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THE K-INDUCTIVE STRUCTURE OF THE NONCOMMUTATIVE FOURIER TRANSFORM

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(Dedicated to Canada on her 150$^{\text{th}}$ birthday)

Abstract

The noncommutative Fourier transform $\sigma(U) = V^{-1}$, $\sigma(V) = U$ of the irrational rotation C*-algebra A_θ (generated by canonical unitaries U, V satisfying $VU = e^{2\pi i\theta}UV$) is shown to have the following K-inductive structure (for a concrete class of irrational parameters, containing dense G_δ's). There are approximately central matrix projections e_1, e_2, f that are σ-invariant and which form a partition of unity in K_0 of the fixed-point orbifold A_θ^σ, where f has the form $f = g + \sigma(g) + \sigma^2(g) + \sigma^3(g)$, and where g is an approximately central matrix projection as well.

1. Introduction

In this paper we prove that the noncommutative Fourier transform of the irrational rotation C*-algebra A_θ has a K-inductive structure for at least a large class of irrationals θ (containing concrete dense G_δ's) – see Theorem 1.2. Let us explain this.

Let $\mathscr{B}$ denote the collection of C*-algebras (which we regard as building blocks) consisting of matrix algebras, matrix algebras over the unit circle, or finite direct sums of these. By a $\mathscr{B}$-type algebra we mean one that is C*-isomorphic to an algebra in the collection $\mathscr{B}$. For example, the Elliott-Evans structure theorem [3] states that the irrational rotation C*-algebra A_θ can be approximated by unital C*-subalgebras of the form $M_q(C(\mathbb{T})) \oplus M_{q'}(C(\mathbb{T}))$, which are in the class $\mathscr{B}$.

Definition 1.1. Let α be an automorphism of a unital C*-algebra A. We say that α is *K-inductive* if, for each $\epsilon > 0$ and each finite subset S of A, there exist a finite number of $\mathscr{B}$-type building block C*-subalgebras $B_1, \dots, B_n$ of A with respective unit projections $e_1, \dots, e_n$, such that

(1) B_j and e_j are α-invariant for each j,

(2) $\|e_j x - x e_j\| < \epsilon$, $\forall x \in S$ and each j,

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(3) $e_j x e_j$ is to within distance ϵ from B_j, $\forall x \in S$ and each j,

(4) $[e_1] + \cdots + [e_n] = [1]$ in $K_0(A^\alpha)$.

Here, A^α is the fixed-point subalgebra of A (i.e., the C*-orbifold under α). Note that the equality of K-classes in condition (4) is stronger than simply requiring it to hold in $K_0(A)$. A projection e is a *matrix projection* in A when it is approximately central and is the unit of a subalgebra $B \cong M_n(\mathbb{C})$ (for some n), and the cut downs exe are close to B for each x in any prescribed finite subset $S \subset A$.

The rotation C*-algebra (or noncommutative 2-torus) A_θ is the universal C*-algebra generated by unitaries U and V enjoying the Heisenberg relation

$$VU = e^{2\pi i\theta} UV.$$

The noncommutative Fourier transform (NCFT) of A_θ is the canonical order four automorphism (or symmetry) σ given by

$$\sigma(U) = V^{-1}, \qquad \sigma(V) = U.$$

We will simply say 'Fourier transform', and drop the adjective 'noncommutative'. (The connection between this C*-Fourier transform and the classical Fourier transform $\widehat{f}$ is aptly expressed in terms of the C*-inner product equation $\sigma(\langle f, g\rangle_D) = \langle \widehat{f}, \widehat{g}\rangle_D$ as in [8], but we will not need this fact here.) The Elliott Fourier Transform problem, which is still open, is the problem of determining the inductive limit structure of the Fourier transform of the irrational rotation C*-algebra A_θ with respect to σ-invariant basic building blocks consisting of finite dimensional algebras and circle algebras. (Or, more generally, in terms of type I C*-subalgebras.)

The main result of this paper is the following theorem, where $\mathcal{G}$ is any of the dense G_δ-sets in (0, 1) constructed in Section 2.5 below.

THEOREM 1.2 (Structure Theorem). *For each irrational number θ in the dense G_δ-set $\mathcal{G}$, the noncommutative Fourier transform σ of A_θ is a K-inductive automorphism with respect to matrix algebras. More specifically, for each $\epsilon > 0$ and each finite subset S of A_θ, there are three $\mathcal{B}$-type building block matrix* C*-*subalgebras*

$$B_1 \cong M_m(\mathbb{C}), \qquad B_2 \cong M_n(\mathbb{C}), \qquad B_3 = M \oplus \sigma(M) \oplus \sigma^2(M) \oplus \sigma^3(M),$$

$M \cong M_\ell(\mathbb{C})$, for some integers ℓ, m and n, with respective unit projections

$$e_1, \qquad e_2, \qquad f = g + \sigma(g) + \sigma^2(g) + \sigma^3(g)$$

where g is the unit projection of M with $g\sigma^j(g) = 0$ (for $j = 1, 2, 3$), such that

1. B_1, B_2, B_3 *and* e_1, e_2 *are* σ*-invariant,*
2. $\|e_j x - x e_j\| < \epsilon$ *and* $\|gx - xg\| < \epsilon$*, for all* $x \in S$ *and* $j = 1, 2$*,*
3. $e_j x e_j$ *and* gxg *are to within distance* ϵ *from* B_j *and* M*, respectively, for all* $x \in S$ *and each* $j = 1, 2$*.*

Further, there exist σ*-invariant unitaries* w *and* z *in* A_θ *satisfying the equation*

$$e_1 + we_2w^* + zfz^* = 1. \tag{1.1}$$

This equation (1.1) is equivalent to condition (4) in the above definition since the orbifold A_θ^σ has the cancellation property.

