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A MAXIMAL TOROIDAL GRAPH WHICH IS NOT A TRIANGULATION

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Abstract.

If G is a planar graph, we may add edges to construct a maximal planar graph H containing G, so that H triangulates the sphere. If G is toroidal, then by adding edges we can extend G to a maximal toroidal graph G_M or to a triangulation G_T of the torus; however G_M and G_T may be different. In particular G_M need not be a triangulation, and G_T need not be a graph, that is, we may have added multiple edges to G so that G_T is a multigraph. The latter phenomenon is easy to construct: for example, K_5 imbeds in the torus but does not triangulate it. It is the main object of this note to construct the unique minimum example of the former phenomenon.

1. Introduction.

For notation and terminology not given here, see Harary [2]. A graph G has *genus* $\gamma(G)=n$ if n is the smallest non-negative integer for which G can be imbedded in the orientable 2-manifold S_n. A graph G is *maximal of genus* n if $\gamma(G)=n$, but for each edge x of $\bar{G}$, $\gamma(G+x)=n+1$. Hence every complete graph is vacuously maximal of its genus. A graph is a *triangulation* of S_n if it can be imbedded in S_n so that every face is a triangle. Every maximal planar graph is a triangulation of the sphere, and conversely.

It is readily seen from the Euler formula that every triangulation of S_n is a maximal graph of genus n, and we have noted that the converse holds for $n=0$. It does not follow, however, that a maximal graph of genus n is a triangulation of S_n. This can fail for the trivial reason that G has too few vertices. For example, K_5 is a maximal toroidal graph which does not triangulate the torus (if it takes at least 7 vertices to do so). We now present the unique smallest noncomplete counterexample, K_8-C_5.

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Since K_7 triangulates S_1, no proper subgraph is maximal toroidal; hence any counterexample must have order at least 8. Duke and Haggard [1] found the genus of every graph of order 8. For H a subgraph of G, let $G-H$ denote G minus the lines of H; let

$$B_1 = K_8 - K_3, \quad B_2 = K_8 - (2K_2 \cup K_{1,2}), \quad B_3 = K_8 - K_{2,3}.$$

A *Kuratowski graph* is homeomorphic to K_5 or $K_{3,3}$. The results of [1] can be stated briefly:

THEOREM 1. *For any subgraph G of K_8,*
(1) $\gamma(G)=0$ *if G does not contain a Kuratowski graph;*
(2) $\gamma(G)=1$ *if G contains a Kuratowski graph but does not contain any B_i, $i=1,2,3$;*
(3) $\gamma(G)=2$ *if G contains any B_i, $i=1,2,3$.*

2. The counterexample.

Let $G=K_8-C_5$. Figure 1 provides a toroidal imbedding of G, where the 5-cycle in the complement $\bar{G}$ is given by $v_1v_2v_3v_4v_5v_1$.

THEOREM 2. *The graph $G=K_8-C_5$ is maximal toroidal but does not triangulate the torus.*

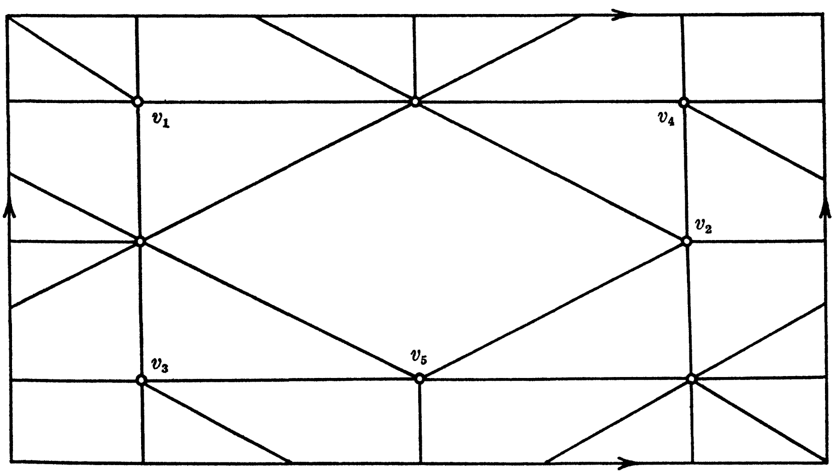

Fig. 1. A toroidal imbedding of K_8-C_5.

