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ON THE HOMOLOGY OF INTERSECTIONS OF COMPLEX PROJECTIVE MANIFOLDS

MOGENS ESROM LARSEN

1. Statement of results.

1.1. This note is concerned with the homology and cohomology of a complex submanifold in a complex projective space, which occurs as an intersection of two high-dimensional complex submanifolds.

Let P_n denote the complex projective space of dimension n, and let $A \subseteqq \mathsf{P}_n$ and $B \subseteqq \mathsf{P}_n$ be submanifolds of dimensions a and b respectively. Suppose, that $2a \geqq n+1$ and $2b \geqq n$, then $a+b>n$ and from [2, proposition 4], $A \cap B$ is connected. Suppose further that $A \cap B$ is a submanifold of P_n. Throughout this note let $s=\min\{2b-n, 2a-n-1\}$. The results are stated in 1.2. and 1.3.

1.2. THEOREM 1. *Let A, B and $A \cap B$ be submanifolds of P_n and* $\dim A = a$, $\dim B = b$, *and* $s=\min\{2b-n, 2a-n-1\}$. *Then the inclusion $A \cap B \subseteqq B$ induces isomorphisms*

$$H^i(B; \mathsf{Z}) \underset{\rightarrow}{\cong} H^i(A \cap B; \mathsf{Z}) \cong \begin{cases} 0 & \text{for } i \text{ odd}, \\ \mathsf{Z} & \text{for } i \text{ even}, \end{cases}$$

$$H_i(A \cap B; \mathsf{Z}) \underset{\rightarrow}{\cong} H_i(B; \mathsf{Z}) \cong \begin{cases} 0 & \text{for } i \text{ odd}, \\ \mathsf{Z} & \text{for } i \text{ even}, \end{cases}$$

for $i \leqq s$. Further the relative groups satisfy

$$H^i(B, A \cap B; \mathsf{Z}) = 0 \quad \text{for } i \leqq s+1,$$

$$H_i(B, A \cap B; \mathsf{Z}) = 0 \quad \text{for } i \leqq s+1.$$

1.3. THEOREM 2. *Under the conditions of theorem* 1 *and further $\pi_1(A \cap B) = 0$, the relative groups*

$$\pi_i(B, A \cap B) = 0 \quad \text{for } i \leqq s+1,$$

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and the inclusion $A \cap B \subseteq B$ *induces isomorphisms*

$$\pi_i(A \cap B) \underset{\rightarrow}{\cong} \pi_i(B) \cong \begin{cases} 0 & \text{for } i \neq 2,\ i \leqq s\,, \\ \mathsf{Z} & \text{for } i = 2\,. \end{cases}$$

1.4. These results generalize the classical theorem of Lefschetz, when A is a hypersurface in P_n, cf. [5, § 7].

2. The Hopf fibration.

2.1 P_n is the set of one-dimensional subspaces of C^{n+1} and

$$S^{2n+1} = \{z \in \mathsf{C}^{n+1} \mid |z| = 1\}\,.$$

The Hopf fibration $h\colon S^{2n+1} \to \mathsf{P}_n$ is the restriction of the obvious map $\mathsf{C}^{n+1} \setminus \{0\} \to \mathsf{P}_n$.

If $X \subseteq \mathsf{P}_n$, we put $\hat{X} = h^{-1}(X) \subseteq S^{2n+1}$. The space $\hat{X}$ is the total space in a fiber bundle over X with fiber S^1.

2.2. The following fact is well-known. For $X \subseteq \mathsf{P}_n$ there is a commutative diagram

$$\begin{array}{ccccccccc}
\ldots \to & \pi_m(S^{2n+1}) & \to & \pi_m(S^{2n+1},\hat{X}) & \to & \pi_{m-1}(\hat{X}) & \to & \pi_{m-1}(S^{2n+1}) & \to \ldots \\
& \downarrow{\scriptstyle \pi_m(h)} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow{\scriptstyle \pi_{m-1}(h)} & \\
\ldots \to & \pi_m(\mathsf{P}_n) & \to & \pi_m(\mathsf{P}_n,X) & \to & \pi_{m-1}(X) & \to & \pi_{m-1}(\mathsf{P}_n) & \to \ldots
\end{array}$$

The map $\pi_m(h)$ is an isomorphism for $m \neq 2$.