We may schematically display the K-inductive structure of the NCFT on A_θ in terms of building blocks as "$\bullet \oplus \bullet \oplus \circ \oplus \circ \oplus \circ \oplus \circ$" where each bullet $\bullet$ represents a σ-invariant matrix algebra and the open bullets $\circ$ are matrix algebras that are cyclically permuted by the NCFT.

The notion of K-inductive is a natural extension of Huaxin Lin's notion of tracially AF for C*-algebras [4] to automorphisms. The one difference is that whereas tracially AF means that there are plenty of finite dimensional projections whose complements are equivalent to some projection in a prescribed hereditary C*-subalgebra, in the case of K-inductive the complement is required to have a rather specific structure – i.e., is required to be invariantly equivalent to other projections of a building block nature.

We believe that a similar result can be proved for any irrational θ. Our choice of the classes $\mathcal{G}$ of irrationals makes our computations far more accessible by avoiding number theoretic complications (and helps to make the paper shorter). A similar approach to that presented in this paper would probably also show that the cubic and hexic transforms of A_θ – namely the canonical order 3 and 6 automorphisms studied in [1], [2], [10] – are K-inductive automorphisms as well, with respect to matrix algebras possibly including circle algebra building blocks. The hoped-for conclusion, then, is that all the canonical finite order automorphisms (the only orders being 2, 3, 4, and 6) are K-inductive automorphisms for all irrational θ.

2. The framework

2.1. Continuous field of Fourier transforms

We write U_t and V_t for the continuous sections of canonical unitaries of the continuous field $\{A_t : 0 < t < 1\}$ of rotation C*-algebras such that $V_tU_t = e(t)U_tV_t$, where we have used the now common notation

$$e(t) := e^{2\pi i t}.$$

(The unitaries U_t, V_t generate A_t for each t.) When dealing with a specific irrational rotation algebra A_θ we often write its unitary generators simply as U and V instead of U_θ and V_θ. On the field $\{A_t\}$ there is a field of noncommutative Fourier transforms σ_t given on the fiber A_t by

$$\sigma_t(U_t) = V_t^{-1}, \qquad \sigma_t(V_t) = U_t.$$

Often we omit the subscript on σ_t and simply write σ since there will be no risk of confusion.

2.2. Basic matrix approximation

We will use the following result from [9, Theorem 1.5] (a result that was originally rooted in [8]).

THEOREM 2.1 ([9, Theorem 1.5]). *Let $\theta > 0$ be an irrational number and $p/q > 0$ be a rational number in reduced form such that $0 < q(q\theta - p) < 1$. Let σ be the Fourier transform of A_θ. Then there exists a Fourier invariant smooth projection e in A_θ of trace $\tau(e) = q(q\theta - p)$ and a Fourier equivariant isomorphism*

$$\eta\colon eA_\theta e \to M_q \otimes A_{\theta'} \qquad \textit{such that} \quad \eta\sigma = (\Sigma \otimes \sigma')\eta, \tag{2.1}$$

where Σ and σ' are Fourier transform automorphisms of M_q and $A_{\theta'}$, respectively, given by

$$\Sigma(u) = v, \qquad \Sigma(v) = u^*, \qquad \sigma'(a) = b^*, \qquad \sigma'(b) = a,$$

where $M_q = C^*(u, v)$ and u, v are order q unitary matrices with $vu = e(p/q)uv$, and $A_{\theta'}$ is generated by unitaries a, b with $ba = e(\theta')ab$, where $\theta' = (c\theta + d)/(q\theta - p)$ is an irrational number in the $\mathrm{GL}(2, \mathbb{Z})$*-orbit of θ. (Here, c and d are integers such that $cp + dq = 1$.)*

Furthermore, given a sequence of rational approximations p/q of θ such that $0 < q(q\theta - p) < \kappa < 1$ for some fixed number κ, the projection e is a matrix projection: e is approximately central and $\eta(eUe)$, $\eta(eVe)$ are close to order q unitary generators of M_q and which is Fourier invariant – the approximations here go to 0 as $q \to \infty$.

It is easy to see that a similar result to this theorem applies for rational approximations of θ such that $0 < q(p - q\theta) < 1$ – simply by replacing θ by $1 - \theta$ and using the canonical isomorphism $A_{1-\theta} \cong A_\theta$ which canonically intertwines the Fourier transform.

Since θ will be fixed throughout the paper, we will write e_q^+ for the above canonical projection of trace $q^2\theta - pq$, since it has positive label (or 'charge'), and write e_q^- for the canonical projection of trace $pq - q^2\theta$ with negative label (see (2.3)). According to the last assertion of Theorem 2.1, we can have Fourier invariant matrix projections of both these types.

