\bar{v})$. Since $\bar{v}$ is a closed $\tilde{x}^0{}_j$-based loop, there exist $t_1, t_2 \in (0,1)$ and a line $L \subseteq L(\tilde{E}_j)$ such that

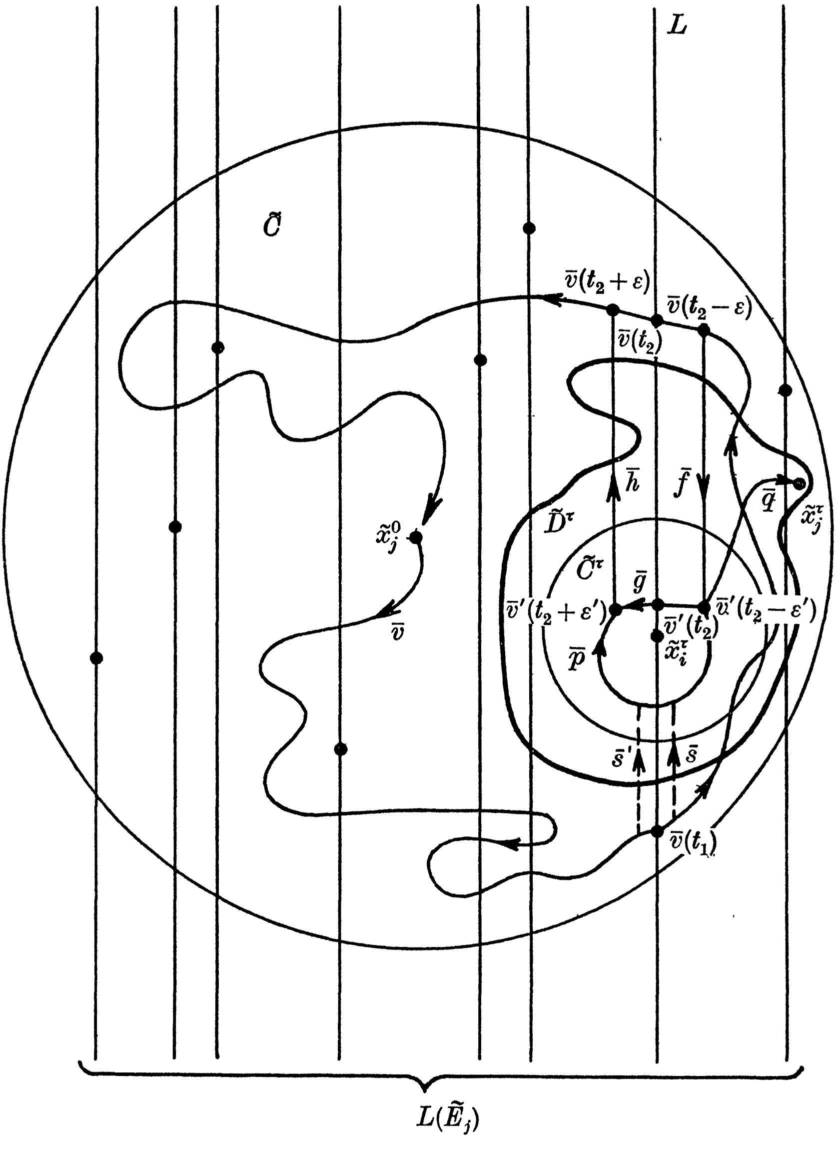

Fig. 1.

$t_1 < t_2$, $\bar{v}(t_1) \in L$, $\bar{v}(t_2) \in L$, and for no $t \in (t_1, t_2)$ does $\bar{v}(t) \in L(\tilde{E}_j)$. For example, if L is the line of $L(\tilde{E}_j)$ which is furthest from $\tilde{x}^0{}_j$, and if t_1 and t_2 are the smallest values of t for which $\bar{v}(t) \in L$, then these conditions are satisfied. Choose an ε small enough that $\bar{v}(t) \notin L(\tilde{E}_j)$ for all

$$t \in [t_2-\varepsilon, t_2+\varepsilon] ,$$

except for $t=t_2$.

Let $\tilde{x}^{\tau}_i$ be the point of $\tilde{E}_j$ which belongs to L, let $\tilde{D}^{\tau}$ be the component of $e^{-1}(D^2)$ which contains $\tilde{x}^{\tau}_i$, and let $\tilde{C}^{\tau}$ be a Euclidean open disk with center $\tilde{x}^{\tau}_i$ and radius so small that $\tilde{C}^{\tau} \subseteq \tilde{C} \cap \tilde{D}^{\tau}$ and $\tilde{C}^{\tau} \cap L(\tilde{E}_j) = \tilde{C}^{\tau} \cap L$. If $\bar{v}(t_2) \notin \tilde{C}^{\tau}$, then $\bar{v}$ is homotopic to a path $\bar{v}'$ with $\bar{v}'(t_2) \in \tilde{C}^{\tau} \cap L$, the homotopy taking place in $\tilde{C}-\tilde{E}_j$ in a small neighborhood of L, and modifying $\bar{v}(t)$ only for $t \in (t_2-\varepsilon, t_2+\varepsilon)$ in such a way that $N(\bar{v}')=N(\bar{v})$. Geometrically, the part of $\bar{v}$ which crosses L slides along L until it is very close to $\tilde{x}^{\tau}_i$. We note that $(\bar{u},\bar{v}') \in W$ and $N(\bar{v}')$ is the smallest possible. Choose ε' small enough such that $\bar{v}'(t) \in \tilde{C}^{\tau}$ for all $t \in [t_2-\varepsilon', t_2+\varepsilon']$. Let $\bar{f}$ be the path