2.3. From 2.2. follows, that if $\pi_m(\hat{X}) = 0$, then $\pi_m(\mathsf{P}_n, X) = 0$ for $1 \leqq m \leqq s+1$.

2.4. From 2.2. follows further, that if $\pi_1(X) = 0$, then $\pi_1(\hat{X})$ is abelian and hence isomorphic to $H_1(\hat{X};\mathsf{Z})$. If further $H_m(\hat{X};\mathsf{Z}) = 0$ for $1 \leqq m \leqq s$, then by the Hurewicz isomorphism theorem $\pi_m(\hat{X}) = 0$ for $1 \leqq m \leqq s$, and hence it follows from 2.3. that $\pi_m(\mathsf{P}_n, X) = 0$ for $1 \leqq m \leqq s+1$.

2.5. Throughout this paper let Z denote one of the groups Z or Z/p for p prime. Let $D \subseteq E \subseteq \mathsf{P}_n$. From the general Gysin cohomology sequence

$$\ldots \to H^{m-2}(E,D;Z) \to H^m(E,D;Z) \to$$
$$\to H^m(\hat{E},\hat{D};Z) \to H^{m-1}(E,D;Z) \to \ldots\,,$$

we deduce that if

$$H^{m-1}(E,D;Z) = H^m(E,D;Z) = 0$$

then $H^m(\hat{E},\hat{D};Z) = 0$. Also if $H^m(\hat{E},\hat{D};Z) = 0$ for $1 \leqq m \leqq s$ then

$$H^{m-2}(E,D;Z) \cong H^m(E,D;Z)$$

for $3 \leqq m \leqq s$. In the absolute case ($D = \emptyset$), we have that $H^m(\hat{E}; Z) = 0$ for $1 \leqq m \leqq s$ implies that

$$H^{m-2}(E; Z) \to H^m(E; Z)$$

is an isomorphism for $2 \leqq m \leqq s$.

2.6. Let

$$\begin{array}{ccc} \hat{N} & \overset{\hat{\varphi}}{\to} & S^{2n+1} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ N & \overset{\varphi}{\to} & \mathsf{P}_n \end{array}$$

be a cartesian diagram. In particular $g: \hat{N} \to N$ is a S^1-bundle. Let $\tilde{D} \subset \tilde{E}$ be a pair in N, and put $\varphi(\tilde{D}) = D$, $\varphi(\tilde{E}) = E$. Then the general Gysin cohomology sequences applied to $h: (\hat{E}, \hat{D}) \to (E, D)$ and $g: g^{-1}((\tilde{E}, \tilde{D})) \to (\tilde{E}, \tilde{D})$ gives a commutative diagram showing that if $\varphi: (\tilde{E}, \tilde{D}) \to (E, D)$ induces isomorphisms

$$H^m(E, D) \overset{\cong}{\to} H^m(\tilde{E}, \tilde{D})$$

in all dimensions, then $\varphi: (g^{-1}(\tilde{E}), g^{-1}(\tilde{D})) \to (\hat{E}, \hat{D})$ induces isomorphisms

$$H^m(\hat{E}, \hat{D}) \overset{\cong}{\to} H^m(g^{-1}(\tilde{E}), g^{-1}(\tilde{D}))$$

in all dimensions.

3. Construction of a ball K in $\mathrm{SU}(n+1)$.

3.1. Let X be a complete Riemannian manifold and $C(y, \varrho)$ denote the closed ball of radius ϱ around $y \in X$ with respect to the Riemannian metric, dist.

LEMMA 1. *Let $M \subset X$ be a compact C^∞-submanifold. Then there exists a positive number $r = r(M)$, such that for $\varrho < r$ and $y \in X$ the intersection $C(y, \varrho) \cap M$ is either empty or homotopy equivalent to a point.*

PROOF. Let $T \subset X$ be a tubular neighbourhood of M with tubular radius r_0. Then for all $y \in T$, there is only one $x \in M$, such that $\operatorname{dist}(y, x) = \operatorname{dist}(y, M)$, and only in this case the geodesic from x to y is orthogonal to $T_x M$. Put $r = r_0$ and let $\varrho < r$.