2.3. Covariant projections

In [11] (but also somewhat evident in [8]) we showed that the Fourier invariant projections e_q^+ of Theorem 2.1 are instances of one and the same continuous field $\mathscr{E}(t)$ of projections of the continuous field $\{A_t\}$ of rotation algebras such that $\tau(\mathscr{E}(t)) = t$. The relation is canonically furnished by equation (2.2) for e_q^+ and (2.3) for its negative charge counterpart e_q^-.

Given an irrational number t and integers n, k, where $n \neq 0$, one has the canonical unital *-morphism

$$\zeta_{n,t}\colon A_{n^2t-k} \to A_t, \qquad \zeta_{n,t}(U_{n^2t-k}) = U_t^n, \qquad \zeta_{n,t}(V_{n^2t-k}) = V_t^n.$$

This map clearly intertwines the Fourier transform

$$\sigma_t\zeta_{n,t} = \zeta_{n,t}\sigma_{n^2t-k}.$$

For rational approximations $p/q < \theta$ such that $0 < q^2\theta - pq < 1$, we have the projection

$$e_q^+ = \zeta_{q,\theta}\mathscr{E}(q^2\theta - pq) \in A_\theta \tag{2.2}$$

whose trace is $q^2\theta - pq$. This is what we mean by saying that e_q^+ is covariant (that it arises from the projection field $\mathscr{E}(t)$ in a natural manner). We could also write down negatively charged projections in A_θ defined by

$$e_b^- = \nu\zeta_{b,1-\theta}\mathscr{E}(ab - b^2\theta) \tag{2.3}$$

of trace $ab - b^2\theta \in (0, 1)$, where ν is the canonical isomorphism

$$\nu\colon A_{1-\theta} \to A_\theta, \qquad \nu(U_{1-\theta}) = V_\theta, \qquad \nu(V_{1-\theta}) = U_\theta. \tag{2.4}$$

(In the mathematical physics literature related to string theory and noncommutative geometry, the Connes-Chern number $-b^2$, for a projection of trace $ab-b^2\theta$, is referred to as the "charge" of the projection, or that of its associated instanton.) We will need to use the parity automorphism γ of A_θ defined by

$$\gamma(U) = -U, \qquad \gamma(V) = -V$$

because it commutes with the Fourier transform and has the nice effect of flipping the signs of two of the topological invariants below (namely, ψ_{11} and ψ_{22}), while preserving the others. (In fact, γ is the only nontrivial of the toral action automorphisms that commutes with the Fourier transform σ.)

2.4. Topological invariants

With U_θ and V_θ denoting the canonical unitaries satisfying

$$V_\theta U_\theta = e(\theta) U_\theta V_\theta$$

and the Fourier transform defined by

$$\sigma(U_\theta) = V_\theta^{-1}, \qquad \sigma(V_\theta) = U_\theta$$

the following are the basic unbounded "trace" functionals defined on the canonical smooth dense *-subalgebra A_θ^∞:

$$\begin{aligned} \psi_{10}^\theta(U_\theta^m V_\theta^n) &= e\big(-\tfrac{\theta}{4}(m+n)^2\big)\delta_2^{m-n}, & \psi_{20}^\theta(U_\theta^m V_\theta^n) &= e\big(-\tfrac{\theta}{2}mn\big)\delta_2^m \delta_2^n, \\ \psi_{11}^\theta(U_\theta^m V_\theta^n) &= e\big(-\tfrac{\theta}{4}(m+n)^2\big)\delta_2^{m-n-1}, & \psi_{21}^\theta(U_\theta^m V_\theta^n) &= e\big(-\tfrac{\theta}{2}mn\big)\delta_2^{m-1} \delta_2^{n-1}, \\ & & \psi_{22}^\theta(U_\theta^m V_\theta^n) &= e\big(-\tfrac{\theta}{2}mn\big)\delta_2^{m-n-1}, \end{aligned} \tag{2.5}$$

where δ_a^b is the *divisor delta function* defined to be 1 if a divides b, and 0 otherwise. These maps were calculated in [6] and used in [7], [8], [11]. (Sometime θ is omitted from the notation ψ_{jk}^θ when there is no risk of confusion.)

The functionals ψ_{1j} are σ-invariant σ-traces and ψ_{2j} are σ-invariant σ^2-traces. Recall that if α is an automorphism of an algebra A (usually a pre-C*-algebra like A_θ^∞), by an α-trace we understand a complex-valued linear map ψ defined on A satisfying the condition

$$\psi(xy) = \psi(\alpha(y)x)$$

for each x, y in A; and we say that ψ is σ-invariant when $\psi\sigma = \psi$. (Clearly, a σ-trace is automatically σ-invariant if its domain contains the identity, but a σ^2-trace need not be σ-invariant.) These unbounded linear functionals induce trace maps on the smooth C*-orbifold $A_\theta^{\infty,\sigma} = A_\theta^\infty \cap A_\theta^\sigma$, thereby inducing homomorphisms on K-theory $\psi_*\colon K_0(A_\theta^\sigma) \to \mathbb{C}$.

