MATH. SCAND. 33 (1973), 130–138

A CAUCHY PROBLEM FOR ANALYTIC FUNCTIONALS

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1. Introduction.

We denote variables in $\mathsf{R} \times \mathsf{C}^n$ by (t,z), $t \in \mathsf{R}$, $z=(z_1,\ldots,z_n) \in \mathsf{C}^n$, and consider a system of N simultaneous linear partial differential equations in the form

$$(1.1) \qquad I\,\partial u/\partial t+\textstyle\sum_{j=1}^n A_j\,\partial u/\partial z_j \;=\; B(t,z)u+f\,.$$

We suppose that the coefficient matrices A_j are constant (complex) and that the entries of B are C^∞-functions on $\mathsf{R} \times \mathsf{C}^n$, entire analytic as functions of z for fixed t. I is the unit matrix.

It follows from work of F. Trèves [6, pp. 53–58] and J. Persson [4] that for every continuous function f from R to the space $(H'(\mathsf{C}^n))^N$ of N-tuples of analytic functionals on C^n, with $f(t)=0$ for $t<0$, there exists a unique C^1-function u from R to $(H'(\mathsf{C}^n))^N$, vanishing for $t<0$, which solves (1.1). In fact their results are more general, and include cases where the A_j are variable, and where the whole situation is considered locally in a neighbourhood of 0. The justification for studying this somewhat unusual form of Cauchy problems is found in its applications, to Trèves' "hyperdifferential operators" and to new proofs of Holmgren's uniqueness theorem. For these applications it may be of some interest to obtain as precise information as possible about the regularity properties and particularly about the size and shape of possible carriers of the solutions. This seems difficult to obtain with the "Ovsjannikov technique" used in the two above-mentioned proofs. The purpose of this note is to show how the much more well-known theory of distribution solutions to symmetric hyperbolic systems can be applied directly to this situation, by means of a nice trick from the theory of partial differential equations with complex variables (see P. R. Garabedian, [1, chapter 16.1]). At the same time we get existence and uniqueness theorems also when f and u are supposed to be vector-valued distributions (instead of functions) with values in $H'(\mathsf{C}^n)^N$.

For simplicity, we only treat the "global" case, where the A_j are con-

Received November 24, 1972.

stant, and B is defined in all of $\mathsf{R} \times \mathsf{C}^n$. It should be clear from the proof (in Section 4) that the same trick applies to "local" situations, and to cases where the A_j may depend on t (but not on z), and also how further information about the solutions (regularity properties) can be transferred from the classical theory of symmetric hyperbolic systems to the situation we consider here.

Our result, and the central part of its proof are found in Section 4, while Sections 2 and 3 contain preliminaries.

2. Symmetric hyperbolic systems.

In this section, we work in R^{1+m}, with variables denoted by (t,x), $t \in \mathsf{R}$, $x=(x_1,\ldots,x_m) \in \mathsf{R}^m$, and we consider a system of N equations in the form

$$(2.1) \qquad I\partial U/\partial t + \textstyle\sum_{j=1}^m A_j \partial U/\partial x_j \;=\; B(t,x)U + F\ .$$

where the entries of the $N \times N$ matrix B are in $C^\infty(\mathsf{R}^{1+m})$, those of the A_j constant.

PROPOSITION 2.1. *Let* $F \in \mathscr{D}'(\mathsf{R}^{1+m})^N$ *be given, and suppose that the set*

$$\operatorname{supp} F \cap \{(t,x) \in \mathsf{R}^{1+m}\ ;\ t \leqq T\}$$

is compact for every real T, *and empty if* $T<0$. *Suppose also that the matrices* $A_1,\ldots,A_m$ *are Hermitian.*

Then there exists a unique $U \in \mathscr{D}'(\mathsf{R}^{1+m})^N$ *with* $\operatorname{supp} U \subset \mathsf{R}_+{}^{1+m} = \{(t,x)\,;\, t \geqq 0\}$ *and solving equation* (2.1).

There also exist a proper convex cone Γ^*, *defined by* (2.2) *and* (2.3) *below such that*

$$\operatorname{supp} U \;\subset\; \Gamma^* + \operatorname{supp} F\ .$$

In particular, the set

$$\operatorname{supp} U \cap \{(t,x) \in \mathsf{R}^{1+m}\ ;\ t \leqq T\}$$

is compact for every $T \in \mathsf{R}$, *empty when* $T<0$.

