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# PARABOLICALLY INDUCED UNITARY REPRESENTATIONS OF THE UNIVERSAL GROUP $U(F)^+$ ARE $C_0$

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## Abstract

We prove that all parabolically induced unitary representations of the Burger-Mozes universal group $U(F)^+$, with $F$ being primitive, are $C_0$. This generalizes the same well-known result for the universal group $U(F)^+$, when $F$ is 2-transitive.


## 1. Introduction

Let $\mathscr{T}$ be a $d$-regular tree, with $d \geq 3$. Let $\mathbb{G} := U(F)^+ \leq \mathrm{Aut}(\mathscr{T})$, with $F \leq \mathrm{Sym}\{1, \dots, d\}$ being primitive, be the universal group introduced by Burger and Mozes in [3, §3]. Given a (strongly continuous) unitary representation $\pi\colon \mathbb{G} \to \mathscr{U}(\mathscr{H})$ on a (infinite-dimensional) complex Hilbert space $(\mathscr{H}, \langle \cdot, \cdot \rangle)$, we are interested in studying the matrix coefficients $c_{v,w}\colon \mathbb{G} \to \mathbb{C}$ given by $c_{v,w}(g) := \langle \pi(g)v, w \rangle$, for every $v, w \in \mathscr{H}$. We say $(\pi, \mathscr{H})$ is a $C_0$ unitary representation of $\mathbb{G}$ if for any of its associated matrix coefficients $c_{v,w}$, the subset $\{g \in \mathbb{G} \mid |c_{v,w}(g)| \geq \epsilon\}$ is compact in $\mathbb{G}$, for every $\epsilon > 0$; equivalently, $\lim_{g\to\infty} |c_{v,w}(g)| = 0$, for every $v, w \in \mathscr{H}$, where $\infty$ represents the one-point compactification of the locally compact group $\mathbb{G}$.

It is a general fact [2, Appendix C, Proposition C.4.6] that the left regular unitary representation of $\mathbb{G}$ is $C_0$. When $F$ is 2-transitive (if and only if $\mathbb{G}$ is 2-transitive on the boundary $\partial\mathscr{T}$) Burger and Mozes [3] showed the Howe-Moore property of $\mathbb{G}$: every unitary representation of $\mathbb{G}$, without non-zero $\mathbb{G}$-invariant vectors, is $C_0$. Still, for $F$ being just primitive, but not 2-transitive, it is difficult to predict when a (non-trivial) unitary representation of $\mathbb{G}$ is $C_0$ or not. Apart from [2, Proposition C.4.6] and the general criterion proven in [6] (and the references therein) providing a unified proof of the Howe-Moore property for all known examples, there are no other known techniques to prove that a unitary representation of a locally compact group is $C_0$.

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When $F$ is primitive, [7] shows $\mathbb{G}$ has a weakening of the Howe-Moore property, namely the relative Howe-Moore property with respect to any *horospherical stabilizer* $\mathbb{G}_\xi^0 := \{g \in \mathbb{G} \mid g(\xi) = \xi,\ g \text{ elliptic}\}$ with $\xi \in \partial\mathcal{T}$. This relative property was introduced and studied in [8]. Another result of [7] shows when $F$ is primitive but not 2-transitive, that the stabilizer $\mathbb{G}_v$ in $\mathbb{G}$ of a non-zero vector $v \in \mathcal{H}$ of a unitary representation $(\pi, \mathcal{H})$ of $\mathbb{G}$ without non-zero $\mathbb{G}$-invariant vectors, either is compact or if it is not compact then it equals $\mathbb{G}_\xi^0$, for some $\xi \in \partial\mathcal{T}$. It is then natural to ask whether the unitary representations of $\mathbb{G}$ induced from closed subgroups of the stabilizer $\mathbb{G}_\xi := \{g \in \mathbb{G} \mid g(\xi) = \xi\}$ for $\xi \in \partial\mathcal{T}$ are $C_0$ or not. The following vanishing result gives the answer to this question. For the theory of induced unitary representations we use notation and the results from [2, Appendices B and E].

THEOREM 1.1. *Let $F \leq \mathrm{Sym}\{1, \dots, d\}$ be primitive and $\xi \in \partial\mathcal{T}$. Let $H$ be a closed subgroup of $\mathbb{G}$ stabilizing $\xi$ and let $(\sigma, \mathcal{K})$ be a unitary representation of $H$. Then the induced unitary representation $(\pi_\sigma, \mathcal{H}_\sigma)$ on $\mathbb{G}$ is $C_0$.*

We emphasise Theorem 1.1 covers the known case when $F$ is 2-transitive (that case being covered by the Howe-Moore property). Still, the proof of Theorem 1.1 is very different from the general one proving the Howe-Moore property. This is firstly, because the group $\mathbb{G}$, when $F$ is primitive but *not* 2-transitive, does not verify the general criterion given in [6]. Secondly, if $F$ is primitive but *not* 2-transitive, it is a direct consequence the quotient $\mathbb{G}/\mathbb{G}_\xi$ is *not* compact anymore and *not* isomorphic to the boundary $\partial\mathcal{T}$, for any choice of $\xi \in \partial\mathcal{T}$. Moreover, by [7], $\mathbb{G}_\xi$ is a closed, still non-compact subgroup, for every $\xi \in \partial\mathcal{T}$.

To prove Theorem 1.1, we follow the lines of the standard argument that the left regular unitary representation is $C_0$. The novelty of the article is the control of the integral given by Remark 3.10 in terms of indices of subgroups and we distinguish three cases in the calculation of the asymptotics of that integral.

## 2. Some properties of $\mathbb{G}$

For the definition and the main properties of the Burger-Mozes universal groups the reader can consult Burger-Mozes [3], Amann [1], Ciobotaru [5].

To fix the notation, let $d_{\mathcal{T}}(\cdot, \cdot)$ be the usual metric on $\mathcal{T}$. Let $\mathrm{Aut}(\mathcal{T})^+$ be the group of all type-preserving automorphisms of $\mathcal{T}$ and by definition $\mathbb{G} \leq \mathrm{Aut}(\mathcal{T})^+$. For every pair of points $x, y \in \mathcal{T} \cup \partial\mathcal{T}$, $[x, y]$ is the unique geodesic between $x$ and $y$ in $\mathcal{T} \cup \partial\mathcal{T}$. For $G \leq \mathrm{Aut}(\mathcal{T})$ and $x, y \in \mathcal{T} \cup \partial\mathcal{T}$ we define $G_{[x,y]} := \{g \in G \mid g \text{ fixes pointwise the geodesic } [x, y]\}$. In particular, $G_x := \{g \in G \mid g(x) = x\}$. For $\xi \in \partial\mathcal{T}$ we have already defined $G_\xi$ and $G_\xi^0$.

Note $G_\xi$ can contain hyperbolic elements; if this is the case then $G_\xi^0 \lneq G_\xi$. If $H \leq G_\xi$ and $x \in \mathscr{T}$ then $H_x$ evidently equals $H_{[x,\xi]}$. For a vertex $x \in \mathscr{T}$ and an edge $e$ in the star of $x$, set $K := G_x$ and let $\mathscr{T}_{x,e}$ be the half-tree of $\mathscr{T}$ emanating from the vertex $x$ and containing the edge $e$. For a hyperbolic element $\gamma \in \mathrm{Aut}(\mathscr{T})$, we write $|\gamma| := \min_{x \in \mathscr{T}}\{\mathrm{d}_{\mathscr{T}}(x, \gamma(x))\}$, which is called the translation length of $\gamma$. Set $\mathrm{Min}(\gamma) = \{x \in \mathscr{T} \mid \mathrm{d}_{\mathscr{T}}(x, \gamma(x)) = |\gamma|\}$.

REMARK 2.1. As $F$ is primitive, given an edge $e' \in E(\mathscr{T})$ at odd distance from $e$, one can construct, using the definition of $\mathbb{G}$, a hyperbolic element in $\mathbb{G}$ translating $e$ to $e'$. Moreover, every hyperbolic element in $\mathbb{G}$ has even translation length, as $\mathbb{G}$ has only type-preserving automorphisms.

LEMMA 2.2 (The $KA^+K$ decomposition). *Let $F$ be primitive. Let $x \in \mathscr{T}$ be a vertex and $e$ an edge in the star of $x$. Then $\mathbb{G}$ admits a $KA^+K$ decomposition, where $A^+ := \{\gamma \in \mathbb{G} \mid e \subset \mathrm{Min}(\gamma), \gamma(e) \subset \mathscr{T}_{x,e}\} \cup \{\mathrm{id}\}$.*

PROOF. Let $g \in \mathbb{G}$. If $g(x) = x$, then $g \in K$. If not, consider the geodesic segment $[x, g(x)]$ in $\mathscr{T}$; denote by $e_1$ the edge of the star of $x$ belonging to $[x, g(x)]$. By type-preserving, $[x, g(x)]$ has even length. As $F$ is also transitive, there is $k \in K$ with $k(e_1) = e$; therefore, $kg(x) \in \mathscr{T}_{x,e}$. By Remark 2.1, there is a hyperbolic element $\gamma \in \mathbb{G}$ of translation length equal to the length of $[x, g(x)]$, translating the edge $e$ inside $\mathscr{T}_{x,e}$ and with $\gamma(x) = kg(x)$; thus $\gamma^{-1}kg \in K$. Note the $KA^+K$ decomposition of an element $g \in \mathbb{G}$ is not unique.

LEMMA 2.3. *Let $F$ be primitive and let $H$ be a closed, non-compact and proper subgroup of $\mathbb{G}$. Then, for every $x \in \mathscr{T}$, $H_x$ does not have finite index in $\mathbb{G}_x$.*

PROOF. By Caprace-De Medts [4, Proposition 4.1] the subgroup $F$ is primitive if and only if every proper open subgroup of $\mathbb{G}$ is compact. $H$ cannot be an open subgroup of $\mathbb{G}$, as otherwise $H$ would be compact, contradicting the hypothesis. Suppose there is an $x \in \mathscr{T}$ with $[\mathbb{G}_x : H_x] < \infty$. As $H_x$ is closed in $\mathbb{G}_x$ and of finite index, $H_x$ is open in $\mathbb{G}_x$ and so also in $\mathbb{G}$. This means $H$ is open in $\mathbb{G}$, obtaining a contradiction.

## 3. Induced unitary representations

We follow [2, Appendices B and E] where all the definitions, notation, proofs and complementary definitions can be found (see also [5]). Fix, in this section, $G$ to be a locally compact group and $H \leq G$ a closed subgroup. All Haar measures used in this paper are considered to be left invariant. We denote by $dx$, respectively, $dh$ the Haar measure on $G$, respectively, $H$. We endow

$G/H$ with the quotient topology: the canonical projection $p: G \to G/H$ is continuous and open.

DEFINITION 3.1 (See [2, Appendix B]). A *rho-function* of $(G, H)$ is a continuous function $\rho: G \to \mathbb{R}_+^*$ satisfying the equality

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(x) \quad \text{for all } x \in G, h \in H, \tag{1}$$

where $\Delta_G$, $\Delta_H$ are the modular functions on $G$, respectively on $H$.

By [2, Theorem B.1.4], there is a correspondence between rho-functions of $(G, H)$ and continuous $G$-quasi-invariant regular Borel (CGQIRB) measures on $G/H$ (see [2, Appendix A.3]), where continuous means the Radon-Nikodym derivative of $\mu$ is continuous.

DEFINITION 3.2. Let $(\sigma, \mathcal{K})$ be a unitary representation of $H$. Suppose $G/H$ is endowed with a CGQIRB-measure $\mu$, with associated rho-function $\rho$ on $G$. The *induced unitary representation* $(\pi_{\sigma,\mu}, \mathcal{H}_{\sigma,\mu})$ of $G$ is defined as follows. For every $g \in G$, we define the unitary operator $\pi_{\sigma,\mu}(g)$ on $\mathcal{A}$ by $\pi_{\sigma,\mu}(g)(\xi)(x) := (\rho(g^{-1}x)/\rho(x))^{1/2}\xi(g^{-1}x)$, where $\xi \in \mathcal{A}$ and $x \in G$, and where $\mathcal{A}$ is a specific dense subset of the Hilbert space $\mathcal{H}_{\sigma,\mu}$. For a complete definition see [2, Appendix E]. Moreover, by [2, Proposition E.1.4], this is a unitary representation of $G$ on the Hilbert space $\mathcal{H}_{\sigma,\mu}$.