In [6] it was shown that $\{\psi_{10}, \psi_{11}\}$ is a basis for the 2-dimensional vector space of all σ-traces on A_θ^∞, and that $\{\psi_{20}, \psi_{21}, \psi_{22}\}$ is a basis for the 3-dimensional vector space of all σ-invariant σ^2-traces on A_θ^∞.

The unbounded traces ψ_{jk} along with the canonical bounded trace τ comprise the associated Connes-Chern character group homomorphism for the fixed-point algebra A_θ^σ:

$$\mathbf{T}\colon K_0(A_\theta^\sigma) \to \mathbb{C}^6, \quad \mathbf{T}(x) = (\tau(x); \psi_{10}(x), \psi_{11}(x); \psi_{20}(x), \psi_{21}(x), \psi_{22}(x)).$$

For the identity one has $\mathbf{T}(1) = (1; 1, 0; 1, 0, 0)$. It will be convenient to write

$$\mathbf{T}(x) = (\tau(x); \mathbf{Top}(x))$$

where

$$\mathbf{Top}(x) := (\psi_{10}(x), \psi_{11}(x); \psi_{20}(x), \psi_{21}(x), \psi_{22}(x))$$

consists of the discrete topological invariants of x. Indeed, in view of [6] and [7], the values of the unbounded traces on projections, and on $K_0(A_\theta^\sigma)$, are quantized, with ψ_{10}, ψ_{11} having range the lattice subgroup $\mathbb{Z} + \mathbb{Z}((1-i)/2)$ of $\mathbb{C}$; ψ_{20}, ψ_{21} range $\frac{1}{2}\mathbb{Z}$; and ψ_{22} range $\mathbb{Z}$. (Cf. Lemma 2.3 below which gives the topological invariants for the field $\mathscr{E}(t)$.)

It is straightforward to check that the parity automorphism γ changes the signs of ψ_{11} and ψ_{22}:

$$\psi_{11}\gamma = -\psi_{11}, \qquad \psi_{22}\gamma = -\psi_{22}$$

and it keeps the other ψ_{jk} unchanged. Thus,

$$\mathbf{Top}(\gamma(x)) = (\psi_{10}(x), -\psi_{11}(x); \psi_{20}(x), \psi_{21}(x), -\psi_{22}(x)).$$

The Connes-Chern map $\mathbf{T}$ was shown to be injective [7] for a dense G_δ-set of irrationals θ, but since $K_0(A_\theta^\sigma) \cong \mathbb{Z}^9$ for all θ by [2] or [5], $\mathbf{T}$ is injective for all irrational θ. This allows us to conclude that since A_θ^σ has the cancellation property for any irrational θ, two projections e and e' in A_θ^σ are unitarily equivalent by a σ-invariant unitary if and only if $\mathbf{T}(e) = \mathbf{T}(e')$.

DEFINITION 2.2. A projection f is called *flat* (or *σ-flat*), when it is an orthogonal sum of the form

$$f = g + \sigma(g) + \sigma^2(g) + \sigma^3(g)$$

for some projection g. We call such projection g a *cyclic* subprojection for f since it is orthogonal to its orbit under σ and its orbit sum gives f.

Another reason we call f "flat" is because its topological invariants vanish:

$$\mathbf{Top}(f) = (0, 0; 0, 0, 0).$$

Indeed, if ψ is any of the two kinds of unbounded traces in (2.5), we have (can assume g is smooth) $\psi(g) = \psi(gg) = \psi(\sigma^j(g)g) = 0$ for $j = 1, 2$, hence $\psi(f) = 0$.

In [11] we proved that the topological invariants of the continuous section $\mathscr{E}(t)$ mentioned in Section 2.3 are given as follows.

LEMMA 2.3 ([11, Theorem 1.7]). *The topological invariants of the projection section $\mathscr{E}(t)$ are*

$$\mathbf{Top}(\mathscr{E}(t)) = \left(\frac{1-i}{2}, \frac{1-i}{2}; \frac{1}{2}, \frac{1}{2}, 1\right)$$

and its trace is $\tau(\mathscr{E}(t)) = t$ *for each* $t \in (0,1)$.

This will allow us to calculate the topological invariants of the canonical projections e_q^+ and e_q^- of Theorem 2.1 using the following lemma.

LEMMA 2.4 ([11, Lemma 3.2]). *Let* $\theta_n = n^2\theta - k$ *where* k, n *are integers. The unbounded traces on* A_{θ_n} *and* A_θ *are related by* $\zeta_{n,\theta}$ *according to the equations*

$$\begin{aligned}
\psi_{10}^{\theta}\zeta_{n,\theta} &= \psi_{10}^{\theta_n} + i^{-k}\delta_2^n\psi_{11}^{\theta_n}, & \psi_{20}^{\theta}\zeta_{n,\theta} &= \psi_{20}^{\theta_n} + (-1)^k\delta_2^n\psi_{21}^{\theta_n} + \delta_2^n\psi_{22}^{\theta_n},\\
\psi_{11}^{\theta}\zeta_{n,\theta} &= i^{-k}\delta_2^{n-1}\psi_{11}^{\theta_n}, & \psi_{21}^{\theta}\zeta_{n,\theta} &= (-1)^k\delta_2^{n-1}\psi_{21}^{\theta_n},\\
& & \psi_{22}^{\theta}\zeta_{n,\theta} &= \delta_2^{n-1}\psi_{22}^{\theta_n}.
\end{aligned}$$

(Lemma 2.4 is in fact easy to check by directly working out both sides on generic unitaries $U_{\theta_n}^m V_{\theta_n}^n$.)