$$\{\bar{v}'(t) : t \in [0, t_2-\varepsilon']\} ,$$

let $\bar{g}$ be the path

$$\{\bar{v}'(t) : t \in [t_2-\varepsilon', t_2-\varepsilon']\} ,$$

and let $\bar{h}$ be the path

$$\{\bar{v}'(t) : t \in [t_2+\varepsilon', 1]\} .$$

Then $\bar{v}' = \bar{f}\cdot\bar{g}\cdot\bar{h}$ and $N(\bar{g})=1$. Let $\bar{q}$ be any path in $\tilde{D}^{\tau}-\tilde{E}_j$ from $\bar{v}'(t_2-\varepsilon')$ to $\tilde{x}^{\tau}_j$, the point of $e^{-1}(x^0_j)$ in $\tilde{D}^{\tau}$. Now $\bar{g}$ is homotopic in $\tilde{C}^{\tau}$ (but not $\tilde{C}-\tilde{E}_j$) to a path $\bar{p}$ such that $\bar{p}(t_2) \in L$, $N(\bar{p})=1$, and that $\bar{p}(t_2)$ and $\bar{g}(t_2)$ lie on opposite sides of $\tilde{x}^{\tau}_i$ in L. Then $\bar{g}\cdot\bar{p}^{-1}$ may be thought of as a small loop encircling $\tilde{x}^{\tau}_i$. With homotopies taking place in $\tilde{C}-\tilde{E}_j$ we have

$$\begin{aligned} \bar{v}' &= \bar{f}\cdot\bar{g}\cdot\bar{h} \\ &\simeq (\bar{f}\cdot\bar{q})\cdot(\bar{q}^{-1}\cdot\bar{g}\cdot\bar{p}^{-1}\cdot\bar{q})\cdot(\bar{q}^{-1}\cdot\bar{f}^{-1})\cdot(\bar{f}\cdot\bar{p}\cdot\bar{h}) . \end{aligned}$$

But $\bar{a}_{r+1} = \bar{f}\cdot\bar{q} \in A_j$ and $\bar{b}_{r+1} = \bar{q}^{-1}\cdot\bar{g}\cdot\bar{p}^{-1}\cdot\bar{q} \in B_j$. We let

$$\bar{u}'' = \bar{u}\cdot(\bar{f}\cdot\bar{q})\cdot(\bar{q}^{-1}\cdot\bar{g}\cdot\bar{p}^{-1}\cdot\bar{q})\cdot(\bar{q}^{-1}\cdot\bar{f}^{-1})$$

and

$$\bar{v}'' = \bar{f}\cdot\bar{p}\cdot\bar{h} .$$

Then

$$\bar{u}'' = \prod_{k=1}^{r+1} \bar{a}_k \bar{b}_k \bar{a}_k^{-1}$$

is of the proper form, and

$$\begin{aligned} \bar{u}''\cdot\bar{v}'' &= \bar{u}\cdot(\bar{f}\cdot\bar{q})\cdot(\bar{q}^{-1}\cdot\bar{g}\cdot\bar{p}^{-1}\cdot\bar{q})\cdot(\bar{q}^{-1}\cdot\bar{f}^{-1})\cdot(\bar{f}\cdot\bar{p}\cdot\bar{h}) \\ &\simeq \bar{u}\bar{v}' \simeq \bar{u}\bar{v} \simeq \bar{y}_j . \end{aligned}$$

Thus $(\bar{u}'',\bar{v}'') \in W$ and $N(\bar{v}'')=N(\bar{v})$ is the smallest possible. Also $\bar{v}''(t) \notin L(\tilde{E}_j)$ for all $t \in (t_1,t_2)$.

Replacing $(\bar{u},\bar{v})$ by $(\bar{u}'',\bar{v}'')$ if necessary, we may assume that $\bar{v}(t_1)$ and $\bar{v}(t_2)$ lie on the same side of $\tilde{x}^{\tau}{}_i$ in L. Choosing ε'' small enough, the path

$$\{\bar{v}(t) : t \in [t_1+\varepsilon'', t_2-\varepsilon'']\}$$

is homotopic in a strip of $\tilde{C}-L(\tilde{E}_j)$ to the line segment $\bar{s}$ from $\bar{v}(t_1+\varepsilon'')$ to $\bar{v}(t_2-\varepsilon'')$ and the path

$$\{\bar{v}(t) : t \in [t_1-\varepsilon'', t_1+\varepsilon'']\}\cdot\bar{s}\cdot\{\bar{v}(t) : t \in [t_2-\varepsilon'', t_2+\varepsilon'']\}$$

is homotopic in $\tilde{C}-\tilde{E}_j$ to the line segment $\bar{s}'$ from $\bar{v}(t_1-\varepsilon'')$ to $\bar{v}(t_2+\varepsilon'')$. Let

$$\bar{v}''' = \{\bar{v}(t) : t \in [0, t_1-\varepsilon'']\}\cdot\bar{s}'\cdot\{\bar{v}(t) : t \in [t_2+\varepsilon'', 1]\}.$$

Then $\bar{v}'''$ is homotopic in $\tilde{C}-\tilde{E}_j$ to $\bar{v}$, and thus $(\bar{u},\bar{v}''') \in W$. But $N(\bar{v}''') = N(\bar{v})-2$, a contradiction to the minimality of $N(\bar{v})$. Thus $N(\bar{v})=0$ and the lemma has been proved.

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