From [5, Lemma 10.3, p. 59] the function $f: X \to \mathsf{R}$ defined by

$$f(x) = \operatorname{dist}(y, x)$$

is differentiable, and

$$C(y, \varrho) \cap M = \{x \in M \mid f(x) \leqq \varrho\}.$$

If $x \in M$ is a critical point of $f|M \cap C(y,\varrho)$, then the geodesic from x to y is orthogonal to T_xM, so only one critical point can exist.

If $C(y,\varrho)\cap M \neq \emptyset$, then it follows from [5, Theorem 3.1, p. 12] that $C(y,\varrho)\cap M$ is homotopy equivalent to a point.

REMARK. Obviously $r(M)=r(u(M))$, if $u: X \to X$ is a transformation, preserving the Riemannian structure.

3.2. Let G denote the special unitary group $\mathrm{SU}(n+1)$. Then

$$G \subseteqq \mathrm{GL}(n+1,\mathsf{C}) \subseteqq \mathsf{C}^{(n+1)^2} = \mathsf{R}^m$$

for $m=2(n+1)^2$. Further the unitary group $\mathrm{U}(n+1)$ is embedded in $\mathsf{C}^{(n+1)^2}$. Now any $\sigma \in \mathrm{U}(n+1)$ gives by matrix multiplication a map

$$u \in \mathsf{C}^{(n+1)^2} \to \sigma u \in \mathsf{C}^{(n+1)^2} ,$$

which preserves the euclidean distance in R^m.

3.3. Define $K(\varrho)$ as $G\cap(C(1),\varrho)$, where 1 is the unit matrix in $\mathrm{GL}(n+1,\mathsf{C})$. Fix $r>0$ so small, that the following two conditions are fulfilled.

1) For any $y \in \mathsf{P}_n$ let G_y be the subgroup of G fixing y. By lemma 1 using G compact we can suppose for $\varrho<r$ that $K(\varrho)\cap\sigma\tau G_y\tau^{-1}$ is either empty or homotopy equivalent to a point for $\sigma,\tau \in \mathrm{U}(n+1)$.

2) Since $\sigma K(\varrho)\sigma^{-1}=K(\varrho)$ for all $\sigma \in \mathrm{U}(n+1)$, and since $\mathrm{U}(n+1)$ operates doubly transitively on P_n, there exists a function $d(\varrho)$, such that for all $z \in \mathsf{P}_n$,

$$K(\varrho)z = \{x \in \mathsf{P}_n \mid \mathrm{dist}(z,x) \leqq d(\varrho)\} .$$

Compare [1, Lemmata 1, 2, 3]. Choose r so small, that for all $\varrho \leqq r$ the set $K(\varrho)A$ is a tubular neighbourhood of A in P_n. Then obviously $K(\varrho)\sigma A$ are tubular neighbourhoods of σA for all $\sigma \in G$.

Put $K=K(\frac{1}{2}r)$.

4. Statement of lemmate 2 and 3.

4.1. G operates transitively on S^{2n+1} and P_n. The Hopf map h is G-equivariant. Let $A \subseteqq \mathsf{P}_n$. We study the maps

$$\hat{\varphi}: G \times \hat{A} \to S^{2n+1} ,$$

$$\varphi: G \times A \to \mathsf{P}_n$$

defined by $\hat{\varphi}(\sigma,\hat{x})=\sigma\hat{x}$ for $\sigma\in G$ and $\hat{x}\in\hat{A}$, and $\varphi(\sigma,x)=\sigma x$ for $\sigma\in G$ and $x\in A$. We have the commutative diagram

$$\begin{array}{ccc} G\times\hat{A} & \xrightarrow{\hat{\varphi}} & S^{2n+1} \\ {\scriptstyle \mathrm{id}\times(h|\hat{A})}\downarrow & & \downarrow{\scriptstyle h} \\ G\times A & \xrightarrow{\varphi} & \mathsf{P}_n \end{array}$$

G operates on $G\times\hat{A}$, respectively $G\times A$, by left translation on the first factor. With respect to this operation, $\hat{\varphi}$ and φ are equivariant.