Combining these two lemmas and applying them to the canonical projection $e_q^+ = \zeta_{q,\theta}\mathscr{E}(q^2\theta - pq)$, we obtain its topological invariants

$$\mathbf{Top}(e_q^+) = \big(\tfrac{1-i}{2}[1+(-1)^{q/2}\delta_2^q],\ \big(\tfrac{1-i}{2}\big)i^{-pq}\delta_2^{q-1};\ \tfrac12+\tfrac32\delta_2^q,\ \tfrac12(-1)^p\delta_2^{q-1},\ \delta_2^{q-1}\big). \tag{2.6}$$

Likewise, the topological invariants of the negatively charged canonical projection e_b^- of trace $ab - b^2\theta$ (where a, b are coprime) are

$$\mathbf{Top}(e_b^-) = \big(\tfrac{1+i}{2}[1+(-1)^{b/2}\delta_2^b],\ \big(\tfrac{1+i}{2}\big)i^{-ab}\delta_2^{b-1};\ \tfrac12+\tfrac32\delta_2^b,\ \tfrac12(-1)^a\delta_2^{b-1},\ \delta_2^{b-1}\big). \tag{2.7}$$

To check the latter invariants of e_b^-, one uses the following relations between the unbounded traces and the canonical isomorphism ν of (2.4):

$$\psi_{10}^{\theta}\nu = (\psi_{10}^{1-\theta})^*, \qquad \psi_{11}^{\theta}\nu = -i(\psi_{11}^{1-\theta})^*$$
$$\psi_{20}^{\theta}\nu = \psi_{20}^{1-\theta}, \qquad \psi_{21}^{\theta}\nu = -\psi_{21}^{1-\theta}, \qquad \psi_{22}^{\theta}\nu = \psi_{22}^{1-\theta},$$

which are straightforward to check by working them out on the unitary elements $U_{1-\theta}^m V_{1-\theta}^n$. (Here, ψ^* is the Hermitian adjoint $\psi^*(x) = \overline{\psi(x^*)}$.)

To verify the first component of $\mathbf{Top}(e_b^-)$, for example, we compute:

$$\psi_{10}^{\theta}(e_b^-) = \psi_{10}^{\theta}\nu\zeta_{b,1-\theta}\mathscr{E}(ab - b^2\theta) = \overline{\psi_{10}^{1-\theta}\zeta_{b,1-\theta}\mathscr{E}(\tilde\theta)},$$

where we have written $\tilde\theta := ab - b^2\theta = b^2(1-\theta) - (b^2 - ab)$. By Lemma 2.4, with θ replaced by $1-\theta$, and by Lemma 2.3 we get

$$\begin{aligned}\psi_{10}^{1-\theta}\zeta_{b,1-\theta}\mathscr{E}(\tilde{\theta}) &= \psi_{10}^{\tilde{\theta}}\mathscr{E}(\tilde{\theta}) + i^{-(b^2-ab)}\delta_2^b\psi_{11}^{\tilde{\theta}}\mathscr{E}(\tilde{\theta}),\\ &= \tfrac{1-i}{2} + i^{-(b^2-ab)}\delta_2^b\tfrac{1-i}{2}\\ &= \tfrac{1-i}{2}(1+i^{ab}\delta_2^b)\\ &= \tfrac{1-i}{2}(1+(-1)^{b/2}\delta_2^b),\end{aligned}$$

since if b is odd the delta term vanishes and when b is even, a has to be odd so can be removed in the power of -1. After conjugating we obtain $\psi_{10}^{\theta}(e_b^-)$, as asserted. The other invariants of e_b^- in (2.7) are similarly checked.

2.5. *Class of irrationals $\mathscr{G}$*

We begin with any dense set D of rational numbers $\frac{k}{m}$ in the open interval $\left(0, \frac{1}{2}\right)$ where $k, m \geq 1$. We form the following integers

$$n = 4mk+1, \quad q = n^2, \quad s = n^2+4m^2, \quad p = 4k^2(2n+1), \quad r = p+2n-3,$$

which can easily be checked to satisfy the modular equation

$$ps - qr = 1 \tag{2.8}$$

(for any k, m). Let κ_2, κ_1 be any fixed pair of positive numbers such that

$$0 < \kappa_2 \leq \frac{1}{2} < \kappa_1 < 1, \qquad 1 < \kappa_1 + \kappa_2. \tag{2.9}$$

One checks (using (2.8)) directly that the following inequality

$$\frac{r}{s} < \frac{pq-\kappa_1}{q^2} < \frac{rs+\kappa_2}{s^2} < \frac{p}{q} \tag{2.10}$$

holds for large enough k. The left inequality holds for large enough k (specifically for k such that $q/s > \kappa_1$, since $q/s \to 1$ as $k \to \infty$). (Indeed, the left inequality holds for k such that $4k^2 \geq \kappa_1/(1-\kappa_1)$.) The middle inequality holds for all k, m by virtue of (2.9), and the right inequality holds always since $\kappa_2 \leq \frac{1}{2}$. (The middle inequality yields the quadratic inequality $\kappa_2 x^2 - x + \kappa_1 > 0$, where $x = q/s$. By (2.9), the quadratic is a decreasing function over the interval [0, 1] and is positive at the endpoints, so it is positive on [0, 1].)