REMARK 3.3. By [2, Proposition E.1.5], induced unitary representations do only depend on the unitary representations $(\sigma, \mathcal{K})$ of $H$ and not on the CGQIRB-measures on $G/H$. If $(\pi_{\sigma,\mu_1}, \mathcal{H}_{\sigma,\mu_1})$ is $C_0$ the same is true for $(\pi_{\sigma,\mu_2}, \mathcal{H}_{\sigma,\mu_2})$.

NOTATION 3.4. By Remark 3.3 it is legitimate to write $(\pi_\sigma, \mathcal{H}_\sigma)$ for the unitary representation of $G$ induced from the unitary representation $(\sigma, \mathcal{K})$ of $H$.

LEMMA 3.5. *For every compact subgroup $K$ of $G$ there exists a CGQIRB-measure $\mu$ on $G/H$ which is left $K$-invariant.*

PROOF. By [2, Theorem B.1.4] let $\mu_1$ be a CGQIRB-measure on $G/H$ with $\rho_1: G \to \mathbb{R}_+^*$ its associated rho-function. Let $\rho: G \to \mathbb{R}_+^*$ be the function defined by $g \in G \mapsto \rho(g) := \int_K \rho_1(kg)\, dk$, with $dk$ the Haar measure on $K$. Then $\rho$ is continuous, satisfies equation (1) from Definition 3.1 and so $\rho$ is a rho-function and left $K$-invariant. By [2, Theorem B.1.4], let $\mu$ be the CGQIRB-measure on $G/H$ associated with $\rho$. As the Radon-Nikodym derivative of $\mu$ satisfies $\frac{dy_\star\mu}{d\mu}(xH) = \frac{\rho(yx)}{\rho(x)}$, for every $x, y \in G$ and because $\rho$ is left $K$-invariant, we obtain $\mu$ is left $K$-invariant.

LEMMA 3.6. *Let $K \leq G$ be compact. Consider on $G/H$ a CGQIRB-measure $\mu$ which is left $K$-invariant and suppose $\mu(KH) \neq 0$. If $K' < K$ is a compact subgroup of infinite index in $K$, with $H \cap K \leq K'$, then the index of $K'$ in $K$ is uncountable and $\mu(K'H) = 0$. In particular, if $H \cap K$ has infinite index in $K$ then the index of $H \cap K$ in $K$ is uncountable and $\mu(H) = 0$.*

PROOF. By Lemma 3.5, we know $G/H$ admits a CGQIRB-measure $\mu$ which is left $K$-invariant. By the definition of a regular Borel measure $\mu(KH) < \infty$. Suppose the index of $K'$ in $K$ is countable; there exist $\{k_n\}_{n\in\mathbb{N}} \subset K \setminus K'$ with $K = \bigsqcup_{n\in\mathbb{N}} k_n K'$. Then $KH = \bigsqcup_{n\in\mathbb{N}} k_n K'H$. Indeed, if $k_n K'H \cap k_m K'H \neq \emptyset$ for some $n \neq m$ we would have $k_n k' = k_m k'' h$, for some $h \in H$ and some $k', k'' \in K'$; so $h \in H \cap K \leq K'$ and thus $k_n K' = k_m K'$, which is a contradiction. Therefore, write $\mu(KH) = \sum_{n\in\mathbb{N}} \mu(k_n K'H) = \sum_{n\in\mathbb{N}} \mu(K'H)$, as $\mu$ is countably additive and left $K$-invariant. Because $\mu(K'H), \mu(KH) < \infty$ we conclude $\mu(K'H)$ must be zero and so $\mu(KH)$ is zero too, which contradicts the hypothesis. Therefore, the index of $K'$ in $K$ must be uncountable. By the countable additivity of $\mu$, $K$-invariance of $\mu$ and $\mu(KH) \neq 0$, one easily obtains $\mu(K'H) = 0$.

REMARK 3.7. By [2, Theorem B.1.4] let $\mu$ be a CGQIRB-measure on $G/H$ with associated rho-function $\rho$. Let $K$ be a compact subset of $G$. Then, for every $g \in G$,

$$g_\star\mu(KH) = \mu(g^{-1}KH) = \int_{G/H} \mathbf{1}_{KH}(xH)\, dg_\star\mu(xH)$$
$$= \int_{KH} \frac{\rho(gx)}{\rho(x)}\, d\mu(xH) \leq C_g \int_{KH} \mathbf{1}_{KH}(x)\, d\mu(xH) = C_g \cdot \mu(KH),$$

where $C_g \geq \max_{x\in K}\{\rho(gx)/\rho(x)\}$.

LEMMA 3.8. *Let $(\sigma, \mathscr{K})$ be a unitary representation of $H$. Assume the induced unitary representation $(\pi_\sigma, \mathscr{H}_\sigma)$ on $G$ is not $C_0$. Then there exist $\eta_1', \eta_2' \in$ $\operatorname{span}(\{\xi_{f,v} \mid f \in C_c(G), v \in \mathscr{K}\})$, $\delta > 0$ and a sequence $\{t_k\}_{k>0} \subset G$, with $t_k \to \infty$, such that $|\langle \pi_\sigma(t_k)\eta_1', \eta_2'\rangle| > \delta$, for every $k > 0$.*

PROOF. Follows from $\operatorname{span}(\{\xi_{f,v} \mid f \in C_c(G), v \in \mathscr{K}\})$ is dense in $\mathscr{H}_\sigma$.

LEMMA 3.9. *Let $K$ be an open-compact neighborhood in $G$ of the identity. Let $(\sigma, \mathscr{K})$ be a unitary representation of $H$ and $\eta_1, \eta_2 \in$ $\operatorname{span}(\{\xi_{f,v} \mid f \in C_c(G), v \in \mathscr{K}\})$. Consider on $G/H$ a CGQIRB-measure $\mu$, with associated rho-function $\rho$ on $G$.*

*Then there exist a constant $C > 0$, numbers $N_1, N_2 \in \mathbb{N}$ and elements $\{h_i\}_{i\in\{1,\dots,N_1\}}$, $\{h_j'\}_{j\in\{1,\dots,N_2\}} \subset G$, all of them depending only on $\eta_1$ and $\eta_2$,* *such that*

$$
\begin{aligned}
|\langle \pi_\sigma(t)\eta_1, \eta_2\rangle| &= |\langle \eta_1, \pi_\sigma(t^{-1})\eta_2\rangle| \\
&\le \sum_{i,j=1}^{N_1,N_2} \int_{t(h_i KH)\cap h'_j KH} \left| \left(\frac{\rho(t^{-1}x)}{\rho(x)}\right)^{1/2} \langle \eta_1(t^{-1}x), \eta_2(x)\rangle_{\mathscr{K}} \right| d\mu(xH) \\
&\le C \sum_{i,j=1}^{N_1,N_2} \int_{t(h_i KH)\cap h'_j KH} \left(\frac{\rho(t^{-1}x)}{\rho(x)}\right)^{1/2} d\mu(xH)
\end{aligned}
$$

*for every $t \in G$. Moreover, we have*

$$
\begin{aligned}
&\int_{t(h_i KH)\cap h'_j KH} \left| \left(\frac{\rho(t^{-1}x)}{\rho(x)}\right)^{1/2} \langle \eta_1(t^{-1}x), \eta_2(x)\rangle_{\mathscr{K}} \right| d\mu(xH) \\
&\qquad = \int_{h_i KH\cap t^{-1}(h'_j KH)} \left| \left(\frac{\rho(ty)}{\rho(y)}\right)^{1/2} \langle \eta_1(y), \eta_2(ty)\rangle_{\mathscr{K}} \right| d\mu(yH).
\end{aligned}
$$

PROOF. Using Notation 3.4 we simply refer to $(\pi_{\sigma,\mu}, \mathscr{H}_{\sigma,\mu})$ as $(\pi_\sigma, \mathscr{H}_\sigma)$.

Let $t \in G$. As $\eta_1, \eta_2 \in \operatorname{span}(\{\xi_{f,v} \mid f \in C_c(G), v \in \mathscr{K}\})$, they only depend on a finite number of functions from $C_c(G)$. Denote by $A, B \subset G$ the union of the support of those functions defining $\eta_1$, respectively, $\eta_2$. $A$ and $B$ are compact subsets of $G$. Cover $A$, respectively, $B$, with open sets of the form $hK$, where $h \in A$, respectively, $h \in B$. From these open covers extract finite ones covering $A$, respectively, $B$. By making a choice and fixing the notation, consider $A \subset \bigcup_{i=1}^{N_1} h_i K$ and $B \subset \bigcup_{j=1}^{N_2} h'_j K$, where $h_i, h'_j \in G$ and $N_1, N_2 \in \mathbb{N}$. We obtain:

$$
\begin{aligned}
|\langle \pi_\sigma(t)\eta_1, \eta_2\rangle| &= \left| \int_{G/H} \left(\frac{\rho(t^{-1}x)}{\rho(x)}\right)^{1/2} \langle \eta_1(t^{-1}x), \eta_2(x)\rangle_{\mathscr{K}}\, d\mu(xH) \right| \\
&\le \sum_{i,j=1}^{N_1,N_2} \int_{t(h_i KH)\cap h'_j KH} \left| \left(\frac{\rho(t^{-1}x)}{\rho(x)}\right)^{1/2} \langle \eta_1(t^{-1}x), \eta_2(x)\rangle_{\mathscr{K}} \right| d\mu(xH).
\end{aligned}
$$

To obtain the last inequality of the lemma and the constant $C$, we use the following. Recall $\eta_1, \eta_2 \in \operatorname{span}(\{\xi_{f,v} \mid f \in C_c(G), v \in \mathscr{K}\})$. We claim the scalar product $\langle \eta_1(t^{-1}x), \eta_2(x)\rangle_{\mathscr{K}}$ is a bounded function in $x \in G$ and this upper-bound depends neither on $t$ nor on the domains $\{t(h_i KH)\cap h'_j KH\}_{h_i, h'_j}$. Indeed, for simplicity, consider $\eta_1 = \xi_{f_1,v_1}$ and $\eta_2 = \xi_{f_2,v_2}$, where $f_1, f_2 \in C_c(G)$ and $v_1, v_2 \in \mathscr{K}$. In this case we have:

$$
\begin{aligned}
&\left|\langle \xi_{f_1,v_1}(t^{-1}x), \xi_{f_2,v_2}(x)\rangle_{\mathscr{H}}\right| \\
&\quad = \left|\int_H \int_H \langle f_1(t^{-1}xh_1)\sigma(h_1)(v_1),\, f_2(xh_2)\sigma(h_2)(v_2)\rangle_{\mathscr{H}} dh_1 dh_2\right| \\
&\quad\quad \leq \int_H \int_H |f_1(t^{-1}xh_1)| \cdot |f_2(xh_2)| \cdot \|v_1\|_{\mathscr{H}} \cdot \|v_2\|_{\mathscr{H}}\, dh_1 dh_2 < C,
\end{aligned}
$$

where $C$ is a constant which does not depend on $t$, but depends on $\eta_1, \eta_2$. From here the conclusion follows. Note the last assertion of the lemma follows using the change of variables $y := t^{-1}x$ and the positivity of $\rho$.

REMARK 3.10. Lemma 3.9 can be used in the following way. In order to show that induced unitary representations are $C_0$, it is enough to evaluate integrals of the form

$$
\int_{t_n(f_1KH)\cap f_2KH} \left(\frac{\rho(t_n^{-1}x)}{\rho(x)}\right)^{1/2} d\mu(xH),
$$

where $f_1, f_2 \in G$ are considered to be fixed and $t_n \to \infty$.