4.2. Let $K\subseteqq G$ be chosen as in 3.3. For any $\sigma\in G$ we have a commutative diagram

$$(*)\qquad \begin{array}{ccc} \{\sigma\}\times A\cap\varphi^{-1}(B) & \to & \sigma A\cap B \\ \downarrow & & \downarrow \\ K\sigma\times A\cap\varphi^{-1}(B) & \to & K\sigma A\cap B \end{array}$$

with inclusions as vertical arrows and restrictions of φ as horisontal arrows. The upper map is a homeomorphism.

4.3. $H^q(\varphi)$ is the map

$$H^q(K\sigma A\cap B,\ \sigma A\cap B;\ Z)\to H^q(K\sigma\times A\cap\varphi^{-1}(B),\ \{\sigma\}\times A\cap\varphi^{-1}(B);\ Z)$$

LEMMA 2. *$H^q(\varphi)$ is an isomorphism for all q.*

REMARK. Lemma 2 remains valid when B is exchanged with $A\cap B$.

LEMMA 3. *$H^q(K\sigma A\cap B,\ \sigma A\cap B;\ Z)=0$ for $0\leqq q\leqq s$.*

REMARK. Lemma 3 remains valid when B is exchanged with $A\cap B$.

4.4. LEMMA 2^1. *$H^q(\hat{\varphi})$ is an isomorphism for all q.*

PROOF. Follows from 2.6.

LEMMA 3^1. *$H^q(K\sigma\hat{A}\cap\hat{B},\ \sigma\hat{A}\cap\hat{B};\ Z)=0$ for $0\leqq q\leqq s$.*

PROOF. Follows from 2.5.

5. Proof of lemma 2.

5.1. The map

$$\varphi\colon\ K\sigma\times A\cap\varphi^{-1}(B)\to K\sigma A\cap B$$

is proper and surjective. In order to show, that φ induces isomorphisms

$$H^q(\varphi)\colon\ H^q(K\sigma A\cap B;\ Z)\to H^q(K\sigma\times A\cap\varphi^{-1}(B);\ Z)$$

for all q, it is by the Leray spectral sequences enough to show, that all fibers have the homotopy type of a point. If $y \in K\sigma A \cap B$, the fiber satisfies

$$\begin{aligned} \varphi^{-1}(y) &= \{(\tau,x) \in K\sigma \times B \mid \tau x = y\} \\ &\cong \{\tau \in K\sigma \mid \tau^{-1} y \in B\} \\ &\cong \{\tau \in K \mid \tau y^1 \in B\}, \end{aligned}$$

where $y^1 = \sigma^{-1} y$.

5.2. Define a map

$$\psi\colon \{\tau \in K \mid \tau y^1 \in B\} \to K y^1 \cap B$$

by $\psi(\tau) = \tau y^1$. Then ψ is surjective and proper. Using lemma 1 on $C(y,\varrho) = K y^1$ and $y = y^1$, we find that $K y^1 \cap B$ has the homotopy type of a point.

5.3. Again by the Leray spectral sequences, it is enough to show, that the fibers of ψ have the homotopy type of a point. If $z \in K y^1 \cap B$, then

$$\begin{aligned} \psi^{-1}(z) &= \{\tau \in K \mid \tau y^1 = z\} \\ &= K \cap \{\sigma \in G \mid \sigma y^1 = z\} \\ &= K \cap \sigma^1 G_{y^1}, \end{aligned}$$

where $\sigma^1 y^1 = z$.

This fiber is homotopy equivalent to a point according to the choice of K in 3.3 having property 1).

5.4. Proof of lemma 2. The mapping φ of the pair

$$(\sigma \times A \cap \varphi^{-1}(B),\ K\sigma \times A \cap \varphi^{-1}(B))$$

onto the pair $(\sigma A \cap B,\ K\sigma A \cap B)$ gives a series of homomorphisms between the cohomology sequences. Now two of each three consecutive homomorphisms are isomorphisms according to 4.2 and 5.1. Hence the five-lemma can be applied to the remaining homomorphisms.

