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A ČECH-HUREWICZ ISOMORPHISM THEOREM FOR MOVABLE METRIC COMPACTA

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In [7] we proved that the condition of stability introduced there gave good exactness results for Čech homology and homotopy theories. Consequently we proved that, at least for metric compacta, stability implied movability [7] and [8]. Hence it seemed natural to investigate the corresponding results for movable metric compacta. R. Overton [5] has managed to obtain the first result in this direction by proving that, for a movable pair of metric compacta, the Čech homology sequence is exact. So of the trio of results singled out in [6] and [7] as applications of stability, there remains the "Čech-Hurewicz Isomorphism Theorem" and the "Čech-Van Kampen Theorem" for movable metric compacta. The methods introduced here give a proof of both relative and absolute forms of the "Čech-Hurewicz Theorem" but, because the methods rely heavily on the groups concerned being abelian, they give no indication of how to prove the Van Kampen Theorem in this case. Incidentally, although not explicitly proved here, the preliminary results obtained here give a proof of the exactness of a large part of the Čech homotopy sequence of a movable pair; again they don't allow the lower, possibly non-abelian, groups to be studied.

1. Movability and Mittag-Leffler.

First we review some definitions. Overton [5] makes the following definition (see also [4]).

A pair of metric compacta (X,A) is *movable* if it is the inverse limit of an ANR(M)-pair sequence

$$\{(X_i,A_i),p_{ij}:\ (X_j,A_j) \to (X_i,A_i)\}$$

with the property that for any i there exists $j \geqq i$ such that if $k \geqq j$ then there is a map $s_{kj}\colon (X_j,A_j) \to (X_k,A_k)$ with $p_{ik}s_{kj}$ homotopic to p_{ij}.

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We will add the restriction that if (X, A, x_0) is a pointed pair, then all maps will be assumed to preserve the relevant base points.

Next we need a definition from the algebraic theory of projective systems. (See [1] or [2].)

Let $(A_\alpha)_{\alpha \in \mathrm{N}}$ be a projective system (indexed by the non-negative integers) of abelian groups, then $(A_\alpha)_{\alpha \in \mathrm{N}}$ is said to satisfy the Mittag-Leffler condition (ML) if for each α, there is a $\beta \geqq \alpha$ such that

$$\mathrm{Im}(A_\beta \to A_\alpha) = \mathrm{Im}(A_\gamma \to A_\alpha)$$

for all $\gamma \geqq \beta$.

If (X, A) is a movable pair of metric compacta then for each $n \geqq 0$ the projective system $(H_n(X_i, A_i))_{i \in \mathrm{N}}$ satisfies the following condition:

Given any $i \in \mathrm{N}$, there is a $j \geqq i$ such that for all $k \geqq j$ there is a homomorphism

$$\sigma_{k,j} : H_n(X_j, A_j) \to H_n(X_k, A_k)$$

such that

$$p_{ij*} = p_{ik*} \sigma_{k,j} .$$

Hence

$$\mathrm{Im}(p_{ij*}) \subset \mathrm{Im}(p_{ik*}) .$$

But by the definition of the maps $(p_{ij})_{j \geqq i}$, we have

$$p_{ik} = p_{ij} p_{jk} ,$$

hence

$$\mathrm{Im}(p_{ik*}) = \mathrm{Im}(p_{ij*} p_{jk*}) \subset \mathrm{Im}(p_{ij*}) ,$$

and so we have the result:

LEMMA 1. *If (X, A) is a movable pair of metric compacta, then the corresponding relative homology systems, which we will denote by*

$$H_n(\boldsymbol{X}, \boldsymbol{A}) \quad \textit{for } n = 0, 1, 2 \ldots$$

satisfy (ML).

Similarly we can prove the absolute version.

LEMMA 2. *If X is a movable metric compactum then $H_n(\boldsymbol{X})$ satisfies* (ML).

In fact since the proofs use only the functoriality of H_n on the homotopy category, we also get

LEMMA 1′. *If* (X, A, x_0) *is a movable pointed pair of metric compacta then*

$$\pi_n(X, A, x_0) \quad (for\ n > 2)$$

satisfies (ML).

LEMMA 2′. *If* X *is a movable pointed metric compactum then* $\pi_n(X, x_0)$ (*for* $n > 1$) *satisfies* (ML).