LEMMA 3.11. *Let $K \leq G$ be open-compact. Let $g, f_1, f_2 \in G$ with $g(f_1KH) \cap f_2KH \neq \emptyset$. Then $g(f_1KH) \cap f_2KH = \bigsqcup_{i\in I} f_2k_iH$, for some $\{k_i\}_{i\in I} \subset K/(K\cap H)$ pairwise different. In addition, for every $i \in I$, there is a unique $k_{k_i} \in K/(K\cap H)$ and a unique $h_i \in H$ with $gf_1k_{k_i} = f_2k_ih_i \in f_2k_iH$.*

PROOF. Let $x \in gf_1KH \cap f_2KH$. Then there exist $k, k' \in K$ and $h, h' \in H$, with $x = gf_1kh = f_2k'h'$; so $xh^{-1} = gf_1k = f_2k'h'h^{-1}$. By taking $k' \in K/(K \cap H)$ we obtain the first part of the lemma. Suppose there are $k, k' \in K/(K \cap H)$ and $h, h' \in H$ with $k \notin k'H$ and $gf_1k = f_2k_ih$, $gf_1k' = f_2k_ih' \in f_2k_iH$. From here we have $k = k'$ and $h = h'$. Note for $i \neq j \in I$, we might have $h_i = h_j$.

LEMMA 3.12. *Let $K \leq G$ be open-compact and $G$ be unimodular. Consider on $G/H$ a CGQIRB-measure $\mu$, with associated rho-function $\rho$ on $G$. Let $g, f_1, f_2 \in G$. Then there is a constant $C > 0$, depending only on $K$, $\rho$ and $f_1, f_2$ with*

$$
\int_{gf_1KH\cap f_2KH} \left(\frac{\rho(g^{-1}x)}{\rho(x)}\right)^{1/2} d\mu(xH) \leq C \int_{\bigsqcup_{i\in I_n} f_2k_iH} \Delta_H(h_i)^{-1/2} d\mu(f_2k_iH),
$$

*where $I$, $k_i$ and $h_i$ are given by Lemma 3.11.*

PROOF. Suppose $gf_1KH \cap f_2KH \neq \emptyset$, as otherwise the conclusion is trivial.

By Lemma 3.11, $gf_1KH \cap f_2KH = \bigsqcup_{i\in I} f_2k_iH$, for some $\{k_i\}_{i\in I} \subset K/(K\cap H)$ pairwise different. Let $x \in gf_1KH \cap f_2KH$. Then, by the same Lemma 3.11, $x = gf_1k_{k_i}h = f_2k_ih_ih$, for some $h \in H$ and some $i \in I$. Therefore,

$$\left(\frac{\rho(g^{-1}x)}{\rho(x)}\right)^{1/2} = \left(\frac{\rho(f_1k_{k_i}h)}{\rho(f_2k_ih_ih)}\right)^{1/2} = \left(\frac{\rho(f_1k_{k_i})\Delta_H(h)}{\rho(f_2k_i)\Delta_H(h_ih)}\right)^{1/2}.$$

As the map $\rho$ is continuous on $G$ and $K$ is compact, there exists a constant $C > 0$ with $0 < (\rho(f_1k)/\rho(f_2k'))^{1/2} \leq C$, for every $k, k' \in K$. We obtain $(\rho(g^{-1}x)/\rho(x))^{1/2} \leq C\Delta_H(h_i)^{-1/2}$, for $x \in f_2k_iH$. The conclusion follows.

Note for $i \neq j \in I$, so for $f_2k_iH$, $f_2k_jH$, one can have $\Delta_H(h_i)^{-1/2} = \Delta_H(h_j)^{-1/2}$. Therefore, the function $\Delta_H(h_i)^{-1/2}$ might be integrated on a bigger subset than $f_2k_iH$, and thus on a subset that might not have measure zero. We summarise below our general strategy to prove induced unitary representations on locally compact groups are $C_0$.

REMARK 3.13 (The strategy: first step). Let $K \leq G$ be open-compact and $G$ be unimodular. Consider on $G/H$ a CGQIRB-measure $\mu$, with associated rho-function $\rho$ on $G$. By Lemma 3.5 and Remark 3.3, $\mu$ and the associated rho-function $\rho$ are both $K$-invariant. From now on consider fixed these $\mu$ and $\rho$. As $K$ is open-compact, $0 \neq \mu(KH) < \infty$. Suppose $\mu(KH) = 1$. Let $(\sigma, \mathscr{K})$ be a unitary representation of $H$ and denote by $(\pi_\sigma, \mathscr{H}_\sigma)$ the induced unitary representation on $G$. Note we have applied Remark 3.3 and Notation 3.4. Assume there exist a sequence $\{t_n\}_{n>0}$ of $G$ and $\eta_1, \eta_2 \in \mathscr{H}_\mu$ with $t_n \to \infty$ and $|\langle \pi_\sigma(t_n)\eta_1, \eta_2\rangle| \nrightarrow 0$, thus the representation $(\pi_\sigma, \mathscr{H}_\sigma)$ is not $C_0$. To the sequence $\{t_n\}_{n>0}$ apply Lemma 3.8 and then Lemma 3.9. By Remark 3.10 it is enough to evaluate the integrals

$$\int_{t_n(f_1KH)\cap f_2KH} \left(\frac{\rho(t_n^{-1}x)}{\rho(x)}\right)^{1/2} d\mu(xH),$$

where $f_1, f_2 \in G$ are fixed and $t_n \to \infty$.

First of all, fix $t_n$. Apply Lemmas 3.11 and 3.12 to $t_n$, $f_1$, $f_2$. One obtains

$$\int_{t_n(f_1KH)\cap f_2KH} \left(\frac{\rho(t_n^{-1}x)}{\rho(x)}\right)^{1/2} d\mu(xH) \leq C \int_{\bigsqcup_{i\in I_n} f_2k_{i,n}H} \Delta_H(h_{i,n})^{-1/2}\, d\mu(f_2k_{i,n}H),$$

where the constant $C > 0$ depends only on $K$, $\rho$, $f_1$ and $f_2$; the set $I_n$ and $k_{i,n}, h_{i,n}$, with $i \in I_n$, depend on $t_n$, $f_1$ and $f_2$. As noticed above, for $i \neq$

$j \in I_n$, so for $f_2k_{i,n}H$, $f_2k_{j,n}H$, one can have $\Delta_H(h_{i,n})^{-1/2} = \Delta_H(h_{j,n})^{-1/2}$. Therefore, the function $\Delta_H(h_{i,n})^{-1/2}$ it might be integrated on a bigger subset than $f_2k_{i,n}H$, and thus on a subset that might not have measure zero. We want to show that integral tends to zero when $n \to \infty$. This would be a contradiction of our assumption $|\langle \pi_\sigma(t_n)\eta_1, \eta_2\rangle| \nrightarrow 0$.

REMARK 3.14. Let $K \leq G$ be open-compact and $G$ be unimodular. Suppose we have been able to evaluate the intersections $gKH \cap KH$, for $g \in G$. It would remain to evaluate the values of $\Delta_H^{-1/2}$. These values strictly depend on the structure of the group $H$. Because of this, we restrict ourself to the case when $G$ is a closed subgroup of $\mathrm{Aut}(\mathcal{T})$ and $H$ is a closed subgroup of $\mathrm{Aut}(\mathcal{T})_\xi$, with $\xi \in \partial\mathcal{T}$. In this case the values of the function $\Delta_H^{-1/2}$ are determined by the hyperbolic elements of $H$, the structure of those being very well understood.

## 4. Vanishing results for the universal group $\mathbb{G}$

In this section we consider parabolically induced unitary representations of the universal group $\mathbb{G}$. Recall by [3] $\mathbb{G}$ is unimodular when $F$ is primitive. We split this study in two parts: when $H \leq \mathbb{G}_\xi$ does not contain hyperbolic elements, and the general case, when $H \leq \mathbb{G}_\xi$ does contain hyperbolic elements. By [7] $\mathbb{G}_\xi^0$ is a closed, non-compact subgroup of $\mathbb{G}$, for every $\xi \in \partial\mathcal{T}$.

### *4.1. The non-hyperbolic case*

REMARK 4.1. Let $\xi \in \partial\mathcal{T}$. If $H \leq \mathrm{Aut}(\mathcal{T})_\xi$ is a closed subgroup not containing hyperbolic elements then $H$ is unimodular. This is because $H$ can be written as a countable union of compact subgroups. Indeed, by [9] as $H$ contains only elliptic elements, each element of $H$ fixes pointwise an infinite geodesic ray of $\mathcal{T}$ with endpoint $\xi$. Thus every element of $H$ is contained in some $H_x$ for some vertex $x$ of $\mathcal{T}$ and $H_x$ is compact (whence unimodular).

LEMMA 4.2. *Let $x \in \mathcal{T}$ and $\xi \in \partial\mathcal{T}$. Let $K \leq \mathrm{Aut}(\mathcal{T})_x$ be closed and let $H$ be a closed, non-compact subgroup of* $\mathrm{Aut}(\mathcal{T})_\xi$*, not containing hyperbolic elements. Let $g \in \mathrm{Aut}(\mathcal{T})^+$. If $gKH \cap KH \neq \emptyset$, then there exists $k_g \in K$ with $gKH \cap KH \subset k_g K_{[x,x_g]}H$, where $x_g \in [x,\xi)$ has the properties* $\mathrm{d}_{\mathcal{T}}(x,x_g) = \frac{1}{2}\mathrm{d}_{\mathcal{T}}(x,g(x))$ *and $k_g$ sends $[x,x_g]$ into the first half of the geodesic segment $[x,g(x)]$.*

PROOF. From $gKH \cap KH \neq \emptyset$, $g = k'hk$, for some $h \in H$ and $k', k \in K$. We want to determine the domain in $K$ of the variable $k'$. From $g = k'hk$ we have:

$$\mathrm{d}_{\mathcal{T}}(x, g(x)) = \mathrm{d}_{\mathcal{T}}(x, k'h(x)) = \mathrm{d}_{\mathcal{T}}(x, h(x)). \quad (2)$$

As $h$ is not hyperbolic, denote by $x_h$ the first vertex of the geodesic ray $[x,\xi)$ fixed by $h$. From equation (2) we obtain $\mathrm{d}_{\mathscr{T}}(x,x_h)=\frac{1}{2}\mathrm{d}_{\mathscr{T}}(x,g(x))$, thus $x_h$ is a precise point on the geodesic ray $[x,\xi)$ determined only by the element $g$ and not by the non-hyperbolic element $h$. So take $x_g:=x_h$. Because $k'([x,h(x)])=[x,g(x)]$, $k'$ sends the geodesic segment $[x,x_g]$ into the first half of the geodesic segment $[x,g(x)]$. We conclude $k'\in k_g K_{[x,x_g]}$, where $k_g\in K$ is a fixed element sending $[x,x_g]$ into the first half of the geodesic segment $[x,g(x)]$.

THEOREM 4.3. *Let $\xi\in\partial\mathscr{T}$, $x\in\mathscr{T}$, $G$ be a closed, non-compact, unimodular subgroup of* $\mathrm{Aut}(\mathscr{T})^+$ *and suppose the index in $K:=G_x$ of $G_{[x,\xi]}$ is infinite. Let $H$ be a closed, non-compact subgroup of $G_\xi$, not containing hyperbolic elements and let $(\sigma,\mathscr{K})$ be a unitary representation of $H$. Then the induced unitary representation $(\pi_\sigma,\mathscr{H}_\sigma)$ on $G$ is $C_0$.*

PROOF. By Remark 3.3 and because $H$ and $G$ are unimodular, it is enough to consider the case when the rho-function $\rho$ is the constant function $\mathbf{1}$ on $G$. Thus, the measure $\mu$ on $G/H$ associated with the rho-function $\mathbf{1}$ on $G$ is $G$-invariant. As $K$ is open and compact with respect to the locally compact topology on $G$, we have $0\neq\mu(KH)<\infty$. Assume there exist a sequence $\{t_n\}_{n>0}$ of $G$ and $\eta_1,\eta_2\in\mathscr{H}_\sigma$ with $t_n\to\infty$ and $|\langle\pi_\sigma(t_n)\eta_1,\eta_2\rangle|\nrightarrow 0$. To the sequence $\{t_n\}_{n>0}$ apply Lemma 3.8 and then Lemma 3.9. Moreover, by Remark 3.10 it is enough to evaluate $\mu(t_n(h_iKH)\cap h'_jKH)$, where $h_i$ and $h'_j$ are considered to be fixed and $t_n\to\infty$. Note $\mu(t_n(h_iKH)\cap h'_jKH)=\mu((h'_j)^{-1}t_nh_iKH\cap KH)$.