It is easy to see that the difference $\frac{p}{q} - \frac{2k}{m}$ goes to 0 for large k, m, hence the set of rationals $\{p/q : k, m \geq 1\}$ is dense in the open interval (0, 1), as also is the set $\{r/s\}$.

We can extend slightly inequality (2.10) to the following

$$\frac{2km-\frac{1}{2}}{m^2} < \frac{r}{s} < \frac{pq-\kappa_1}{q^2} < \theta < \frac{rs+\kappa_2}{s^2} < \frac{p}{q} < \frac{2k}{m}, \tag{2.11}$$

where θ will be the type of irrational that we'll be interested in. The leftmost and rightmost inequalities here can be checked to hold for all k, m since they follow from the equalities

$$rm^2 - s\big(2km - \tfrac{1}{2}\big) = 4k^2m^2 + m^2 + 2km + \tfrac{1}{2}, \quad 2kq - pm = 4k^2m + 2k.$$

Of course, the remaining inequalities in (2.11) hold for large enough k, m depending on choice of κ_1, κ_2 satisfying (2.9).

The above leads to the construction of various dense G_δ-sets of irrational numbers θ in $(0, 1)$ for each choice of κ_1, κ_2 satisfying (2.9) and choice of dense set D of rational numbers in $\big(0, \frac{1}{2}\big)$. Such irrationals θ possess infinitely many pairs of integers k, m, and associated rational approximations $\frac{2k}{m}, \frac{p}{q}, \frac{r}{s}$ satisfying (2.11). For example, based on the inner inequality in (2.11), one takes a countable intersection of the dense unions of open intervals

$$\mathscr{G} = \mathscr{G}_{\kappa_1,\kappa_2} = \bigcap_{N\geq 1} \bigcup_{\substack{k,m\geq N \\ k/m\in D}} \left(\frac{pq-\kappa_1}{q^2}, \frac{rs+\kappa_2}{s^2}\right). \tag{2.12}$$

One could conceivably construct specific irrationals in the class $\mathscr{G}$.

3. Proof of structure theorem

We begin the proof with the following lemma. If B is a C*-subalgebra of A and $x \in A$, we use the standard notation $d(x, B)$ for the norm distance between x and B: $d(x, B) = \inf\{\|x - y\| : y \in B\}$.

LEMMA 3.1. *Let $\theta > 0$ be an irrational number and M, N positive coprime integers such that $0 < N(N\theta - M) < 1$. Then for each $t \in (0, 1) \cap (\mathbb{Z} + \mathbb{Z}\theta)$ such that*

$$t < \frac{1}{4}(N\theta - M)$$

there exists a cyclic projection h (i.e., $h\sigma^j(h) = 0$ for $j = 1, 2, 3$) of trace

$$\tau(h) = Nt.$$

If, in addition, there is a sequence of rationals M/N such that $0 < N(N\theta - M) < \kappa < 1$ for some fixed κ, then for each $\epsilon > 0$, there are N, M large enough such that

(1) $\|hU - Uh\| < \epsilon$, $\|hV - Vh\| < \epsilon$,

(2) *there is a matrix* C*-*subalgebra* $\mathfrak{M}$ *of* A_θ *having* h *as its unit such that*

$$d(hUh, \mathfrak{M}) < \epsilon, \qquad d(hVh, \mathfrak{M}) < \epsilon.$$

PROOF. Consider the canonical Fourier invariant projection e in A_θ of trace $\tau(e) = N(N\theta - M)$ given by Theorem 2.1 and corresponding isomorphism

$$\eta\colon eA_\theta e \to M_N \otimes A_{\theta'}, \qquad \text{where} \quad \theta' = \frac{c\theta + d}{N\theta - M}$$

and c, d are integers such that $cM + dN = 1$. Write $t = m + n\theta$, for some integers m, n, and let $K = Mn + Nm$ and $L = dn - cm$. Then

$$K\theta' + L = \frac{(Mn + Nm)(c\theta + d) + (dn - cm)(N\theta - M)}{N\theta - M} = \frac{t}{N\theta - M} < \frac{1}{4}$$

so that $K\theta' + L$ is in $\left(0, \frac{1}{4}\right) \cap (\mathbb{Z} + \mathbb{Z}\theta')$. By Theorem 1.6 of [9] (or Theorem 1.5 in [12]), there exists a σ'-cyclic projection h' in $A_{\theta'}$ of trace $K\theta' + L = t/(N\theta - M)$, where σ' is the Fourier transform of $A_{\theta'}$ in Theorem 2.1. This gives the cyclic projection

$$h := \eta^{-1}(I_N \otimes h')$$

of trace

$$\tau(h) = N(N\theta - M) \cdot \frac{t}{N\theta - M} = Nt.$$

Since the isomorphism η is Fourier covariant, as expressed by (2.1), the projection h is a cyclic subprojection of e.

