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BORSUK'S THEORY OF SHAPE AND ČECH HOMOTOPY

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K. Borsuk introduced his theory of shapes of metric compacta in his papers [2] and [3]. More recently S. Mardešić and J. Segal have extended his theory to include compact Hausdorff spaces by using ANR(M)-systems, [9]. They also showed in [10] that, for metric compacta, the two approaches were equivalent. In this paper, we shall investigate the relationship between the ANR(M)-systems approach to "shape" and the constructions and results of [12], [13] and [14].

Whether detailed calculations of "shape" invariants will be possible using the connections revealed in this paper remains to be seen.

First we must review some of the ideas from [9] and [12]. By an ANR(M) we mean a compact absolute neighbourhood retract for metric spaces. (In fact we shall really only be dealing with the case when the ANR(M)'s are polyhedra).

In [9] a construction is given (Theorem 7) which associates to each compact Hausdorff space, X, an inverse system $\boldsymbol{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ such that

(i) X_α is a polyhedron for each $\alpha \in A$,
(ii) A is a closure-finite directed set with cardinality not greater than the weight of X,
(iii) If $\alpha' \geq \alpha$, then $p_{\alpha\alpha'}\colon X_{\alpha'} \to X_\alpha$ is continuous,
(iv) $X = \lim_{\alpha \in A} X_\alpha$,
(v) The choice of $\boldsymbol{X}$ is unique up to homotopy type of ANR(M)-systems.

This homotopy type is called the *shape of* X (denoted by $\mathrm{Sh}(X)$).

A continuous map $f\colon X \to Y$ "lifts" to give a map of ANR(M)-systems $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$.

The category of pro-objects in a category, C, was introduced by Grothendieck [8] and has been used with good effect by Artin and Mazur [1]. It is usually denoted by $\mathbf{pro}(C)$. We shall need this in the cases where C is the category, K, of Kan complexes and homotopy classes of simplicial maps (e.g. [7]) and when C is H, the homotopy category of CW-complexes. The categories $\mathbf{pro}(K)$ and $\mathbf{pro}(H)$ are equivalent.

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Given any space X, the nerve or Čech complex of X ([12]) is regarded as an object of $\mathbf{pro}(K)$ and is denoted by $C(X;\ )$. In the following we obtain a link between shape theory, and "Čech homotopy" by comparing $C(X;\ )$ and $\boldsymbol{X}$ as objects of $\mathbf{pro}(H)$. (Strictly speaking the bonding maps or structure maps of objects in $\mathbf{pro}(H)$ are homotopy classes of maps and not continuous maps as in $\boldsymbol{X}$.)

LEMMA 1. *For any compact Hausdorff space* X,

$$|C(X;\ )| \approx \boldsymbol{X}$$

in $\mathbf{pro}(H)$

PROOF. Let $\boldsymbol{X}=\{X_\alpha,p_{\alpha\alpha'},A\}$ be as above and let $p_\alpha\colon X \to X_\alpha$ be the canonical projection. We will need the following result from [6] (Lemmas X,3.7 and 3.8 and the proof of Theorem X, 3.1).

The finite open covers, γ, of X of the form $\gamma = p_\alpha{}^{-1}(\delta)$ for some finite open cover, δ, of X_α and such that

$$C(X;\gamma) \approx C(X_\alpha;\delta)$$

form a cofinal subcategory of $\mathrm{Cov}(X)$, the category of open covers of X. Moreover the above isomorphism can be chosen to be natural with respect to refinements.

If we regard $\boldsymbol{X}$ as a functor,

$$\boldsymbol{X}\colon A \to H$$

we can form an interwoven system of polyhedra, $C(\boldsymbol{X};\ )$, defined on the indexing category

$$\tilde{A} = \{(\alpha,\delta) : \delta \text{ is a finite open cover of } X_\alpha \text{ and } \alpha \in A\}$$

where $(\alpha_1,\delta_1) \geq (\alpha_2,\delta_2)$ if and only if $\alpha_1 \geq \alpha_2$ and $\delta_1 \geq p^{-1}_{\alpha_2\alpha_1}(\delta_2)$. $C(\boldsymbol{X};\ )$ is defined by

$$C(\boldsymbol{X};(\alpha,\delta)) = C(X_\alpha;\delta)\,.$$

The above result can be summarized as

$$C(X;\ ) \approx C(\boldsymbol{X};\ ) \quad \text{in } \mathbf{pro}(K)\,.$$

Also if, for any polyhedron, we consider the cofinal sequence, of open star covers of triangulations [6] we get a sequence of open covers of X and the system of simplicial sets, obtained by interweaving the induced open star covers of X for the various triangulations of the polyhedra in $\boldsymbol{X}$, gives a cofinal system of covers of X. Thus $C(X;\ )$ is isomorphic in $\mathbf{pro}(K)$ to the interwoven Čech system $C(\boldsymbol{X};\ )$ of $\boldsymbol{X}$. It follows that

$|C(X;\ )|$ and $|C(\mathbf{X};\ )|$ are isomorphic in $\mathbf{pro}(H)$, however since $|C(\mathbf{X};\ )|$ consists cofinally of "realisations of triangulations" of $\mathbf{X}$, it collapses to something isomorphic to $\mathbf{X}$. This completes the proof.