LEMMA 3. *If* $A = (A_i)_{i \in \mathsf{N}}$ *is a projective system of abelian groups satisfying* (ML) *and* $\varprojlim A = 0$, *then* A *is isomorphic to* $\mathbf{0}$ *in* pro (Ab) (*see* [3]) *the category of projective systems of abelian groups.*

PROOF. $\varprojlim A = 0$, so for each i the natural map

$$\varprojlim A \to A_i$$

is a monomorphism. By a result of Verdier [10, Proposition 7] A is essentially constant, i.e. is isomorphic in pro (Ab) to a projective system indexed by a one-element ordered set. This projective system must therefore be the zero system, $\mathbf{0}$.

Although it seems probable that Lemma 3 holds for arbitrary inverse systems of groups, I have been unable to find a proof of this or to prove it directly myself.

2. The Čech form of the Hurewicz Isomorphism Theorem.

As lemma 3 requires that the A_i are abelian groups, we cannot (without a much deeper analysis of the background to this result) use it for handling the low dimensions in this theorem ($n = 1$ in the absolute case and $n = 2$ for the relative case). To avoid this difficulty we shall assume that in the absolute case $\pi_1(X_i, x_{0,i}) = 0$ for each i and in the relative case $\pi_2(X_i, A_i, x_{0,i}) = 0$ and $\pi_1(A_i, x_{0,i}) = 0$ for each i. [In fact if X is 1-stable or (X, A) is 2-stable and A_i 1-stable (see [7]) then this will follow from $\check{\pi}_1(X, x_0) = 0$ and $\check{\pi}_2(X, A, x_0) = 0$ and $\check{\pi}_1(A, x_0) = 0$.] (For a definition of the Čech homotopy groups of a space see [7, part I].)

THEOREM 1. *If* (X, A, x_0) *is a connected movable, pointed pair of metric compacta* (*satisfying the condition above*) *then, if there is an* $n > 2$ *such that* $\check{\pi}_i(X, A, x_0) = 0$ *for* $2 \leqq i < n$, *the Hurewicz homomorphism*

$$h : \check{\pi}_n(X, A, x_0) \to \check{H}_n(X, A; \mathsf{Z})$$

is an isomorphism and $\check{H}_i(X, A; \mathsf{Z}) = 0$ *for* $2 \leqq i < n$.

THEOREM 2. *If* (X,x_0) *is a connected movable pointed metric compactum (satisfying the condition mentioned above) then, if there is a* $n>1$ *such that* $\check{\pi}_i(X,x_0)=0$ *for* $1\leqq i<n$, *the Hurewicz homomorphism*

$$h: \check{\pi}_n(X,x_0)\to \check{H}_n(X;\mathsf{Z})$$

is an isomorphism and $\check{H}_i(X;\mathsf{Z})=0$ *for* $1\leqq i<n$.

PROOF OF THEOREM 1. We suppose that $(\boldsymbol{X},\boldsymbol{A},\boldsymbol{x}_0)$ is a pointed ANR(M)-pair sequence, satisfying

(i) $\check{\pi}_2(X_i,A_i,x_{0,i})=0$ for all i and $\pi_1(A_i,x_{0,i})=0$;
(ii) $(\boldsymbol{X},\boldsymbol{A},\boldsymbol{x}_0)$ is movable.

Since $\check{\pi}_r(X,A,x_0)=0$ for $r=2,\ldots,n-1$, lemma 3 and (ii) tells us that given any $i\in\mathsf{N}$, there is a $j\geqq i$ such that for each $k\geqq j$, we have

$$p_{ik}\cong p_{ik}s_{kj}p_{jk}$$

and that we can choose k so that for all $l\geqq k$ the map

$$p_{jl\#}: \pi_r(X_l,A_l,x_{0,l})\to\pi_r(X_j,A_j,x_{0,j})$$

is the zero map for $r=2,\ldots,n-1$. (Here we are tacitly using the continuity of the Čech homotopy group function and the fact that on ANR(M)'s π_r and $\check{\pi}_r$ agree for all r.)