PROOF. Since C_5 is not a subgraph of K_3, $2K_2 \cup K_{1,2}$, or $K_{2,3}$, $\gamma(G) < 2$ by Theorem 1. Since K_5 is a subgraph of G, $\gamma(G) > 0$; hence $\gamma(G) = 1$. Let G' be the graph formed by adding any new edge to G. Then $G' = K_8 - P_5$, and since the path P_5 is a subgraph of $K_{2,3}$, B_3 is a subgraph of G', so that $\gamma(G') = 2$, by Theorem 1. Hence G is maximal toroidal. In any triangulation of S_n by a graph H, twice the number of edges of H must equal three times the number of faces. But G has exactly 23 edges; hence G cannot triangulate the torus. Note that, in Figure 1, the only non-triangular face is a quadrilateral and both potential diagonals already lie in G.

Using Theorem 1, it is straightforward to show that the graph G of Theorem 2 is unique among all graphs of order 8.

3. A short proof of Heawood's Inequality.

We have already pointed out that, while any graph G with $\gamma(G) = n$ can be extended by the addition of new edges to a triangulation G_T of S_n, the resulting G_T may be a multigraph but not a graph. Several proofs of Heawood's Inequality which occur in the literature make use of the false assumption that G_T must be a graph. We provide a short proof of Heawood's Inequality which avoids this pitfall.

LEMMA 1. *Let G be a connected graph with $\gamma(G) = n$. Then G can be extended to a triangulation G_T of S_n, where G_T is a multigraph.*

PROOF. Since G is connected and $\gamma(G) = n$, all of the faces produced by an imbedding of G in S_n are simply connected (see König [3, p. 198]). Now add diagonal lines to each of these faces, until a triangulation G_T results.

Note that when $n > 0$, the above process may introduce multiple edges, as in Figure 1, when either diagonal is added within the quadrilateral face.

For $n \geqq 0$, we define

$$H(n) = [\tfrac{1}{2}(7 + (1 + 48n)^{\frac{1}{2}})],$$

and

$$\chi(S_n) = \max\{\chi(G) : \gamma(G) \leqq n, G \text{ a graph}\},$$

where $\chi(G)$ denotes the chromatic number of G.

THEOREM 3. *For* $n > 0$, $\chi(S_n) \leqq H(n)$.

PROOF. We show that for $n > 0$ and $\gamma(G) = n$, $\chi(G) \leqq H(n)$. For if $\chi(G) = k \leqq n$, then $\chi(G) \leqq H(k) \leqq H(n)$ for $k > 0$, whereas $\chi(G) \leqq 5 < H(n)$ for $k = 0$. We proceed by induction on p, the order of G. First, define $\alpha(n)$ to be the positive root of the equation

$$x^2 - 7x - 12(n-1) = 0 ;$$

thus

$$\alpha(n) = \tfrac{1}{2}(7 + (1 + 48n)^{\frac{1}{2}}) .$$

If $p \leqq \alpha(n)$, then $\chi(G) \leqq p \leqq [\alpha(n)] = H(n)$. Suppose now that $p > \alpha(n)$ and that the assertion is true for graphs of order $p-1$. Consider G with $\gamma(G) = n$. We may assume G to be connected, for otherwise G has a component G_1 such that $\chi(G) = \chi(G_1) \leqq H(n)$, by the inductive hypothesis. Thus we may use Lemma 1 to extend G to a multigraph G_T with p vertices with triangulates S_n. Let d_T and d denote the average degree of the points of G_T and G respectively. Clearly, $d_T \geqq d$.

Since $n = \gamma(G) \leqq \gamma(G_T) \leqq n$, $\gamma(G_T) = n$. By a standard argument based on Euler's formula as applied to multigraphs, we compute that

$$d_T = 6 + 12(n-1)/p .$$

If $n = 1$, $d_T = 6 = [\alpha(1)] - 1$. On the other hand, if $n > 1$, then since $p > \alpha(n)$,

$$d_T < 6 + 12(n-1)/\alpha(n) = \alpha(n) - 1 ,$$

so again $d_T \leqq [\alpha(n)] - 1$. Hence, for $n > 0$,

$$d \leqq d_T \leqq [\alpha(n)] - 1 = H(n) - 1 .$$

Choose a point v of G with degree $\leqq H(n) - 1$ and consider the graph $G - v$. Since $G - v$ has order $p - 1$, we see by induction that

$$\chi(G - v) \leqq H(\gamma(G - v)) \leqq H(n) .$$

Hence $G - v$ can be $H(n)$-colored, with at least one color available for use at v, so that $\chi(G) \leqq H(n)$.

CONJECTURE. If M is the set of integers n for which there exist maximal noncomplete graphs of genus n which are not triangulations, then M is the set of all positive integers. We now know only that $0 \notin M$ and $1 \in M$.

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