As yet there seems no obvious way of extending Lemma 1 to give an analogous result for metrizable spaces (see Borsuk's paper, [5]).

Lemma 2. *Let $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ be a map of* ANR(M)*-systems, where $\mathbf{X}$ is associated with a compactum X and $\mathbf{Y}$ with a compactum Y. Then $\mathbf{f}$ induces a map*

$$C(f): C(X;\ ) \to C(Y;\ )\,.$$

Proof. If $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ is a map of ANR(M)-systems in the sense of [9], then $\mathbf{f}$ induces a map

$$C(\mathbf{f}): C(\mathbf{X};\ ) \to C(\mathbf{Y};\ )$$

in $\mathbf{pro}(K)$. This map composed with the two isomorphisms

$$C(X;\ ) \to C(\mathbf{X};\ ),\qquad C(\mathbf{Y};\ ) \to C(Y;\ )$$

gives the required map

$$C(f): C(X;\ ) \to C(Y;\ )$$

Remarks. 1. If $\mathbf{f} \simeq \mathbf{g}$, then $C(f) = C(g)$ in $\mathbf{pro}(K)$. Thus lemma 2 shows that the induced map of Čech cohomology

$$\mathbf{f}^*:\ \check{H}^q(Y; G) \to \check{H}^q(X; G)$$

of Theorem 15 in [9] and induced map of Čech homology of Theorem 11.6 in [2] can be realised at the semi-simplicial level and does not rely on the algebraic constructs used in those papers.

2. In lemma 3 below, we shall prove a partial converse of lemma 2 in the case that X is a metric compactum. This restriction seems somewhat ridiculous, but as yet I have found no way around it.

Lemma 3. *Let $f: C(X;\ ) \to C(Y;\ )$ be a map in* $\mathbf{pro}(K)$, *where X is a compact metric space. If $\mathbf{X}, \mathbf{Y}$ are* ANR(M)*-systems associated to X, Y respectively then there is a map*

$$\mathbf{f}: \mathbf{X} \to \mathbf{Y}$$

of ANR(M)*-systems corresponding to f.*

Proof. The map $|f|$ belongs to $\mathbf{pro}(H)$ and since $\mathbf{X}$ and $\mathbf{Y}$ are cofinally equivalent to $|C(X;\ )|$ and $|C(Y;\ )|$ in $\mathbf{pro}(H)$, $|f|$ can be thought of as a map $f\colon \boldsymbol{X} \to \boldsymbol{Y}$, again in **pro**$(H)$, but not, as yet, in ANR(M)-systems.

If $\boldsymbol{X}=\{X_i;\, p_{ij}, \mathsf{N}\}$ and $\boldsymbol{Y}=\{Y_\beta;\, q_{\beta\beta'}, B\}$, then $|f|$ is defined by a family $\{f_\beta\}_{\beta\in B}$ where

$$f_\beta \in \lim_{i\in \mathsf{N}}[X_i, Y_\beta]\,.$$

If $\beta' \geq \beta$ and $j \geq i$ are such that there is an $f_{\beta i}$ and $f_{\beta' j}$ representing f_β and $f_{\beta'}$ respectively, then there is a $k \geq j, i$ such that

$$f_{\beta i}p_{ik} \simeq q_{\beta'\beta}f_{\beta' j}p_{jk}\,.$$

Now we define an increasing map $\varphi\colon B \to N$ by

$$\varphi(\beta) = \inf\{i \in \mathsf{N} : \exists f_{\beta i} \in [X_i, Y_\beta] \text{ representing } f_\beta$$
$$\text{and for all } \beta' \leq \beta,\, j \leq i \text{ such that } f_{\beta j} \text{ exists } f_{\beta' j}p_{ji} \simeq q_{\beta\beta'}f_{\beta i}\}\,.$$

φ is well defined because B is closure finite, so the second condition has only to be satisfied for finitely many β'.

The map φ is increasing, but not necessarily strictly increasing. (The definition of φ when X is not metric has so far defeated me. We will need the fact that there is a map $f_{\beta\varphi(\beta)}$ representing f_β and this cannot be guaranteed if the above definition is used in the general case.)