If $(h'_j)^{-1}t_nh_iKH\cap KH\neq\emptyset$ apply Lemma 4.2 to $g_n:=(h'_j)^{-1}t_nh_i$. We obtain $g_nKH\cap KH\subset k_{g_n}G_{[x,x_{g_n}]}H$, where $x_{g_n}\in[x,\xi)$ with one of the properties being $\mathrm{d}_{\mathscr{T}}(x,x_{g_n})=\frac{1}{2}\mathrm{d}_{\mathscr{T}}(x,g_n(x))$. As $t_n\to\infty$, we also have $g_n\to\infty$ ($h_i,h'_j$ being fixed); in addition, $\mathrm{d}_{\mathscr{T}}(x,x_{g_n})\to\infty$ when $n\to\infty$. To evaluate $\mu(g_nKH\cap KH)$ it is enough to compute $\mu(k_{g_n}G_{[x,x_{g_n}]}H)=\mu(G_{[x,x_{g_n}]}H)$, where $\mathrm{d}_{\mathscr{T}}(x,x_{g_n})\to\infty$ as $g_n\to\infty$. We claim $\lim_{g_n\to\infty}\mu(G_{[x,x_{g_n}]}H)=0$, giving a contradiction. Indeed, there are two cases that should be considered: either for every $y\in(x,\xi)$ the index of $G_{[x,y]}$ in $K$ is finite or there exists $y\in(x,\xi)$ with the index of $G_{[x,y]}$ in $K$ is infinite. Consider the first case; so the index in $K$ of $G_{[x,x_{g_n}]}$ is finite for every $g_n$. Moreover, since $[K:G_{[x,\xi]}]=\infty$, $[K:G_{[x,x_{g_n}]}]\to\infty$ as $g_n\to\infty$. As $\mu(KH)<\infty$, $\mu$ is $G$-invariant, and so $K$-invariant, the claim follows. Consider the second case; so there exists $N>0$ such that for every $n\geq N$ we have the index of $G_{[x,x_{g_n}]}$ in $K$ is infinite. By Lemma 3.6 applied to $K'=G_{[x,x_{g_n}]}$, we have $\mu(G_{[x,x_{g_n}]}H)=0$, for every $n\geq N$. The theorem follows.

COROLLARY 4.4. *Let $F$ be primitive and let $\xi\in\partial\mathscr{T}$. Let $H$ be a closed, non-compact subgroup of $\mathbb{G}_\xi$, not containing hyperbolic elements and let $(\sigma,\mathscr{K})$*

*be a unitary representation of $H$. Then the induced unitary representation $(\pi_\sigma, \mathscr{H}_\sigma)$ on $\mathbb{G}$ is $C_0$.*

PROOF. The hypotheses of Theorem 4.3 are fulfilled: let $K := \mathbb{G}_x$ and by Lemma 2.3 applied to $\mathbb{G}_\xi^0$ we have $[K : K_{[x,\xi]}] = \infty$.

### *4.2. The hyperbolic case*

Let $\xi \in \partial\mathscr{T}$. In this subsection we consider $H$ a closed subgroup of $\mathrm{Aut}(\mathscr{T})_\xi$ containing hyperbolic elements. This implies $H$ is not compact.

#### *4.2.1. Structure and modular function of parabolic subgroups.*

LEMMA 4.5. *Let $\xi \in \partial\mathscr{T}$ and $H \leq \mathrm{Aut}(\mathscr{T})_\xi$ be a closed subgroup containing hyperbolic elements. Then there exists a hyperbolic element $\gamma \in H$, of attracting endpoint $\xi$, that is minimal, in the sense any other hyperbolic element $\gamma' \in H$ is written $\gamma' = \gamma^n h$, where $n \in \mathbb{Z}$, $|n||\gamma| = |\gamma'|$ and $h \in H \cap \mathrm{Aut}(T)_\xi^0$.*

PROOF. Let $\mathrm{Hyp}(H) := \{\gamma \in H \mid \gamma \text{ is hyperbolic}\}$. Let $\mathrm{hyp}_H := \min_{\gamma\in H}(|\gamma|)$. Note $\mathrm{hyp}_H$ exists and $\mathrm{hyp}_H \geq 1$. Let fix $\gamma \in H$ with $|\gamma| = \mathrm{hyp}_H$. Fix also a vertex $x$ in $\mathrm{Min}(\gamma)$. Moreover, consider the attracting endpoint of $\gamma$ is $\xi$; if not take $\gamma^{-1}$. Let $\gamma' \in \mathrm{Hyp}(H)$ and let $x_{\gamma'}$ be the first vertex of $[x, \xi)$ contained in $\mathrm{Min}(\gamma')$. By minimality $|\gamma'|$ is a multiple of $|\gamma|$. If the attracting endpoint of $\gamma'$ is $\xi$, then $\gamma^{-n}\gamma'(x_\gamma) = x_{\gamma'}$, where $n|\gamma| = |\gamma'|$. Thus, $\gamma' = \gamma^n h$, where $h \in H_{x_{\gamma'}}$. If $\gamma'$ has $\xi$ as a repelling endpoint, then $\gamma^n\gamma'((\gamma')^{-1}(x_{\gamma'})) = (\gamma')^{-1}(x_{\gamma'})$, where $n|\gamma| = |\gamma'|$. Thus $\gamma' = \gamma^{-n}h$, where now $h$ is in $H_{(\gamma')^{-1}(x_{\gamma'})}$.

LEMMA 4.6. *Let $\xi \in \partial\mathscr{T}$ and $H \leq \mathrm{Aut}(\mathscr{T})_\xi$ be a closed subgroup containing hyperbolic elements. Let $\gamma$ be a hyperbolic element of $H$ with attracting endpoint $\xi$ and let $x$ be a vertex of $\mathrm{Min}(\gamma)$. Then $1/(d-1)^{|\gamma|} \leq \Delta_H(\gamma) = 1/[H_{\gamma(x)} : H_x] \leq 1$. In particular, $H$ is unimodular if and only if $H_x = H_y$, for every $y \in \mathrm{Min}(\gamma)$.*

PROOF. By Remark 4.1, for every $h \in H \cap \mathrm{Aut}(\mathscr{T})_\xi^0$, $\Delta_H(h) = 1$. Note the following facts. Firstly, $H_x = H_{[x,\xi]} \leq H_{\gamma(x)}$ are compact subgroups and secondly, the index $[H_{\gamma(x)} : H_x] \leq (d-1)^{|\gamma|}$, where $d$ is the regularity of the tree $\mathscr{T}$. Moreover, $H_{\gamma(x)} = \gamma H_x \gamma^{-1}$. Let $dh$ denote the left Haar measure on $H$. Then $dh(H_{\gamma(x)}) = dh(\gamma H_x \gamma^{-1}) = dh(H_x\gamma^{-1}) = \Delta_H(\gamma^{-1})dh(H_x)$. As $\Delta_H(h) = 1$ for $h \in H \cap \mathrm{Aut}(\mathscr{T})_\xi^0$, we have $dh(H_{\gamma(x)}) = dh(H_x)\cdot[H_{\gamma(x)} : H_x]$. From the above two equalities we obtain $1 \leq \Delta_H(\gamma^{-1}) = [H_{\gamma(x)} : H_x] \leq (d-1)^{|\gamma|}$. In particular, $1 = \Delta_H(\gamma^{-1}) = [H_{\gamma(x)} : H_x]$ if and only if $H_x = H_y$, for every $y \in \mathrm{Min}(\gamma)$; thus $H_y = H_{\xi_-}$ for every $y \in \mathrm{Min}(\gamma)$, where $\xi_-$ is the repelling endpoint of $\gamma$.

LEMMA 4.7. *Let $\xi \in \partial\mathcal{T}$ and $H \leq \mathrm{Aut}(\mathcal{T})_\xi$ be a closed subgroup containing hyperbolic elements. Let $\gamma$ be a hyperbolic element of $H$ with attracting endpoint $\xi$ and $x$ be a vertex of $\mathrm{Min}(\gamma)$. Then we have the following properties:*

(1) $[H_{\gamma(x)} : H_x] = [H_{\gamma^2(x)} : H_{\gamma(x)}]$;

(2) $[H_{\gamma^n(x)} : H_x] = [H_{\gamma^{n-m}(x)} : H_x] \cdot [H_{\gamma^m(x)} : H_x]$ *for every* $0 \leq m \leq n$;

(3) $[H_{\gamma^m(x)} : H_x] \leq [H_{\gamma^n(x)} : H_x]$ *for every* $0 \leq m \leq n$.

PROOF. Note assertion (3) is a consequence of assertion (2) and the latter one follows from assertion (1) and from $H_x \leq H_{\gamma^m(x)} \leq H_{\gamma^n(x)}$, for $0 \leq m \leq n$. The first assertion follows as $H_{\gamma(x)} = \gamma H_x \gamma^{-1}$ and $H_{\gamma^2(x)} = \gamma H_{\gamma(x)} \gamma^{-1}$. Moreover, for every coset $hH_x$ of $H_{\gamma(x)}/H_x$ we have $\gamma h \gamma^{-1} \gamma H_x \gamma^{-1}$ is a coset of $H_{\gamma^2(x)}/H_{\gamma(x)}$ and vice versa. The lemma is proved.

For $F \leq \mathrm{Sym}\{1, \dots, d\}$ and $e$ an edge of $\mathcal{T}$ we abuse notation and use $F_e$ to denote the stabiliser in $F$ of the colour from $\{1, \dots, d\}$ of the edge $e$.

LEMMA 4.8. *Let $F$ be transitive and let $\gamma \in \mathbb{G}$ be hyperbolic. Denote by $\xi_+, \xi_- \in \partial\mathcal{T}$ the attracting and respectively, the repelling endpoints of $\gamma$. Take $x \in (\xi_-, \xi_+)$, the edges $e_-, e_+$ in the star of $x$ with $e_+ \in [x, \xi_+)$, $e_- \in (\xi_-, x]$ and $K := \mathbb{G}_x$. Then we have:*

(1) $[\mathbb{G}_{[\gamma(x),\xi_+]} : \mathbb{G}_{[x,\xi_+]}] = \Delta_{\mathbb{G}_{\xi_+}}(\gamma^{-1}) = [K : \mathbb{G}_{[x,\gamma^{-1}(x)]}] \cdot k_1/d$, *where $d$ is the regularity of $\mathcal{T}$ and $k_1$ is the number of orbits of the edge $e_-$ in $\{1, \dots, d\}$ under the stabilizer subgroup $F_{e_+} \leq F$;*

(2) $[\mathbb{G}_{[x,\xi_-]} : \mathbb{G}_{[\gamma(x),\xi_-]}] = \Delta_{\mathbb{G}_{\xi_-}}(\gamma) = [K : \mathbb{G}_{[x,\gamma(x)]}] \cdot k_2/d$ *where $k_2$ is the number of orbits of the edge $e_+$ in $\{1, \dots, d\}$ under the stabilizer subgroup $F_{e_-} \leq F$;*

(3) $[K : \mathbb{G}_{[\gamma^{-1}(x),x]}] = [K : \mathbb{G}_{[x,\gamma(x)]}] = [\mathbb{G}_{[\gamma(x),\xi_+]} : \mathbb{G}_{[x,\xi_+]}] \cdot d/k_1[4] = [\mathbb{G}_{[x,\xi_-]} : \mathbb{G}_{[\gamma(x),\xi_-]}] \cdot d/k_2$.