6. Real and complex index of functions.

6.1. Let M be a complex n-dimensional manifold and $f\colon M \to \mathbb{R}$ a C^2-function. Then for any $p \in M$ and coordinate system $z_j = x_j + i x_{n+j}$ $j = 1,\ldots,n$, around p, the quadratic Levi form is

$$L_f(p,w) = \sum_{j,k} \frac{\partial^2 f(p)}{\partial \bar{z}_k \partial z_j} w_j \bar{w}_k, \quad w \in \mathbb{C}^n .$$

This form is known to be independent of coordinates chosen and to be real valued. So we can define the complex index $\mathrm{Index}_{\mathsf{C}}(f,p)$ as the maximum dimension of a complex subspace of C^n on which $L_f(p)$ is negative definite.

6.2. If we consider M as a real $2n$-dimensional manifold with coordinates x_j, $j=1,\ldots,2n$, we have the quadratic Hessian

$$H_f(p,v) = \sum_{j,k} \frac{\partial^2 f(p)}{\partial x_k \partial x_j} v_k v_j, \quad v \in \mathsf{R}^{2n}.$$

This form is known to be independent of the coordinates chosen, when $df(p)=0$. So we can define the real index $\mathrm{Index}_{\mathsf{R}}(f,p)$ as the maximum dimension of a real subspace of R^{2n} on which $H_f(p)$ is negative definite.

6.3. LEMMA 4. $\mathrm{Index}_{\mathsf{R}}(f,p) \geqq \mathrm{Index}_{\mathsf{C}}(f,p)$.

PROOF. Define

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $w_j = v_j + iv_{n+j}$ and $z_j = x_j + ix_{n+j}$. Then the formula

$$L_f(p,w) = \tfrac{1}{4}(H_f(p,v) + (E^{-1}H_fE)(p,v))$$

holds for all $p \in M$. If we compute dimensions of the subspaces where the forms are positively semi-definite, we see, that

$$2n - 2\,\mathrm{Index}_{\mathsf{C}}(f,p) \geqq 2(2n - \mathrm{Index}_{\mathsf{R}}(f,p)) - 2n$$

and this proves the lemma.

7. Proof of lemma 3.

Define $M(\varrho) = K(\varrho)\sigma A \cap A$ for $\varrho \leqq r$.

There exists a function $f\colon M(r) \to \mathsf{R}$ such that the Levi form of f has at least s negative eigenvalues, and for all $\varrho, 0 \leqq \varrho \leqq r$,

$$M(\varrho) = \{x \in M(r) \mid f(x) \leqq \alpha(\varrho)\},$$

[1, Satz 1]. Further for any ϱ_0, $0 < \varrho_0 < r$, there exists a $k > 0$, such that

1) The Levi form of $-e^{-kt}$ has at least $s+1$ negative eigenvalues in $M(r) \setminus M(\varrho_0)$,
2) For all $\varrho, 0 \leqq \varrho \leqq r$,

$$M(\varrho) = \{x \in M(r) \mid -e^{-kf(x)} \leqq -e^{-k\alpha(\varrho)}\}.$$

Let $\varepsilon > 0$ be small enough. From [5, lemma 22.4, p. 119] we find g_ε approximating f on the set

$$\{x \in M(r) \mid \varrho_0 + \varepsilon \leqq f(x) \leqq \alpha(\tfrac{1}{2}r) + \varepsilon\}$$

so well, that for all x we have $|f(x) - g_\varepsilon(x)| < \varepsilon$ and

$$\mathrm{Index}_{\mathsf{R}}(g_\varepsilon, x) = \mathrm{Index}_{\mathsf{R}}(f, x) .$$

Lemma 4 says, that this index is at least $s+1$. Define

$$K_0(\varepsilon) = \{x \in M(r) \mid g_\varepsilon \leqq \alpha(\varrho_0) + \varepsilon\}$$

$$K(\varepsilon) = \{x \in M(r) \mid g_\varepsilon(x) \leqq \alpha(\tfrac{1}{2}r) + \varepsilon\} .$$