We now choose for each $\beta \in B$ a continuous map

$$\varphi_\beta\colon X_{\varphi(\beta)} \to Y$$

in the homotopy class $f_{\beta\varphi(\beta)}$ which exists by the definition. The diagram (for $\beta' \geq \beta$)

$$\begin{array}{ccc} X_{\varphi(\beta)} & \xrightarrow{\ \varphi_\beta\ } & Y_\beta \\ \uparrow{\scriptstyle p_{\varphi(\beta)\varphi(\beta')}} & & \uparrow{\scriptstyle q_{\beta\beta'}} \\ X_{\varphi(\beta')} & \xrightarrow[\ \varphi_{\beta'}\ ]{} & Y \end{array}$$

commutes up to homotopy. Clearly the φ_β define a map $\boldsymbol{f}$ of ANR(M)-systems in the sense of [9].

Although $\boldsymbol{f}$ is not unique, its homotopy class is clearly uniquely determined by f.

THEOREM 1. *If X and Y are metric compacta, then they have the same shape if and only if*

$$C(X;\ ) \approx C(Y;\ )$$

in **pro**(K) *(or equivalently* $|C(X;\ )| \approx |C(Y;\ )|$ in **pro**(H)*).*

PROOF. Suppose $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, then there are maps

$$\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y} \quad \text{and} \quad \boldsymbol{g}\colon \boldsymbol{Y} \to \boldsymbol{X}$$

of ANR(M)-systems $\boldsymbol{X}$ and $\boldsymbol{Y}$ associated with X and Y, such that $\boldsymbol{fg} \cong \mathbf{1}_{\boldsymbol{Y}}$ and $\boldsymbol{gf} \cong \mathbf{1}_{\boldsymbol{X}}$. Using lemma 2 this implies that there are maps

$$C(f)\colon C(X;\ ) \to C(Y;\ ), \quad C(g)\colon C(Y;\ ) \to C(X;\ ).$$

It is easy to check that this construction is functorial and hence

$$C(f)C(g) = C(fg) = C(1_X) = 1_{C(X;\ )}$$

similarly

$$C(g)C(f) = 1_{C(Y;\ )}$$

so

$$C(X;\ ) \approx C(Y;\ )$$

in $\mathbf{pro}(K)$. (Note this half of the result does not require that X and Y be metric.)

Now suppose there are maps

$$f\colon C(X;\ ) \to C(Y;\ ), \quad g\colon C(Y;\ ) \to C(X;\ )$$

such that fg is cofinally equivalent to $1_{C(X;\ )}$ and gf is cofinally equivalent to $1_{C(Y;\ )}$. Lemma 3 gives us maps

$$\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y} \quad \text{and} \quad \boldsymbol{g}\colon \boldsymbol{Y} \to \boldsymbol{X}.$$

Further $\boldsymbol{fg} \cong \mathbf{1}_{\boldsymbol{X}}$, $\boldsymbol{gf} \cong \mathbf{1}_{\boldsymbol{Y}}$, by construction of $\boldsymbol{f}$ and $\boldsymbol{g}$, the definitions of isomorphism in $\mathbf{pro}(K)$ (see [1]) and "equivalence" of ANR(M)-systems.

We are now in a position to start a comparison of some of the other ideas of shape theory, in particular the idea of a movable compactum (see [4] and [11]) with some of the concepts from [13] and [14].

THEOREM 2. *An ∞-stable metric compactum has the same shape as a polyhedron.*

PROOF. X is ∞-stable implies that the bonding maps in $C(X;\ )$ are cofinally isomorphisms in K. Hence for some finite open cover, α, of X, $C(X;\ )$, and the constant system $C(X;\alpha)$ are isomorphic in $\mathbf{pro}(K)$ and hence by lemma 1, $\boldsymbol{X}$ and the polyhedron $|C(X;\alpha)|$ are isomorphic in $\mathbf{pro}(H)$. Using the proof of lemma 3, we see that $\boldsymbol{X}$ and $|C(X;\alpha)|$ have the same shape.

COROLLARY. *An ∞-stable metric compactum, X, is a fundamental absolute neighbourhood retract (in the sense of Borsuk) and hence is movable.*

PROOF. This follows from the fact that X has the shape of a polyhedron.

THEOREM 3. *If a compactum has the same shape as a polyhedron then it is stable.*

PROOF. There is a polyhedron, P, such that $X \approx P$. Thus $|C(X;\ )| \approx |C(P;\ )|$ in **pro**(H) and so $C(X;\ ) \approx C(P;\ )$. Since P is stable, so is X.

It would seem likely that if X is stable then X is movable but as yet there is no way to ensure that $C(X;\ )$ "stabilises" to a polyhedral simplicial set. If this conjecture is true it will clarify the connection between "movable" and "stable", two ideas which are intuitively closely connected. There remain, of course, two other unresolved questions:

1. *Is there a movable (metric?) compactum which is not stable?*

2. *Is there a characterization of stability in terms of the intrinsic topological properties of the space.*

An answer to this second question would go a long way towards solving problem (6.2) in [3].

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