PROOF. First, it is easy to see that

$$\gamma \mathbb{G}_{[\gamma^{-1}(x),x]} \gamma^{-1} = \mathbb{G}_{\gamma([\gamma^{-1}(x),x])} = \mathbb{G}_{[x,\gamma(x)]}.$$

Let $m$ be the left Haar measure on $\mathbb{G}$. Then we have

$$m(K) = m(\mathbb{G}_{[x,\gamma(x)]}) \cdot [K : \mathbb{G}_{[x,\gamma(x)]}] = m(\mathbb{G}_{[\gamma^{-1}(x),x]}) \cdot [K : \mathbb{G}_{[\gamma^{-1}(x),x]}].$$

By a standard computation we have $m(\mathbb{G}_{[x,\gamma(x)]}) = \Delta_{\mathbb{G}}(\gamma^{-1}) m(\mathbb{G}_{[\gamma^{-1}(x),x]})$. As $\mathbb{G}$ is unimodular we obtain

$$\Delta_{\mathbb{G}}(\gamma^{-1}) \cdot [K : \mathbb{G}_{[x,\gamma(x)]}] = [K : \mathbb{G}_{[x,\gamma(x)]}] = [K : \mathbb{G}_{[\gamma^{-1}(x),x]}]. \qquad (3)$$

Let us prove assertion (1) of the lemma. First, by Lemma 4.6 applied to $\mathbb{G}_{\xi_+}$ we have $[\mathbb{G}_{[\gamma(x),\xi_+]} : \mathbb{G}_{[x,\xi_+]}] = \Delta_{\mathbb{G}_{\xi_+}}(\gamma^{-1}) = [\mathbb{G}_{[x,\xi_+]} : \mathbb{G}_{[\gamma^{-1}(x),\xi_+]}]$. As

$\gamma$ is translating along the axis $(\xi_-, \xi_+)$, $F_{\gamma(e_+)}$ is isomorphic to $F_{e_+}$, thus the number of $F_{\gamma(e_+)}$-orbits of $\gamma(e_-)$ in $\{1, \dots, d\}$ is the same as the $F_{e_+}$-orbits of $e_-$ in $\{1, \dots, d\}$, which is $k_1$. As $F$ is transitive on $\{1, \dots, d\}$, we have $[K : \mathbb{G}_{[\gamma^{-1}(x),x]}] = d \cdot [\mathbb{G}_{e_-} : \mathbb{G}_{[\gamma^{-1}(x),x]}]$. Also as $\mathbb{G}$ has Tits' independence property [3], [1], $[\mathbb{G}_{[x,\xi_+]} : \mathbb{G}_{[\gamma^{-1}(x),\xi_+]}] = k_1 \cdot [\mathbb{G}_{e_-} : \mathbb{G}_{[\gamma^{-1}(x),x]}]$. We conclude indeed

$$[\mathbb{G}_{[\gamma(x),\xi_+]} : \mathbb{G}_{[x,\xi_+]}] = \frac{[K : \mathbb{G}_{[x,\gamma^{-1}(x)]}] \cdot k_1}{d}$$

and part (1) of the lemma is proved. The assertion (2) of the lemma goes in the same way. The assertion (3) of the lemma is a consequence of assertions (1), (2) and relation (3).

*4.2.2. The evaluation of $gKH \cap KH$.* By Remark 3.13, the next step is the evaluation of $gKH \cap KH$. This is because we need to integrate the modular function $\Delta_H^{-1/2}$ on the intersection $gf_1KH \cap f_2KH = \bigsqcup_{i \in I} f_2k_iH$, for $g, f_1, f_2 \in G$. We are able to evaluate $gKH \cap KH$ for the universal group $\mathbb{G}$ and not in a more general case. This is due to the $KA^+K$ decomposition of $\mathbb{G}$ proven in Lemma 2.2, making our task easier. That decomposition might not hold in a more general situation. Using the $KA^+K$ decomposition, we only evaluate $gKH \cap KH$ when $g \in A^+$. This is given by the next technical proposition. We state the proposition as generally as possible, making use of the following general definition.

DEFINITION 4.9. Let $G$ be a closed subgroup of $\mathrm{Aut}(\mathcal{T})^+$, $x \in \mathcal{T}$ and $e$ be an edge of the star of $x$. Set $K := G_x$ and define

$$A^+ := \{\gamma \in G \mid e \subset \mathrm{Min}(\gamma), \gamma(e) \subset \mathcal{T}_{x,e}\} \cup \{\mathrm{id}\}.$$

Let $\xi$ be an endpoint in $\partial\mathcal{T}_{x,e}$. Define the map $\mathrm{proj}_{(x,\xi]}\colon A^+ \to (x, \xi]$ by $\mathrm{proj}_{(x,\xi]}(g)$ is the vertex or the endpoint $\xi$ with the property $[x, \xi_{g,+}] \cap [x, \xi] = [x, \mathrm{proj}_{(x,\xi]}(g)]$, where $\xi_{g,+}$ is the attracting endpoint of $g$. As $g \in A^+$, note $\mathrm{proj}_{(x,\xi]}(g)$ is indeed a point in $(x, \xi]$. Let now $g \in G$ be a hyperbolic element translating the vertex $x$. Consider its $K$-double coset $KgK$ and set $\mathrm{proj}_{(x,\xi]}(KgK) := \max_{g' \in A^+ \cap KgK}\{\mathrm{proj}_{(x,\xi]}(g')\}$.

PROPOSITION 4.10. *Let $G$ be a closed subgroup of $\mathrm{Aut}(\mathcal{T})^+$ and let $\xi \in \partial\mathcal{T}$. Assume $G_\xi$ contains hyperbolic elements. Let $H < G_\xi$ be a closed subgroup containing also hyperbolic elements. Let $\gamma$ be a minimal hyperbolic element of $H$ given by Lemma 4.5, with attracting endpoint $\xi$, and let $x$ be a vertex of $\mathrm{Min}(\gamma)$. Set $K := G_x$. Choose the edge $e$ in the star of $x$ and define $A^+$ such that $\gamma \in A^+$.*

*Let $g \in A^+$. Assume $\mathrm{proj}_{(x,\xi]}(KgK) = \mathrm{proj}_{(x,\xi]}(g)$. Assume there also exist $k_2 \in K \setminus \{H \cap K\}$, $k_1 \in K$ and $h \in H$ with $k_1gk_2 = h = \gamma^n h_0$, where $h_0 \in H \cap G_\xi^0$ and $n \in \mathbb{Z}$.*

*Then* $0 \leq |n| \leq \mathrm{d}_{\mathscr{T}}(x, g(x))/|\gamma|$ *and* $k_1 \in G_{[x,x_h]}$, *where* $x_h \in [x, \mathrm{proj}_{(x,\xi]}(g)]$ *is with* $\mathrm{d}_{\mathscr{T}}(x, x_h) = \frac{1}{2}(\mathrm{d}_{\mathscr{T}}(x, g(x)) + \mathrm{sign}(n)|\gamma^n|)$, *where* $\mathrm{sign}(0) = 0$.

PROOF. Denote by $e$ the edge of the star of $x$ with $\xi \in \partial\mathscr{T}_{x,e}$. In particular, $A^+$ is defined using $x$ and $e$. Let $\xi_+$ and $\xi_-$ be the attracting and the repelling endpoints of $g$. As $k_2$ is not fixing $\xi$, we denote $x_{k_2}$ the vertex of the geodesic line $(\xi_-, \xi)$ with the property $[x, k_2(\xi)) \cap (\xi_-, \xi) = [x, x_{k_2}]$. We have three cases: either $x_{k_2} \in [x, \xi)$ or $x_{k_2} \in (x, g^{-1}(x))$ or $x_{k_2} \in [g^{-1}(x), \xi_-)$.

Suppose $x_{k_2} \in [x, \xi)$. Because $k_1 g k_2(\xi) = \xi$, $k_1 g k_2(e)$ is an edge of $\mathscr{T}_{x,e}$ and the orientation of $k_1 g k_2(e)$ induced from $e$ points towards the boundary $\partial\mathscr{T}_{x,e}$, like $e$. Therefore, $k_1 g k_2 \in A^+$. As $k_1 g k_2 \in H$, we have $h = k_1 g k_2 \in A^+ \cap H$. As by hypothesis $\mathrm{proj}_{(x,\xi]}(KgK) = \mathrm{proj}_{(x,\xi]}(g)$, we conclude $g \in A^+ \cap H$. In addition, by Lemma 4.5 we have $h = \gamma^n h_0$, where $h_0 \in K \cap H$; thus $|g| = |h| = n|\gamma|$. As $k_1 g k_2(\xi) = \xi$ and because $g$ is hyperbolic, with attracting endpoint $\xi$ and with $x \in \mathrm{Min}(g)$, $k_1$ must fix at least the vertex $g(x) \in (x, \xi)$. Therefore, $k_1 \in G_{[x,x_h]}$, where $x_h = g(x)$ and the conclusion follows.

Suppose $x_{k_2} \in [g^{-1}(x), \xi_-)$. Then $g k_2(e)$ is an edge of $\mathscr{T}_{x,e}$ and the orientation of $g k_2(e)$ induced from $e$ points outwards the boundary $\partial T_{x,e}$, thus towards $e$. Because $x_{k_2} \in [g^{-1}(x), \xi_-)$, by applying $k_1$ to $g k_2$, $k_1(\mathscr{T}_{x,e}) \cap \mathscr{T}_{x,e} = \{x\}$ and the edge $k_1 g k_2(e)$ points towards the edge $e$. Therefore $k_1 g k_2$ must be a hyperbolic element (of $H$) translating the vertex $x$ outwards the half-tree $\mathscr{T}_{x,e}$. Consequently, $\xi$ is the repelling endpoint of $k_1 g k_2$, as $k_1 g k_2(\xi) = \xi$. Otherwise saying, $\xi$ is the attracting endpoint of the hyperbolic element $(k_1 g k_2)^{-1} = h^{-1} \in H$ and $x \in \mathrm{Min}(h^{-1})$. We have $|h| = |h^{-1}| = \mathrm{d}_{\mathscr{T}}(x, g(x)) = |n||\gamma|$. Although we can say more, we do not impose any restriction for $k_1$, so $k_1 \in G_{[x,x_h]}$ where $x_h = x$. The conclusion of the proposition is still valid in this case.

Suppose now $x_{k_2} \in (g^{-1}(x), x)$. We claim $g(x_{k_2}) \in [x, \mathrm{proj}_{(x,\xi]}(g)]$. Indeed, supposing the contrary we have $\mathrm{proj}_{(x,\xi]}(g) \in (x, g(x_{k_2}))$. Then $g(x_{k_2}) \notin [x, \xi)$. As the geodesic ray $[x_{k_2}, k_2(\xi))$ is sent by $g$ into the geodesic ray $[g(x_{k_2}), g k_2(\xi))$, $[g(x_{k_2}), g k_2(\xi))$ does not intersect $[x, \xi)$. However, by applying $k_1$, we must have $k_1 g(x_{k_2}) \in [x, \xi)$, as $k_1 g k_2(\xi) = \xi$. This is a contradiction with $\mathrm{proj}_{(x,\xi]}(KgK) = \mathrm{proj}_{(x,\xi]}(g)$ and the claim follows. As $k_1 g k_2(\xi) = \xi$, from the latter claim we immediately have $k_1 \in G_{[x,g(x_{k_2})]}$. From here we deduce the following two facts:

(1) the segment $[x, k_2^{-1}(x_{k_2}))$, where $k_2^{-1}(x_{k_2}) \in (x, \xi)$, is sent by $h = k_1 g k_2$ into the segment $(g(x_{k_2}), k_1 g(x)] \subset \mathscr{T}_{x,e} \setminus \{[x, \xi)\}$, and the orientation is reversed;
(2) the edge $k_1 g k_2(e)$ belongs to $\mathscr{T}_{x,e}$ and the orientation of $k_1 g k_2(e)$ induced from $e$ would point outwards the boundary $\partial T_{x,e}$, thus towards $e$. Therefore, either $k_1 g k_2$ is elliptic, or $k_1 g k_2$ is hyperbolic in $H$, with translation length strictly smaller than $\mathrm{d}_{\mathscr{T}}(x, g(x))$.