Then from [5, Theorem 3.2, p. 14] $H^m(K(\varepsilon), K_0(\varepsilon); Z) = 0$ for $0 \leqq m \leqq s$. Now

$$\textstyle\bigcap_{\varepsilon > 0} K(\varepsilon) = M(\tfrac{1}{2}r) \quad \text{and} \quad \bigcap_{\varepsilon > 0} K_0(\varepsilon) = M(\varrho_0) ,$$

so when $\varepsilon \to 0$, we get $H^m(M(\tfrac{1}{2}r), M(\varrho_0); Z) = 0$ for $0 \leqq m \leqq s$. Finally letting $\varrho_0 \to 0$ we get $\bigcap_{\varrho_0 > 0} M(\varrho_0) = M(0)$, and hence

$$H^m(M(\tfrac{1}{2}r), M(0); Z) = 0$$

for $0 \leqq m \leqq s$, that is

$$H^m(B\sigma A \cap A, \sigma A \cap A; Z) = 0$$

for $0 \leqq m \leqq s$.

8. Proof of the theorems.

8.1. From lemmate 2^1 and 3^1 the homomorphisms j^q and k^q are isomorphisms for $0 \leqq q \leqq s$

$$j^q\colon H^q(K\sigma \times \hat{A} \cap \hat{\varphi}^{-1}(\hat{B}); Z) \to H^q(\{\sigma\} \times \hat{A} \cap \hat{\varphi}^{-1}(\hat{B}); Z) ,$$

$$k^q\colon H^q(K\sigma \times \hat{A} \cap \hat{\varphi}^{-1}(\hat{A} \cap \hat{B}); Z) \to H^q(\{\sigma\} \times \hat{A} \cap \hat{\varphi}^{-1}(\hat{A} \cap \hat{B}); Z) ,$$

and they are injective for $q = s+1$.

Let $p\colon G \times \hat{A} \to G$ be the projection on G, and p^1 and p^{11} the restrictions of p to $\hat{\varphi}^{-1}(\hat{B})$ and $\hat{\varphi}^{-1}(\hat{A} \cap \hat{B})$ respectively, both mapping onto G. Because j^q and k^q are isomorphisms, the sheafs $R^q p^1{}_* Z$ and $R^q p^{11}{}_* Z$ are locally constant for $q \leqq s$, and because $\pi_1(G) = 0$, they are constant for $q \leqq s$.

Further the maps

$$H^0(G, R^{s+1} p^1{}_* Z) \to H^{s+1}(\hat{A} \cap \hat{B}; Z) ,$$

$$H^0(G, R^{s+1} p^{11}{}_* Z) \to H^{s+1}(\hat{A} \cap \hat{B}; Z)$$

are injective, because j^{s+1} and k^{s+1} are injective. So in the following commutative diagram defined by the inclusion $\hat{\varphi}^{-1}(\hat{A}\cap\hat{B})\subseteqq\hat{\varphi}^{-1}(\hat{B})$,

$$\begin{array}{ccc} H^0(G,R^{s+1}p^1{}_*Z) & \to & H^0(G,R^{s+1}p^{11}{}_*Z) \\ \downarrow & & \downarrow \\ H^{s+1}(\hat{A}\cap\hat{B};Z) & \underset{\to}{\cong} & H^{s+1}(\hat{A}\cap\hat{B};Z) \end{array}$$

both vertical maps are injective, hence also the upper map must be injective.

8.2. Lemma 5. *If the inclusion* $\hat{\varphi}^{-1}(\hat{A}\cap\hat{B})\subseteqq\hat{\varphi}^{-1}(\hat{B})$ *induces isomorphisms*

$$H^j(\hat{\varphi}^{-1}(\hat{B});Z)\underset{\to}{\cong}H^j(\hat{\varphi}^{-1}(\hat{A}\cap\hat{B});Z)\quad \textit{for } j<i$$

and $H_j(\hat{A}\cap\hat{B};Z)=0$ *for* $1\leqq j\leqq i$, *then the inclusion induces an isomorphism for* $j=i\leqq s$, *and a monomorphism for* $j=i=s+1$.