Following Spanier [9, p. 391] we let $\Delta(X_i)$ denote the singular chain complex of X_i and $\Delta(X_i,A_i,x_{0,i})^{(r)}$ the subcomplex of $\Delta(X_i)$ generated by those singular simplexes $\sigma\colon\Delta^q\to X_i$ with

$$(\mathrm{sk}_0(\Delta^q)) = \{x_{0,i}\}$$

and

$$(\mathrm{sk}_r(\Delta^q)) \subset A_i\,,$$

where sk_n denotes the nth skeleton functor.

We denote by $H_q^{(r)}(X_i,A_i,x_{0,i})$ the homology of the pair

$$(\Delta(X_i,A_i,x_{0,i})^{(r)},\Delta(X_i,A_i,x_{0,i})^{(r)}\cap\Delta(A_i))\,.$$

By Spanier's results [9, $B_n\Rightarrow\Phi_n$ p. 397] we have

$$\pi_n(X_i,A_i,x_{0,i})\approx H_n^{(n-1)}(X_i,A_i;\mathsf{Z})\,.$$

This result "homotopy additivity lemma $\Rightarrow\varphi''$ is an isomorphism" does not depend on the connectivity assumptions. (The conditions we have placed on (X,A,x_0) imply that we can avoid Spanier's π_n' notation.) Thus using continuity of $\check{\pi}_n$, we have

$$\check{\pi}_n(X,A,x_0)\approx \varprojlim_{i\in\mathsf{N}} H_n^{(n-1)}(X_iA_i;\mathsf{Z})\,.$$

It therefore remains only to prove the analogue of Theorem 7.4.8 in [9], namely that for (X,A,x_0) as above

$$\lim_{\leftarrow} H_n^{(n-1)}(X_i,A_i;\mathsf{Z}) \approx \check{H}_n(X,A;\mathsf{Z})\ .$$

The proof given for the result in [9] is easily generalised to this situation as follows:

We define, for each q and each $\sigma: \Delta^q \to X_l$, a map $\tau(\sigma): \Delta^q \to X_j$ with $\tau(\sigma) \in \Delta(X_j,A_j,x_{0,j})^{(n-1)}$ and with a fixed homotopy $P(\sigma)$ from $p_{jl}\sigma$ to $\tau(\sigma)$.

If $q=0$ then $\sigma: \Delta^0 \to X_l$ is a point and since X_l is path connected there is a map $P(\sigma): \Delta^0 \times I \to X_j$ with

$$P(\sigma)(\Delta^0 \times 0) = p_{jl}\sigma(\Delta^0) \quad \text{and} \quad P(\sigma)(\Delta^0 \times 1) = x_{0,j}\ .$$

[If $\sigma(\Delta)=x_{0,l}$, we take $P(\sigma)$ to be the constant map to $x_{0,j}$.]

Assume $0<q<n$ and that $P(\sigma)$ has been defined for all σ of degree $<q$ and that for each of these σ, $P(\sigma)$ has the following properties:

a) $P(\sigma)\,|\,\Delta^q \times 0 = p_{jl}\sigma$.

b) $\tau(\sigma) = P(\sigma)\,|\,\Delta^q \times 1$ is in $\Delta(X_j,A_j,x_{0,j})^{(n-1)}$ and if $\sigma \in \Delta(X_l,A_l,x_{0,l})^{(n-1)}$, $P(\sigma)$ is the map

$$\Delta^q \times I \xrightarrow{P} \Delta^q \xrightarrow{\sigma} X_l \xrightarrow{p_{jl}} X_j\ .$$

c) If $e_q^{\,i}: \Delta^{q-1} \to \Delta^q$ omits the ith vertex, then

$$P(\sigma)(e_q^{\,i} \times 1) = P(\sigma^{(i)})\ .$$

Now assume σ has degree q. If $\sigma \in \Delta(X_l,A_l,x_{0,l})^{(n-1)}$, define $P(\sigma)$ to be the map

$$\Delta^q \times I \xrightarrow{P} \Delta^q \xrightarrow{\sigma} X_l \xrightarrow{p_{jl}} X_j\ .$$