Our next claim is $h$ is elliptic if and only if $\mathrm{d}_{\mathscr{T}}(x, x_{k_2}) = \frac{1}{2}\mathrm{d}_{\mathscr{T}}(x, g^{-1}(x))$. Suppose $h = k_1 g k_2$ is elliptic. Then by the above fact (1) we know the segment $h([x, k_2^{-1}(x_{k_2})))$ does not intersect $[x, \xi)$. As $h \in H$ is elliptic, $h$ fixes the midpoint of the segment $[k_2^{-1}(x_{k_2}), h(k_2^{-1}(x_{k_2}))] = [k_2^{-1}(x_{k_2}), g(x_{k_2})]$. We deduce $k_2^{-1}(x_{k_2}) = h(k_2^{-1}(x_{k_2})) = g(x_{k_2})$, from where $\mathrm{d}_{\mathscr{T}}(x, x_{k_2}) = \frac{1}{2}\mathrm{d}_{\mathscr{T}}(x, g^{-1}(x))$. Suppose now $\mathrm{d}_{\mathscr{T}}(x, x_{k_2}) = \frac{1}{2}\mathrm{d}_{\mathscr{T}}(x, g^{-1}(x))$, so we need to prove $h$ is elliptic. Indeed, $k_2^{-1}(x_{k_2}) = g(x_{k_2})$. As $k_1 \in G_{[x,g(x_{k_2})]}$, we conclude $h(k_2^{-1}(x_{k_2})) = k_1 g k_2(k_2^{-1}(x_{k_2})) = g(x_{k_2}) = k_2^{-1}(x_{k_2})$, so $h$ is elliptic. The equivalence follows.

For $h$ elliptic, we resume the following: $k_1 g k_2 \in H \cap G_\xi^0$, so $n = 0$, and $k_1 \in G_{[x,x_h]}$, where $x_h := g(x_{k_2}) \in [x, \mathrm{proj}_{(x,\xi]}(g)]$, with $\mathrm{d}_{\mathscr{T}}(x, g(x_{k_2})) = \frac{1}{2}\mathrm{d}_{\mathscr{T}}(x, g(x))$. If $h = k_1 g k_2$ is hyperbolic, then $\mathrm{d}_{\mathscr{T}}(x, x_{k_2}) \neq \frac{1}{2}\mathrm{d}_{\mathscr{T}}(x, g^{-1}(x))$.

Suppose $\mathrm{d}_{\mathscr{T}}(x, x_{k_2}) < \frac{1}{2}\mathrm{d}_{\mathscr{T}}(x, g^{-1}(x))$, this implies $\frac{1}{2}\mathrm{d}_{\mathscr{T}}(x, g(x)) < \mathrm{d}_{\mathscr{T}}(x, g(x_{k_2}))$. Moreover, using the above fact (1) and $h$ is hyperbolic fixing $\xi$, we conclude $\xi$ is the attracting endpoint of $h$ and $h$ translates the vertex $k_2^{-1}(x_{k_2}) \in (x, \xi)$ to $k_1 g(x_{k_2}) = g(x_{k_2}) \in (x, \xi)$. By Lemma 4.5, $h = \gamma^n h_0$, for some $h_0 \in H \cap G_\xi^0$, and $n$ is such that $n|\gamma| = |h| = \mathrm{d}_{\mathscr{T}}(x, g(x)) - 2\,\mathrm{d}_{\mathscr{T}}(x, k_2^{-1}(x_{k_2})) < \mathrm{d}_{\mathscr{T}}(x, g(x))$. In addition, $k_1 \in G_{[x,x_h]}$, where $x_h = g(x_{k_2}) \in [x, \mathrm{proj}_{(x,\xi]}(g)]$ and indeed $\mathrm{d}_{\mathscr{T}}(x, x_h) = \frac{1}{2}(\mathrm{d}_{\mathscr{T}}(x, g(x)) - |\gamma^n|) + |\gamma^n|$.

Suppose now $\mathrm{d}_{\mathscr{T}}(x, x_{k_2}) > \frac{1}{2}\mathrm{d}_{\mathscr{T}}(x, g^{-1}(x))$, this implies $\frac{1}{2}\mathrm{d}_{\mathscr{T}}(x, g(x)) > \mathrm{d}_{\mathscr{T}}(x, g(x_{k_2}))$. As before, using the above fact (1) and $h$ is hyperbolic fixing $\xi$, we conclude $\xi$ must be the repelling endpoint of $h$ and $h^{-1}$ translates the vertex $g(x_{k_2}) \in \mathrm{Min}(h) \cap (x, \xi)$ to $k_2^{-1}(x_{k_2}) \in (x, \xi)$. By Lemma 4.5, we have that $h = \gamma^{-n} h_0$, for some $h_0 \in H \cap G_\xi^0$, and $n > 0$ is such that $n|\gamma| = |h| = \mathrm{d}_{\mathscr{T}}(x, g(x)) - 2\,\mathrm{d}_{\mathscr{T}}(x, g(x_{k_2})) < \mathrm{d}_{\mathscr{T}}(x, g(x))$. In addition, $k_1 \in G_{[x,x_h]}$, where $x_h = g(x_{k_2}) \in [x, \mathrm{proj}_{(x,\xi]}(g)]$ and indeed $\mathrm{d}_{\mathscr{T}}(x, x_h) = \frac{1}{2}(\mathrm{d}_{\mathscr{T}}(x, g(x)) - |\gamma^{-n}|)$. The proposition is proven.

When $H$ is unimodular, we obtain the following.

COROLLARY 4.11. *Let $G$ be a closed subgroup of* $\mathrm{Aut}(\mathscr{T})^+$ *and let $\xi \in \partial\mathscr{T}$. Assume $G_\xi$ contains hyperbolic elements. Let $H < G_\xi$ be a closed, unimodular, subgroup containing also hyperbolic elements. Let $\gamma$ be a minimal hyperbolic element of $H$ given by Lemma 4.5, with attracting endpoint $\xi$, and let $x$ be a vertex of* $\mathrm{Min}(\gamma)$. *Set $K := G_x$. Choose the edge $e$ in the star of $x$ such that $\gamma \in A^+$.*

*Let $g \in A^+$. Assume* $\mathrm{proj}_{(x,\xi]}(KgK) = \mathrm{proj}_{(x,\xi]}(g)$. *Assume there also exist $k_2 \in K \setminus \{H \cap K\}$, $k_1 \in K$ and $h \in H$ with $k_1 g k_2 = h = \gamma^n h_0$, where $h_0 \in H \cap G_\xi^0$ and $n \in \mathbb{Z}$. Then $h_0 \in K \cap H$ and $|n| = \mathrm{d}_{\mathscr{T}}(x, g(x))/|\gamma|$. If $n > 0$ then $k_1 \in G_{[x,x_h]}$, where $x_h \in [x, \xi]$ is with $\mathrm{d}_{\mathscr{T}}(x, x_h) = \mathrm{d}_{\mathscr{T}}(x, g(x))$. If $n < 0$ then $k_1^{-1} \in kG_{[x,\gamma^n(x)]}$, where $k \in K$ with $[x, g(x)] = k([x, \gamma^n(x)])$.*

Proof. We keep all the notation from the proof of Proposition 4.10. As $H$ is unimodular, by Lemma 4.6 we have $H_x = H_y = H_{\xi_-}$, for every $y \in \mathrm{Min}(\gamma)$, where $\xi_- \in \partial\mathcal{T}$ is the repelling endpoint of $\gamma$. This proves $h_0 \in K \cap H$ and so $\mathrm{d}_{\mathcal{T}}(x, h(x)) = \mathrm{d}_{\mathcal{T}}(x, g(x)) = \mathrm{d}_{\mathcal{T}}(x, \gamma^n(x)) = |n||\gamma|$. If $n > 0$, by applying Proposition 4.10, we directly obtain $k_1 \in G_{[x,x_h]}$, where $x_h \in [x, \mathrm{proj}_{(x,\xi]}(g)]$ is with $\mathrm{d}_{\mathcal{T}}(x, x_h) = \mathrm{d}_{\mathcal{T}}(x, g(x))$. It remains the case $n < 0$. By the proof of Proposition 4.10, the case $n < 0$ with $\mathrm{d}_{\mathcal{T}}(x, g(x)) = |n||\gamma|$ can occur only when $x_{k_2} \in [g^{-1}(x), \xi_-)$. Let us compute $g(\xi) = (k_1)^{-1}\gamma^n h_0(k_2)^{-1}(\xi)$. As $x_{k_2} \in [g^{-1}(x), \xi_-)$, we have $(k_2)^{-1}(\xi) \notin \partial\mathcal{T}_{x,e}$. Then $h_0(k_2)^{-1}(\xi)$ is still a point in $\{\partial\mathcal{T} \setminus \partial\mathcal{T}_{x,e}\}$ as $h_0 \in H_{[x,\xi]}$. By applying $\gamma^n$ to $h_0(k_2)^{-1}(\xi)$ and because $n < 0$ we have $\gamma^n h_0(k_2)^{-1}(\xi)$ in $\{\partial\mathcal{T} \setminus \partial\mathcal{T}_{\gamma^n(x),\gamma^n(e)}\}$. Note $g(\xi) \in \mathcal{T}_{g(x),g(e)} \subsetneq \mathcal{T}_{x,e}$, as $g \in A^+$. By applying $k_1$ to $g(\xi)$ we must have $k_1([x, g(x)]) = [x, \gamma^n(x)]$. We obtain $k_1^{-1} \in k\mathbb{G}_{[x,\gamma^n(x)]}$, where $k \in K$ with $[x, g(x)] = k([x, \gamma^n(x)])$.

*4.2.3. The proof.* We are now ready to prove parabolically induced unitary representations on the universal group $\mathbb{G}$, induced from closed subgroups $H \leq \mathbb{G}_\xi$ containing hyperbolic elements, are $C_0$. We distinguish two cases: either $H$ is unimodular or $H$ is not unimodular.

Remark 4.12 (The strategy: second step). Let $G$ be a closed subgroup of $\mathrm{Aut}(\mathcal{T})^+$, $\xi \in \partial\mathcal{T}$ and $H$ be a closed subgroup of $G_\xi$ containing hyperbolic elements. Applying Remark 3.13 it remains to integrate the modular function $\Delta_H^{-1/2}$ on the intersection $t_n(f_1KH) \cap f_2KH = \bigsqcup_{i\in I_n} f_2k_{i,n}H$, for $t_n, f_1, f_2 \in G$. In order to do that, we need to investigate more closely the set $\{h_{i,n}\}_{i\in I_n}$, given by Lemma 3.11. Even if $h_{i,n}$ is uniquely determined by $k_{i,n}$, for every $i \in I_n$, we might still have two $h_{i,n}, h_{j,n}$, with $i \neq j \in I_n$, belonging to the same right coset of $H/(H \cap G_\xi^0)$, thus $\Delta_H(h_{i,n}) = \Delta_H(h_{j,n})$ by Remark 4.1.

The evaluation of the set of all right cosets $[h_{i,n}] \in H/(H \cap \mathbb{G}_\xi^0)$ follows from Proposition 4.10. Indeed, for simplicity set $g_n := f_2^{-1}t_nf_1$. By Lemma 2.2, one can write $g_n = k\gamma_nk'$, where $k, k' \in K$ and $\gamma_n \in A^+$ and there is a liberty to choose such $\gamma_n \in A^+$ and $k, k' \in K$. We can choose $\gamma_n$ with $\mathrm{proj}_{(x,\xi]}(Kg_nK) = \mathrm{proj}_{(x,\xi]}(\gamma_n)$. Fix such $\gamma_n, k, k'$ with $g_n = k\gamma_nk'$ and $\mathrm{proj}_{(x,\xi]}(Kg_nK) = \mathrm{proj}_{(x,\xi]}(\gamma_n)$.