Proof. Consider the Leray spectral sequences for p^1 and p^{11} with mappings induced by inclusion

$$\begin{array}{ccccc} H^i(\hat{\varphi}^{-1}(\hat{B}),Z) & \Leftarrow & E_2{}^{rq} & = & H^r(G,R^qp^1{}_*Z) \\ \downarrow & & & & \downarrow \\ H^i(\hat{\varphi}^{-1}(\hat{A}\cap\hat{B}),Z) & \Leftarrow & \tilde{E}_2{}^{rq} & = & H^r(G,R^qp^{11}{}_*Z)\,. \end{array}$$

From 8.1. follows, that we have the following commutative diagram

$$\begin{array}{ccc} E_2{}^{rq} & \to & H^r(G,H^q(\hat{A}\cap\hat{B};Z)) \\ \downarrow & & \downarrow\cong \\ \tilde{E}_2{}^{rq} & \to & H^r(G,H^q(\hat{A}\cap\hat{B};Z)) \end{array}$$

with isomorphisms for $q\leqq s$, and injective maps for $q=s+1$ and $r=0$. We have exact rows in the commutative diagram

$$\begin{array}{ccccccc} 0\to E_2{}^{i,0} & \to & H^i(\hat{\varphi}^{-1}(\hat{B});Z) & \to & E_2{}^{0,i} & \to & E_2{}^{i+1,0} \\ \downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong \\ 0\to \tilde{E}_2{}^{i,0} & \to & H^i(\hat{\varphi}^{-1}(\hat{A}\cap\hat{B});Z) & \to & \tilde{E}_2{}^{0,i} & \to & \tilde{E}_2{}^{i+1,0} \end{array}$$

and three maps are isomorphisms for $i\leqq s$, hence so is the fourth. For $i=s+1$, one of the three maps is injective only, but then the fourth map is injective too.

Remark. If lemma 5 is stated without ^ the proof is still valid.

8.3. Lemma 6. *If the inclusion* $\hat{\varphi}^{-1}(\hat{A}\cap\hat{B})\subseteqq\hat{\varphi}^{-1}(\hat{B})$ *induces isomorphisms*

$$H^j(\hat{\varphi}^{-1}(\hat{B}),Z)\underset{\to}{\cong}H^j(\hat{\varphi}^{-1}(\hat{A}\cap\hat{B}),Z)\quad \textit{for } j\leqq i$$

and $H^j(\hat{A}\cap\hat{B}; Z)=0$ *for* $1\leqq j<i$, *and* $H^0(\hat{A}\cap\hat{B}; Z)=Z$, *for some* $i\leqq s+1$, *then* $H^i(\hat{A}\cap\hat{B}; Z)=0$. *If only*

$$H^i(\hat{\varphi}^{-1}(\hat{B}); Z) \to H^i(\hat{\varphi}^{-1}(\hat{A}\cap\hat{B}); Z)$$

is injective, we get only an injective map

$$H^i(\hat{B}; Z) \to H^i(\hat{A}\cap\hat{B}; Z)$$

induced by the inclusion $\hat{A}\cap\hat{B}\subseteqq\hat{B}$.

PROOF. We have $\hat{\varphi}: G\times\hat{A}\to S^{2n+1}$. Let $\hat{\varphi}': \hat{\varphi}^{-1}(\hat{B})\to\hat{B}$ and $\hat{\varphi}'': \hat{\varphi}^{-1}(\hat{A}\cap\hat{B})\to\hat{A}\cap\hat{B}$ be the restrictions of $\hat{\varphi}$. The spectral sequences for $\hat{\varphi}'$ and $\hat{\varphi}''$ are

$$\begin{array}{ccc} H^i(\hat{\varphi}^{-1}(\hat{B}); Z) & \Leftarrow E_2^{rq} = & H^r(\hat{B}, R^q\hat{\varphi}'_* Z) \\ \downarrow\cong & & \downarrow \\ H^i(\hat{\varphi}^{-1}(\hat{A}\cap\hat{B}); Z) & \Leftarrow E_2^{rq} = & H^r(\hat{A}\cap\hat{B}, R^q\hat{\varphi}''_* Z) \end{array}$$

Because $\hat{\varphi}'$ and $\hat{\varphi}''$ are fiber bundles with the same fiber, F, we have the following commutative diagram

$$\begin{array}{ccc} E_2^{rq} & \underset{\to}{\cong} & H^r(\hat{B}, H^q(F,Z)) \\ \downarrow & & \downarrow \\ E_2^{rq} & \underset{\to}{\cong} & H^r(\hat{A}\cap\hat{B}, H^q(F,Z)) \,. \end{array}$$