To prove the second assertion of the lemma, assume we have an infinite sequence of rationals M/N such that $0 < N(N\theta - M) < \kappa < 1$ for some fixed κ. In view of the second part of Theorem 2.1, given $\epsilon > 0$ there is N large enough so that $\eta(eUe)$ and $\eta(eVe)$ are to within ϵ of some elements of the matrix algebra M_N. Then

$$\mathfrak{M} := \eta^{-1}(M_N \otimes h') = h\eta^{-1}(M_N)h$$

is a matrix C*-subalgebra of A_θ with identity element h. (So the algebra $\mathfrak{M}$ is cyclic under σ.) As $\eta(h) = I_N \otimes h'$ commutes with $M_N \otimes h'$, the cut downs $hUh = heUeh$ and $hVh = heVeh$ are to within ϵ of elements of $\mathfrak{M}$, hence condition (2) holds, and $\|ex - xe\| < \epsilon$, for $x = U, V$. To see that h is approximately central, let $x = U, V$ and write

$$hx - xh = h(ex - xe) + (ex - xe)h + hexe - exeh$$

so that from $\|ex - xe\| < \epsilon$ one gets $\|hx - xh\| < 2\epsilon + \|hexe - exeh\|$. Further, since η is an isometry we get

$$\|hexe - exeh\| = \|\eta(h)\eta(exe) - \eta(exe)\eta(h)\|$$

and since $\eta(exe)$ is to within ϵ of an element of $M_N \otimes 1$, with which $\eta(h)$ commutes, one gets $\|hexe - exeh\| < 2\epsilon$. Therefore, $\|hx - xh\| < 4\epsilon$ and h is approximately central.

REMARK 3.2. We point out that the proof of this lemma can be modified slightly to give approximately central Fourier invariant projections h of trace Nt (with the $1/4$ factor removed from the hypothesis on t).

We now have the groundwork necessary in order to proceed with the proof of Theorem 1.2.

Fix an irrational θ in the class $\mathcal{G}$ given by (2.12).

The inequalities (2.11) give three rational convergents of θ and three respective numbers

$$0 < 2km - m^2\theta < \tfrac{1}{2}, \qquad 0 < pq - q^2\theta < \kappa_1, \qquad 0 < s^2\theta - rs < \kappa_2.$$

We are interested in the following approximately central canonical matrix projections

$$e_m^-, \qquad e_q^-, \qquad e_s^+$$

with respective traces $2km - m^2\theta$, $pq - q^2\theta$ and $s^2\theta - rs$. From (2.6) and (2.7), we obtain the topological invariants of the last two to be

$$\mathbf{Top}(e_q^-) = \left(\tfrac{1+i}{2}, \tfrac{1+i}{2} i^{-pq}; \tfrac{1}{2}, \tfrac{1}{2}(-1)^p, 1\right)$$
$$\mathbf{Top}(e_s^+) = \left(\tfrac{1-i}{2}, \tfrac{1-i}{2} i^{-rs}; \tfrac{1}{2}, \tfrac{1}{2}(-1)^r, 1\right)$$

as q and s are odd. Since $p \equiv 0 \bmod 4$ and $rs \equiv -1 \bmod 4$ (see first paragraph of Section 2.5), these become

$$\mathbf{Top}(e_q^-) = \left(\tfrac{1+i}{2}, \tfrac{1+i}{2}; \tfrac{1}{2}, \tfrac{1}{2}, 1\right)$$
$$\mathbf{Top}(e_s^+) = \left(\tfrac{1-i}{2}, \tfrac{1+i}{2}; \tfrac{1}{2}, -\tfrac{1}{2}, 1\right).$$

Taking the parity γ of e_s^+ gives

$$\mathbf{Top}(\gamma e_s^+) = \left(\tfrac{1-i}{2}, -\left(\tfrac{1+i}{2}\right); \tfrac{1}{2}, -\tfrac{1}{2}, -1\right)$$

and adding gives

$$\mathbf{Top}(e_q^-) + \mathbf{Top}(\gamma e_s^+) = (1, 0; 1, 0, 0) = \mathbf{Top}(1).$$

Therefore

$$\mathbf{T}(1) - \mathbf{T}(e_q^-) - \mathbf{T}(\gamma e_s^+) = (\tau_0; 0, 0; 0, 0, 0) \tag{3.1}$$

where the trace value τ_0 here is

$$\tau_0 = 1 - (pq - q^2\theta) - (s^2\theta - rs) = (1 + rs - pq) - (s^2 - q^2)\theta.$$