Theorem 4.13. *Let $F$ be primitive and let $\xi \in \partial\mathcal{T}$. Let $H$ be a closed, unimodular, subgroup of $\mathbb{G}_\xi$, containing hyperbolic elements and let $(\sigma, \mathcal{K})$ be a unitary representation of $H$. Then the induced unitary representation $(\pi_\sigma, \mathcal{H}_\sigma)$ on $\mathbb{G}$ is $C_0$.*

Proof. By Lemma 4.5, let $\gamma$ be a minimal hyperbolic element of $H$. Fix for what follows a vertex $x \in \mathrm{Min}(\gamma)$ and set $K := \mathbb{G}_x$. By Lemma 4.6, $H_x = H_y = H_{[\xi_-,\xi]}$, for every $y \in \mathrm{Min}(\gamma)$, where $\xi_- \in \partial\mathcal{T}$ is the repelling endpoint of $\gamma$. By Lemma 3.5 and Lemma 3.3, we can consider, without loss of generality, the rho-function $\rho$ equals the constant function $\mathbf{1}$ on $\mathbb{G}$. In this particular case, the measure $\mu$ on $\mathbb{G}/H$ associated with the rho-function $\mathbf{1}$ on $\mathbb{G}$ is $\mathbb{G}$-invariant. Apply Remark 3.13 and then Remakr 4.12, keeping all the notation there. By Lemma 3.11 and Corollary 4.11, applied to $\gamma_n$, we have, for every $i \in I_n$, $k_{i,n}^{-1} g_n k_{k_{i,n}} = k_{i,n}^{-1} k\gamma_n k' k_{k_{i,n}} = h_{i,n} = \gamma^{m_i} h_0$, with $|m_i| = \mathrm{d}_{\mathcal{T}}(x, \gamma_n(x))/|\gamma|$ and $h_0 \in H \cap K$. Evaluate now the solutions for the equation

$$k_1 g_n k_2 = k_1 k \gamma_n k' k_2 = h, \tag{4}$$

for a given right coset $[h] \in \{[h_{i,n}] \mid i \in I_n\} \subset H/(H \cap \mathbb{G}_\xi^0)$ and where $k_1 k \in K$ and $k' k_2 \in K \setminus (K \cap H)$. Note for any element $h \in H$ satisfying equation (4) we have

$$\mathrm{d}_{\mathcal{T}}(x, h(x)) = \mathrm{d}_{\mathcal{T}}(x, g_n(x)) = \mathrm{d}_{\mathcal{T}}(x, \gamma_n(x)) = |m| \, \mathrm{d}_{\mathcal{T}}(x, \gamma(x)), \tag{5}$$

where $h = \gamma^m h_0$, with $h_0 \in H \cap K = H_x$. Apply again Corollary 4.11. We obtain for a given right coset $[h = \gamma^m] \in \{[h_{i,n}] \mid i \in I_n\} \subset H/(H \cap \mathbb{G}_\xi^0)$ we have: (1) if $m > 0$ then $k_1^{-1} \in k\mathbb{G}_{[x,x_h]}$, where $x_h \in [x, \xi]$ with $\mathrm{d}_{\mathcal{T}}(x, x_h) = \mathrm{d}_{\mathcal{T}}(x, \gamma_n(x))$; (2) if $m < 0$ then $k_1^{-1} \in kk_3\mathbb{G}_{[x,\gamma^m(x)]}$, where $k_3 \in K$ with $[x, \gamma_n(x)] = k_3([x, \gamma^m(x)])$.

To resume, for a fixed $n > 0$ we have:

$$\begin{aligned}
\int_{\bigsqcup_{i\in I_n} f_2 k_{i,n} H} & \Delta_H(h_{i,n})^{-1/2} \, d\mu(f_2 k_{i,n} H) \\
&= \int_{\bigsqcup_{i\in I_n} f_2 k_{i,n} H} \mathbf{1} \, d\mu(f_2 k_{i,n} H) \\
&\le \int_{kG_{[x,x_h]}H} \mathbf{1} \, d\mu(f_2 k \mathbb{G}_{[x,x_h]} H) + \int_{kk_3 G_{[x,\gamma^m(x)]}H} \mathbf{1} \, d\mu(f_2 k k_3 \mathbb{G}_{[x,\gamma^m(x)]} H) \\
&= \mu(f_2 k \mathbb{G}_{[x,x_h]} H) + \mu(f_2 k k_3 \mathbb{G}_{[x,\gamma^m(x)]} H).
\end{aligned}$$

As $t_n \to \infty$ as $n \to \infty$, we also have $g_n \to \infty$; thus by relation (5) $\mathrm{d}_{\mathcal{T}}(x, x_h) = \mathrm{d}_{\mathcal{T}}(x, \gamma_n(x)) = |m| \, \mathrm{d}_{\mathcal{T}}(x, \gamma(x)) \to \infty$, when $n \to \infty$. By hypothesis, $\mathbb{G}_\xi$, $\mathbb{G}_{\xi_-}$ are closed, non-compact and proper subgroups of $\mathbb{G}$. By Lemma 2.3 applied to $\mathbb{G}_\xi$ and $\mathbb{G}_{\xi_-}$, we have $[K : \mathbb{G}_{[x,\xi]}] = \infty = [K : \mathbb{G}_{[x,\xi_-]}]$. Therefore $[K : G_{[x,\gamma^m(x)]}] \to \infty$ as $m \to \infty$ and $[K : G_{[x,x_h]}] \to \infty$, when $n \to \infty$. By the $\mathbb{G}$-invariance of $\mu$ and because we have supposed $\mu(KH) = 1$ we claim:

$$\begin{aligned}
\mu(f_2 k \mathbb{G}_{[x,x_h]} H) + \mu(f_2 k k_3 \mathbb{G}_{[x,\gamma^m(x)]} H) &= \mu(\mathbb{G}_{[x,x_h]} H) + \mu(\mathbb{G}_{[x,\gamma^m(x)]} H) \\
&= [K : G_{[x,\gamma^m(x)]}]^{-1} + [K : G_{[x,x_h]}]^{-1} \xrightarrow[t_n \to \infty]{} 0. \quad (6)
\end{aligned}$$

Indeed, we only need to prove if $K = \bigsqcup_j k_j \mathbb{G}_{[x,y]}$, for some $y \in (\xi_-, \xi)$, then $KH = \bigsqcup_j k_j \mathbb{G}_{[x,y]} H$. Suppose this is not the case, then there exist $j_1 \neq j_2$ with $(k_{j_1}\mathbb{G}_{[x,y]}H) \cap (k_{j_2}\mathbb{G}_{[x,y]}H) \neq \emptyset$. So $k_{j_1}k = k_{j_2}k'h$, for some $k, k' \in \mathbb{G}_{[x,y]} \leq K$ and $h \in H$. Then $h \in K \cap H = H_{[\xi_-,\xi]} \subset \mathbb{G}_{[x,y]}$. Thus $k_{j_1}\mathbb{G}_{[x,y]} = k_{j_2}\mathbb{G}_{[x,y]}$, which is a contradiction. The claim follows. Relation (6) is a contradiction of our initial assumption $|\langle \pi_\sigma(t_n)\eta_1, \eta_2\rangle| \nrightarrow 0$ and the theorem stands proven.

THEOREM 4.14. *Let $F$ be primitive and let $\xi \in \partial\mathcal{T}$. Let $H$ be a closed, non-unimodular, subgroup of $\mathbb{G}_\xi$, containing hyperbolic elements and let $(\sigma, \mathcal{K})$ be a unitary representation of $H$. Then the induced unitary representation $(\pi_\sigma, \mathcal{H}_\sigma)$ on $\mathbb{G}$ is $C_0$.*

PROOF. By Lemma 4.5, let $\gamma$ be a minimal hyperbolic element of $H$. Fix for what follows a vertex $x \in \mathrm{Min}(\gamma)$ and set $K := \mathbb{G}_x$. Apply Remark 3.13 and then Remark 4.12, keeping all the notation there. By Lemma 3.11 and Proposition 4.10, applied to $\gamma_n$, we have, for every $i \in I_n$, $k_{i,n}^{-1} g_n k_{k_{i,n}} = k_{i,n}^{-1} k\gamma_n k' k_{k_{i,n}} = h_{i,n} = \gamma^{m_i} h_0$, with $0 \leq |m_i| \leq \mathrm{d}_{\mathcal{T}}(x, \gamma_n(x))/|\gamma|$ and $h_0 \in H \cap \mathbb{G}_\xi^0$. Evaluate now the solutions for the equation

$$k_1 g_n k_2 = k_1 k \gamma_n k' k_2 = h, \tag{7}$$

for a given right coset $[h] \in \{[h_{i,n}] \mid i \in I_n\} \subset H/(H \cap \mathbb{G}_\xi^0)$ and where $k_1 k \in K$ and $k' k_2 \in K \setminus (K \cap H)$. Note for any element $h \in H$, satisfying equation (7), we have

$$\mathrm{d}_{\mathcal{T}}(x, h(x)) = \mathrm{d}_{\mathcal{T}}(x, g_n(x)) = \mathrm{d}_{\mathcal{T}}(x, \gamma_n(x)). \tag{8}$$

Apply again Proposition 4.10. For a given right coset $[h = \gamma^m] \in \{[h_{i,n}] \mid i \in I_n\} \subset H/(H \cap \mathbb{G}_\xi^0)$ we have: (1) if $m > 0$ then $k_1^{-1} \in k\mathbb{G}_{[x,x_m]}$, where $x_m \in [x, \xi]$ is with $\mathrm{d}_{\mathcal{T}}(x, x_m) = \frac{1}{2}(\mathrm{d}_{\mathcal{T}}(x, g_n(x)) + m|\gamma|)$; (2) if $m = 0$ then $k_1^{-1} \in k\mathbb{G}_{[x,x_0]}$, where $x_0 \in [x, \xi]$ is with $\mathrm{d}_{\mathcal{T}}(x, x_0) = \frac{1}{2}\mathrm{d}_{\mathcal{T}}(x, g_n(x))$; (3) if $m < 0$ then $k_1^{-1} \in k\mathbb{G}_{[x,x_m]}$, where $x_m \in [x, \xi]$ is with $\mathrm{d}_{\mathcal{T}}(x, x_m) = \frac{1}{2}(\mathrm{d}_{\mathcal{T}}(x, g_n(x)) - |m| \cdot |\gamma|)$. To resume, for a fixed $n > 0$ and for $N := \mathrm{d}_{\mathcal{T}}(x, g_n(x))/|\gamma|$, we have:

$$\begin{aligned} &\int_{\bigsqcup_{i \in I_n} f_2 k_{i,n} H} \Delta_H(h_{i,n})^{-1/2} \, d\mu(f_2 k_{i,n} H) \\ &\quad \leq \sum_{m=-N}^{-1} \int_{kG_{[x,x_m]}H} \Delta_H(\gamma^m)^{-1/2} \, d\mu(f_2 k \mathbb{G}_{[x,x_m]} H) + \mu(f_2 k \mathbb{G}_{[x,x_0]} H) \\ &\qquad + \sum_{m=1}^{N} \int_{kG_{[x,x_m]}H} \Delta_H(\gamma^m)^{-1/2} d\mu(f_2 k \mathbb{G}_{[x,x_m]} H) \end{aligned}$$

$$= \mu(f_2 k \mathbb{G}_{[x,x_0]} H) + \sum_{m=-N}^{-1} \Delta_H(\gamma)^{-m/2} \cdot \mu(f_2 k \mathbb{G}_{[x,x_m]} H) + \sum_{m=1}^{N} \Delta_H(\gamma)^{-m/2} \cdot \mu(f_2 k \mathbb{G}_{[x,x_m]} H).$$