For $i=1$ we get a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0\to & H^1(\hat{B}, H^0(F,Z)) & \to & H^1(\hat{\varphi}^{-1}(\hat{B}), Z) & \to & H^1(F,Z) \\ & \downarrow & & \downarrow\cong & & \downarrow\cong \\ 0\to & H^1(\hat{A}\cap\hat{B}, H^0(F,Z)) & \to & H^1(\hat{\varphi}^{-1}(\hat{A}\cap\hat{B}), Z) & \to & H^1(F,Z)\,, \end{array}$$

so the 5-lemma and $H^1(\hat{B},Z)=0$ from [4, Proposition] give $H^1(\hat{A}\cap\hat{B},Z)=0$.

For $i>1$ we get another commutative diagram of exact sequences

$$\begin{array}{ccccccc} \to & H^i(\hat{B}, H^0(F,Z)) & \to & H^i(\check{\varphi}^{-1}(\hat{B}), Z) & \to & H^i(F,Z) & \to \\ & \downarrow & & \downarrow\cong & & \downarrow\cong & \\ \to & H^i(\hat{A}\cap\hat{B}, H^0(F,Z)) & \to & H^i(\hat{\varphi}^{-1}(\hat{A}\cap\hat{B}), Z) & \to & H^i(F,Z) & \to \end{array}$$

so the 5-lemma and $H^i(\hat{B},Z)=0$ from [4, Proposition] give $H^i(\hat{A}\cap\hat{B},Z)=0$ for $i\leqq s$. If $i=s+1$ and the middle vertical map is injective, we get induced an injective map

$$H^{s+1}(\hat{B},Z)\to H^{s+1}(\hat{A}\cap\hat{B},Z)\,.$$

REMARK. If lemma 6 is stated without ^ the proof is still valid.

8.4. PROPOSITION. $H^i(\hat{B},\hat{A}\cap\hat{B};Z)=0$ *for* $i\leqq s+1$.

PROOF. Induction using lemmate 5 and 6 gives $H^i(\hat{A}\cap\hat{B},Z)=0$ for $1\leqq i\leqq s$ and

$$H^{s+1}(\hat{B},Z)\to H^{s+1}(\hat{A}\cap\hat{B},Z)$$

injective. Because $H^i(\hat{B},Z)=0$ for $1\leqq i\leqq s$ by [4] and because $H^0(\hat{A}\cap\hat{B},Z)=Z$ by [2, Proposition 4], the proposition now follows.

8.5. PROOF OF THEOREM 1. Induction using the remarks following lemmate 5 and 6 gives $H^1(\hat{A}\cap\hat{B};Z)=0$ and an isomorphism if $s\geqq 2$

$$H^2(B,Z)\underset{\to}{\cong}H^2(A\cap B,Z)\,.$$

By [2, Proposition 4], $H^0(A\cap B,Z)=Z$ and by [4, Theorem] $H^2(B,Z)=Z$. The exact sequence for the pair $A\cap B\subseteqq B$ is

$$H^0(B,Z)\underset{\to}{\cong}H^0(A\cap B,Z)\to H^1(B,A\cap B,Z)\to H^1(B,Z)\underset{\to}{\cong}$$
$$\underset{\to}{\cong}H^1(A\cap B,Z)\to H^2(B,A\cap B,Z)\to H^2(B,Z)\to H^2(A\cap B,Z)$$

with isomorphisms $\underset{\to}{\cong}$ and the last map injective. Hence $H^1(B,A\cap B,Z)=0$ and $H^2(B,A\cap B,Z)=0$.

From the proposition and 2.5 follows, that $H^i(B,A\cap B,Z)=0$ for $1\leqq i\leqq s+1$, and the universal coefficient theorem then gives

$$H_i(B,A\cap B,\mathsf{Z})=0\quad\text{for } 1\leqq i\leqq s+1\,.$$

8.6. PROOF OF THEOREM 2. Follows from 2.4, proposition and 2.3.

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UNIVERSITY OF COPENHAGEN, DENMARK