If σ is not in $\Delta(X_l,A_l,x_{0,l})^{(n-1)}$, conditions a) and c) above define $P(\sigma)$ on $\Delta^q \times 0 \cup \dot{\Delta}^q \times I$ and we let

$$f: \Delta^q \times 0 \cup \dot{\Delta}^q \times I \to X_j$$

be this map. Let

$$h: E^q \times I \to \Delta^q \times I$$

be a homeomorphism, as in [9, p. 392–393]. Then we can define a map

$$f': (E^q, S^{q-1}) \to (X_l, A_l)$$

by

$$f'(z) = s_{lj}\big(f(h(z))\big)\ .$$

But $\operatorname{Im} p_{jl\#} = 0$ so there is a homotopy

$$H: (E^q \times I, S^{q-1} \times I) \to (X_j, A_j)$$

from $p_{jl}f'$ to some map of E^q into A_j we take the composite

$$\Delta^q \times I \overset{h^{-1}}{\to} E^q \times I \overset{H}{\to} X_j$$

as $P(\sigma)$. Note that by the properties of $s_{lj}, p_{jl}s_{lj} \simeq 1_{X_l}$, so we have only changed the "old" $P(\sigma)$'s by a homotopy.

In this way we define $P(\sigma)$ for all degrees $q < n$ and, on noting that a singular simplex, σ, of degree $> n$ is mapped to $\Delta(X_j, A_j, x_{0,j})^{(n-1)}$ if and only if all proper faces are in $\Delta(X_j, A_j, x_{0,j})^{(n-1)}$ we see that any map $P(\sigma): \Delta^q \times I \to X$ satisfying conditions a) and c) above will automatically satisfy condition b). Such maps will exist by the homotopy extension property.

Using these methods we arrive at a chain map

$$\tau: \Delta(X_l) \to \Delta(X_j, A_j, x_{0,j})$$

such that $p_{jl\#} \simeq \tau$ and hence $s_{lj\#}p_{jl\#} \simeq s_{lj\#}\tau$.

$$s_{lj\#}\tau: \Delta(X_l) \to \Delta(X_l, A_l, x_{0,l})^{(n-1)}$$

is not necessarily a chain homotopy inverse for the inclusion map

$$j_l: \Delta(X_l, A_l, x_{0,l})^{(n-1)} \subsetneq \Delta(X_l)$$

but

$$p_{il\#}^{(n-1)} \circ s_{lj\#} \circ \tau \circ j_l \simeq p_{il\#} \circ s_{lj\#} \circ p_{jl\#} \circ j_l \simeq p_{il\#} \circ j_l$$

where

$$p_{il\#}^{(n-1)} = p_{il\#} \,|\, \Delta(X_l, A_l, x_{0,l})^{(n-1)}$$

hence $p_{il\#} \circ j_l$ is homotopic to a map into $\Delta(X_i, A_i, x_{0,i})^{(n-1)}$, that is,

$$p_{il*} \circ j_{l*} : H_q^{(n-1)}(X_l, A_l; \mathsf{Z}) \to H_q(X_i, A_i; \mathsf{Z})$$

has image contained in $H_q^{(n-1)}(X_i, A_i; \mathsf{Z})$ and the diagram

$$\begin{array}{ccc} H_q^{(n-1)}(X_i, A_i; \mathsf{Z}) & \supsetneq & \operatorname{Im}(p_{il*} \circ j_*) \\ \uparrow {\scriptstyle p_{il*}^{(n-1)}} & & \uparrow {\scriptstyle p_{il*} \circ j_l} \\ H_q^{(n-1)}(X_l, A_l; \mathsf{Z}) & \text{———} & \end{array}$$

commutes.

Similarly $p_{il\#} \circ j_l \circ s_{lj\#} \circ \tau \simeq p_{il\#}$. Hence we have an isomorphism

$$\underleftarrow{\lim}_{\mathsf{N}} H_q^{(n-1)}(X_i, A_i; \mathsf{Z}) \approx \check{H}_q(X, A; \mathsf{Z})$$

using the continuity of $\check{H}_q$ and its agreement with H_q on ANR(M)'s.

The proof of Theorem 2 is similar but easier.

Addendum.

K. Kuperberg [Fund. Math. 77 (1972), 21–32] has obtained a result similar to Theorem 2 of this paper. At the time of writing, I have not seen a copy of that paper, so cannot say if a result similar to Theorem 1 is to be found there. Also J. B. Quigley: *Equivalence of fundamental and approaching groups of movable pointed compacta* (preprint) has obtained the exactness of the homotopy sequence mentioned in the introduction by a completely different approach.

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