Computing these in terms of the parameters k, m, one gets

$$s^2 - q^2 = 8m^2(2m^2 + n^2) = 8m^2(16k^2m^2 + 8km + 2m^2 + 1) = 4m^2 B$$

and

$$1 + rs - pq = 4m^2(64k^3m + 8km + 24k^2 - 1) = 4m^2 A,$$

where

$$A = 64k^3m + 8km + 24k^2 - 1, \qquad B = 2(16k^2m^2 + 8km + 2m^2 + 1).$$

Thus, we can write

$$\tau_0 = 4m^2(A - B\theta).$$

We now claim that τ_0 is the trace of an approximately central flat projection

$$f = g + \sigma(g) + \sigma^2(g) + \sigma^3(g)$$

whose cyclic subprojection g is approximately central as well. First, it is straightforward to check that

$$2ks - m(r + 4) = 4k^2m + 2k - 3m > 0$$

is positive (for all $k, m \geq 1$), and that one has the equality

$$sA - Br = 1.$$

These give the inequality

$$\frac{4mA - 2k}{4mB - m} < \frac{r}{s} < \theta,$$

from which we get

$$t := m(A - B\theta) < \tfrac{1}{4}(2k - m\theta) < 1. \tag{3.2}$$

To be sure that $A - B\theta > 0$, in view of (2.11) it is enough to see that

$$\theta < \frac{rs + \kappa_2}{s^2} < \frac{A}{B}.$$

Cross multiplying the last inequality here reduces it to $\kappa_2 < \frac{s}{B}$ (again using $sA - Br = 1$) which holds since $\kappa_2 \leq \frac{1}{2}$ and $\frac{1}{2} < \frac{s}{B}$ follows from $2s = B + 4m^2$.

To establish the claim just made, apply Lemma 3.1 with

$$N(N(1-\theta) - M) = m(2k - m\theta) = \tau(e_m^-),$$

i.e., with $N = m$ and $M = m - 2k$, and with $t = m(A - B\theta)$. The hypothesis of this lemma that $t < \frac{1}{4}(2k - m\theta)$ has already been checked in (3.2). Therefore, by Lemma 3.1 there exists an approximately central cyclic projection g of trace

$$\tau(g) = mt = m^2(A - B\theta).$$

The second part of Lemma 3.1 (where the "κ" there can be taken to be $\frac{1}{2}$ in view of the inequalities (2.11) relating θ and $2k/m$) gives the matrix cut down approximation for g. The corresponding flat projection is then

$$f = g + \sigma(g) + \sigma^2(g) + \sigma^3(g)$$

with trace

$$\tau(f) = 4m^2(A - B\theta) = \tau_0.$$

Therefore (3.1) becomes

$$\mathbf{T}(e_q^-) + \mathbf{T}(\gamma e_s^+) + \mathbf{T}(f) = \mathbf{T}(1).$$

where all the underlying projections e_q^-, e_s^+, g and f are approximately central matrix projections. Since the Connes-Chern map $\mathbf{T}$ is injective we get the following equality of classes in $K_0(A_\theta^\sigma)$

$$[e_q^-] + [\gamma e_s^+] + [f] = [1]$$

as required by Definition 1.1. Since the orbifold C*-algebra A_θ^σ has the cancellation property, this equation of K-classes gives equation (1.1) of Theorem 1.2 for some Fourier invariant unitaries w, z – namely, $e_q^- + w\gamma e_s^+ w^* + zfz^* = 1$.

This completes the proof of Theorem 1.2 that the Fourier transform σ is K-inductive on the irrational rotation C*-algebra A_θ.

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REFERENCES

1. Buck, J., and Walters, S., *Connes-Chern characters of hexic and cubic modules*, J. Operator Theory 57 (2007), no. 1, 35–65.
2. Echterhoff, S., Lück, W., Phillips, N. C., and Walters, S., *The structure of crossed products of irrational rotation algebras by finite subgroups of* $SL_2(\mathbb{Z})$, J. Reine Angew. Math. 639 (2010), 173–221.
3. Elliott, G. A., and Evans, D. E., *The structure of the irrational rotation C^*-algebra*, Ann. of Math. (2) 138 (1993), no. 3, 477–501.
4. Lin, H., *Classification of simple tracially AF C^*-algebras*, Canad. J. Math. 53 (2001), no. 1, 161–194.
5. Polishchuk, A., *Holomorphic bundles on* 2*-dimensional noncommutative toric orbifolds*, in "Noncommutative geometry and number theory", Aspects Math., E37, Friedr. Vieweg, Wiesbaden, 2006, pp. 341–359.
6. Walters, S. G., *Chern characters of Fourier modules*, Canad. J. Math. 52 (2000), no. 3, 633–672.
7. Walters, S. G., *K-theory of non-commutative spheres arising from the Fourier automorphism*, Canad. J. Math. 53 (2001), no. 3, 631–672.
8. Walters, S. G., *The AF structure of non commutative toroidal $\mathbb{Z}/4\mathbb{Z}$ orbifolds*, J. Reine Angew. Math. 568 (2004), 139–196.
9. Walters, S. G., *Decomposable projections related to the Fourier and flip automorphisms*, Math. Scand. 107 (2010), no. 2, 174–197.
10. Walters, S. G., *Toroidal orbifolds of $\mathbb{Z}_3$ and $\mathbb{Z}_6$ symmetries of noncommutative tori*, Nuclear Phys. B 894 (2015), 496–526.
11. Walters, S. G., *Continuous fields of projections and orthogonality relations*, J. Operator Theory 77 (2017), no. 1, 191–203.
12. Walters, S. G., *Semiflat orbifold projections*, Houston J. Math. 44 (2018), no. 2, 645–663.

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