Note by Remark 3.7 there is a constant $C_1 > 0$ depending only on $f_2 k$, $K$ and the rho-function of $\mu$, with $\mu(f_2 k \mathbb{G}_{[x,x_m]} H)\Delta_H(\gamma)^{-m/2} \leq C_1 \mu(\mathbb{G}_{[x,x_m]} H)\Delta_H(\gamma)^{-m/2}$, for every $m \in [-N, N]$. Thus:

$$\begin{aligned}\int_{\bigsqcup_{i\in I_n} f_2 k_{i,n} H} & \Delta_H(h_{i,n})^{-1/2} d\mu(f_2 k_{i,n} H) \\ &\leq C_1 \left( \sum_{m=-N}^{-1} \Delta_H(\gamma)^{-m/2} \cdot \mu(\mathbb{G}_{[x,x_m]} H) + \mu(\mathbb{G}_{[x,x_0]} H) \right) \\ &\quad + C_1 \left( \sum_{m=1}^{N} \Delta_H(\gamma)^{-m/2} \cdot \mu(\mathbb{G}_{[x,x_m]} H) \right) \\ &= C_1 \left( \sum_{m=-N}^{-1} \Delta_H(\gamma)^{-m/2} \cdot [K : \mathbb{G}_{[x,x_m]}]^{-1} + [K : \mathbb{G}_{[x,x_0]}]^{-1} \right) \\ &\quad + C_1 \left( \sum_{m=1}^{N} \Delta_H(\gamma)^{-m/2} \cdot [K : \mathbb{G}_{[x,x_m]}]^{-1} \right).\end{aligned}$$

The last equality follows because $[K \colon \mathbb{G}_{[x,x_m]}]^{-1} = \mu(\mathbb{G}_{[x,x_m]} H)$, for every $m \in \{-N, N\}$. Indeed, we only need to prove if $K = \bigsqcup_j k_j \mathbb{G}_{[x,y]}$, for some $y \in [x, \xi]$, then $KH = \bigsqcup_j k_j \mathbb{G}_{[x,y]} H$. Suppose this is not the case, then there exist $j_1 \neq j_2$ with $(k_{j_1} \mathbb{G}_{[x,y]} H) \cap (k_{j_2} \mathbb{G}_{[x,y]} H) \neq \emptyset$. So $k_{j_1} k = k_{j_2} k' h$, for some $k, k' \in \mathbb{G}_{[x,y]} \leq K$ and $h \in H$. Then $h \in K \cap H = H_{[x,\xi]} \subset \mathbb{G}_{[x,y]}$. Thus $k_{j_1} \mathbb{G}_{[x,y]} = k_{j_2} \mathbb{G}_{[x,y]}$, which is a contradiction. Note as $t_n \to \infty$ when $n \to \infty$, we also have $g_n \to \infty$; thus by (8) $\mathrm{d}_{\mathcal{T}}(x, x_0) = \frac{1}{2}\mathrm{d}_{\mathcal{T}}(x, g_n(x)) = \frac{1}{2}\mathrm{d}_{\mathcal{T}}(x, \gamma_n(x)) \to \infty$ when $n \to \infty$. By hypothesis, $\mathbb{G}_\xi$ is a closed, non-compact and proper subgroup of $\mathbb{G}$. By Lemma 2.3 applied to $\mathbb{G}_\xi$ we have $[K : \mathbb{G}_{[x,\xi]}] = \infty$. Therefore we must have $[K : \mathbb{G}_{[x,\gamma^\ell(x)]}] \to \infty$ when $\ell \to \infty$, and $[K \colon \mathbb{G}_{[x,x_0]}] \to \infty$ when $n \to \infty$. Apply Lemma 4.15 below and we will contradict our initial assumption $|\langle \pi_\sigma(t_n)\eta_1, \eta_2 \rangle| \nrightarrow 0$ and the theorem stands proven.

LEMMA 4.15. *Using the same notation as in the proof of Theorem 4.14 we have:*

(1) $\lim_{N\to\infty}\sum_{m=-N}^{-1}\Delta_H(\gamma)^{-m/2}\cdot[K:\mathbb{G}_{[x,x_m]}]^{-1}=0$,

(2) $\lim_{N\to\infty}\sum_{m=1}^{N}\Delta_H(\gamma)^{-m/2}\cdot[K:\mathbb{G}_{[x,x_m]}]^{-1}=0$.

PROOF. Recall

$$N:=\mathrm{d}_{\mathcal{T}}(x,g_n(x))/|\gamma| \quad\text{and}\quad \tfrac{1}{2}\mathrm{d}_{\mathcal{T}}(x,g_n(x))=\tfrac{1}{2}\mathrm{d}_{\mathcal{T}}(x,\gamma_n(x))\to\infty$$

as $n\to\infty$. Moreover, if $m>0$ then $\mathrm{d}_{\mathcal{T}}(x,x_m)=\frac{1}{2}(\mathrm{d}_{\mathcal{T}}(x,g_n(x))+m|\gamma|)$ and if $m<0$ then $\mathrm{d}_{\mathcal{T}}(x,x_m)=\frac{1}{2}(\mathrm{d}_{\mathcal{T}}(x,g_n(x))-|m|\cdot|\gamma|)$. By Lemma 4.6 and the hypothesis $H$ is non-unimodular we have $t:=\Delta_H(\gamma)=1/[H_{[\gamma(x),\xi]}:H_{[x,\xi]}]<1$. As $H\le\mathbb{G}_\xi\le\mathbb{G}$ we also have $[H_{[\gamma^m(x),\xi]}:H_{[x,\xi]}]\le[\mathbb{G}_{[\gamma^m(x),\xi]}:\mathbb{G}_{[x,\xi]}]$, for every $m\ge 0$. By Lemma 4.8, we have $[K:\mathbb{G}_{[x,\gamma^m(x)]}]=[\mathbb{G}_{[\gamma^m(x),\xi]}:\mathbb{G}_{[x,\xi]}]\cdot d/k_1$, for every $m>0$, where $k_1$ is the number of orbits of the edge $e_-$ in $\{1,\dots,d\}$ under the stabilizer subgroup $F_{e_+}\le F$. Let $0\le\ell(m):=\lfloor m/2+N/2\rfloor$ the integer value of $(\mathrm{sign}(m)|\gamma^m(x)|+|\gamma_n(x)|)/(2\cdot|\gamma|)=m/2+N/2$, for every $m\ge -N$. Thus, for every $m\ge -N$:

$$\begin{aligned}[K:\mathbb{G}_{[x,\gamma^{\ell(m)}(x)]}]\le[K:\mathbb{G}_{[x,x_m]}]&=[K:\mathbb{G}_{[x,\frac{1}{2}(\mathrm{sign}(m)|\gamma^m(x)|+|\gamma_n(x)|)]}]\\&\le[K:\mathbb{G}_{[x,\gamma^{\ell(m)+1}(x)]}].\end{aligned}\tag{9}$$

Let us prove the assertion (2). By Lemma 4.6, $t^{-m/2}=[H_{[\gamma^m(x),\xi]}:H_{[x,\xi]}]^{1/2}$, for every $m\ge 0$. By Lemma 4.7 applied to $\mathbb{G}_\xi$, for every $0\le m$ we have

$$[\mathbb{G}_{[\gamma^m(x),\xi]}:\mathbb{G}_{[x,\xi]}]\le[\mathbb{G}_{[\gamma^{\lfloor m/2\rfloor+1}(x),\xi]}:\mathbb{G}_{[x,\xi]}]^2.\tag{10}$$

Using (9) and (10), the assertion (2) becomes:

$$\begin{aligned}&\sum_{m=1}^{N}\Delta_H(\gamma)^{-m/2}\cdot[K:\mathbb{G}_{[x,x_m]}]^{-1}\\
&\quad=\sum_{m=1}^{N}[H_{[\gamma^m(x),\xi]}:H_{[x,\xi]}]^{1/2}\cdot[K:\mathbb{G}_{[x,x_m]}]^{-1}\\
&\quad\le\sum_{m=1}^{N}\frac{[\mathbb{G}_{[\gamma^m(x),\xi]}:\mathbb{G}_{[x,\xi]}]^{1/2}}{[K:\mathbb{G}_{[x,x_m]}]}\le\sum_{m=1}^{N}\frac{[\mathbb{G}_{[\gamma^{\lfloor m/2\rfloor+1}(x),\xi]}:\mathbb{G}_{[x,\xi]}]}{[K:\mathbb{G}_{[x,\gamma^{\ell(m)}(x)]}]}\\
&\quad=\frac{k_1}{d}\sum_{m=1}^{N}\frac{[\mathbb{G}_{[\gamma^{\lfloor m/2\rfloor+1}(x),\xi]}:\mathbb{G}_{[x,\xi]}]}{[\mathbb{G}_{[\gamma^{\ell(m)}(x),\xi]}:\mathbb{G}_{[x,\xi]}]}\le\frac{k_1}{d}\cdot\frac{N}{[\mathbb{G}_{[\gamma^{\lfloor N/2\rfloor-1}(x),\xi]}:\mathbb{G}_{[x,\xi]}]}\\
&\quad=\frac{k_1}{d}\cdot N\cdot\Delta_{\mathbb{G}_\xi}(\gamma)^{\lfloor N/2\rfloor-1}\xrightarrow[N\to\infty]{}0.\end{aligned}$$

Using (9), the assertion (1) becomes:

$$\begin{aligned}
\lim_{N\to\infty}\sum_{m=-N}^{-1}\Delta_H(\gamma)^{-m/2}\cdot[K:\mathbb{G}_{[x,x_m]}]^{-1} &= \lim_{N\to\infty}\sum_{m=-N}^{-1}t^{|m|/2}\cdot[K:\mathbb{G}_{[x,x_m]}]^{-1}\\
&\le \lim_{N\to\infty}\sum_{m=-N}^{-1}t^{|m|/2}\cdot[K:\mathbb{G}_{[x,\gamma^{\ell(m)}(x)]}]^{-1}\\
&= \lim_{N\to\infty}\left(t^{N/2}\cdot 1+\sum_{m=-N+1}^{-1}t^{|m|/2}\cdot[K:\mathbb{G}_{[x,\gamma^{\ell(m)}(x)]}]^{-1}\right)\\
&\le \lim_{N\to\infty}\sum_{m=-N+1}^{(-N+1)/2}t^{|m|/2}\cdot[K:\mathbb{G}_{[x,\gamma^{\ell(m)}(x)]}]^{-1}\\
&\quad+\lim_{N\to\infty}\sum_{m=(-N+1)/2}^{-1}t^{|m|/2}\cdot[K:\mathbb{G}_{[x,\gamma^{\ell(m)}(x)]}]^{-1}\\
&\le \lim_{N\to\infty}\sum_{m=-N+1}^{(-N+1)/2}t^{|m|/2}+\lim_{N\to\infty}\sum_{m=(-N+1)/2}^{-1}t^{|m|/2}\cdot[K:\mathbb{G}_{[x,\gamma^{\ell(m)}(x)]}]^{-1}\\
&\le \lim_{N\to\infty}\frac{N+1}{2}\cdot t^{(N-1)/2}\\
&\quad+\lim_{N\to\infty}[K:\mathbb{G}_{[x,\gamma^{\ell((-N+1)/2)}(x)]}]^{-1}\cdot\sum_{m=(-N+1)/2}^{0}t^{|m|/2}.
\end{aligned}$$

As $t<1$, one has $\lim_{N\to\infty}(N+1)/2\cdot t^{(N-1)/2}=0$. Moreover, $\lim_{N\to\infty}[K:\mathbb{G}_{[x,\gamma^{\ell((-N+1)/2)}(x)]}]^{-1}=0$ and $\sum_{m=(-N+1)/2}^{0}t^{|m|/2}=1/(1-t^{1/2})$. Thus $\lim_{N\to\infty}[K:\mathbb{G}_{[x,\gamma^{\ell((-N+1)/2)}(x)]}]^{-1}\cdot\sum_{m=(-N+1)/2}^{0}t^{|m|/2}=0$.

## 5. The main theorem

Corollary 4.4, Theorem 4.3 and Theorem 4.14 give us the aimed result of this article:

THEOREM 5.1. *Let $F$ be primitive and let $\xi\in\partial\mathscr{T}$. Let $H$ be a closed subgroup of $\mathbb{G}_\xi$ and let $(\sigma,\mathscr{K})$ be a unitary representation of $H$. Then the induced unitary representation $(\pi_\sigma,\mathscr{H}_\sigma)$ on $\mathbb{G}$ is $C_0$.*

PROOF. It remains to consider the case when $H$ is a compact subgroup of $\mathbb{G}_\xi$. This is a particular case of the well-known general fact that all unitary representations of a locally compact subgroup that are induced from compact subgroups are $C_0$. For the idea of the proof the reader can consult the book of Bekka-de la Harpe-Valette [2, Proposition C.4.6].

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