(We use the notation

$$\operatorname{supp} F \;=\; \textstyle\bigcup_{k=1}^n \operatorname{supp} F_k$$

when $F=(F_1,\ldots,F_N) \in (\mathscr{D}')^N$.)

REMARK. This result seems to be well-known. However, a proof will be sketched below, because some extra information which can be extracted from it will be needed in Section 4. The proof is a slightly modified version of one presented by F. John, in [3], for the "classical" case, where F and U are supposed to be continuous functions.

SKETCH OF PROOF. To prove the existence of U, we first note that it is enough to consider F with compact support. Indeed, suppose that the proposition has been proved under this additional hypothesis, and let $F \in \mathscr{D}'(\mathsf{R}^{1+m})^N$ be given. Using a partition of unity on R, we can write $F = F_{(1)} + F_{(2)} + \ldots$, where

$$\operatorname{supp} F_{(j)} \subset \{(t,x) \,;\, j-1 \leqq t \leqq j+1\}$$

and it follows from our assumptions on F that $\operatorname{supp} F_{(j)}$ is compact for every j. For each j we get a solution $U_{(j)}$ of (2.1) with F replaced by $F_{(j)}$, and we get

$$\operatorname{supp} U_{(j)} \subset \operatorname{supp} F_{(j)} + \Gamma^* \subset \{(t,x) \,;\, t \geqq j-1\}\,.$$

Then $U = \sum_{j=1}^{\infty} U_{(j)}$ is well defined (since the sum is locally finite), and solves (2.1), and $\operatorname{supp} U \subset \operatorname{supp} F + \Gamma^*$.

Next, we recall that the constant coefficient operator

$$P(\partial/\partial t, \partial/\partial x) = I\,\partial/\partial t + \textstyle\sum_{j=1}^{m} A_j \,\partial/\partial x_j$$

has a unique fundamental solution, E, with support in

$$\{(t,x) \,;\, t \geqq 0\} = \mathsf{R}_+^{1+m}\,.$$

The suppoıt of E is contained in the proper convex cone Γ^* defined by

$$\text{(2.2)} \qquad \Gamma^* = \{(t,x) \in \mathsf{R}^{1+m} \,;\, t\tau + \textstyle\sum x_j \xi_j \geqq 0 \text{ for } (\tau,\xi) \in \Gamma(P)\}$$

where

$$\text{(2.3)} \qquad \Gamma(P) = \{(\tau,\xi) \in \mathsf{R}^{1+m} \,;\, \tau \geqq 0, \text{ and}$$
$$\gamma \in \mathsf{C},\ \det[I(\tau+\gamma) + \textstyle\sum A_j \xi_j] = 0 \Rightarrow \gamma < 0\}\,.$$

Further, the partial Fourier transform of E with respect to x is given by

$$\text{(2.4)} \qquad \hat{E}(t,\xi) = \exp(-itA(\xi))\, Y(t)$$

where $A(\xi) = \sum_{j=1}^{m} A_j \xi_j$ and Y is the Heaviside function on R. Formula (2.4) is easily verified by direct computation, but can be found in [3, p. 84]. The information about $\operatorname{supp} E$ can also be extracted from [3] but are more easily taken out of Hörmander's book [2, Theorem 5.6.3. combined with Section 3.8].

To solve (2.1) by iteration, we define

$$W^1 = E * F, \qquad W^{\nu+1} = E * (BW^\nu), \qquad \nu = 1,2,\ldots\,.$$

Assuming for a moment that $\sum_{\nu=1}^{\infty} W^{\nu} = U$ converges in $\mathscr{D}'(\mathsf{R}^{1+m})$, we find that

$$P(\partial/\partial t, \partial/\partial x)U = BU + F ,$$

and since we have, for $\nu = 1, 2, \ldots,$

$$\operatorname{supp} W^{\nu} \subset \Gamma^*(P) + \operatorname{supp} F ,$$

the same will be the case for U. So it remains to be proved that $\sum_{\nu} W^{\nu}$ converges.

We define a duality between $\mathscr{D}'(\mathsf{R}^{1+m})^N$ and $C_0^{\infty}(\mathsf{R}^{1+m})^N$ by

$$\langle W, \varphi \rangle = \textstyle\sum_{k=1}^{N} W_k \varphi_k$$

where $W = (W_1, \ldots, W_N)$ and $\varphi = (\varphi_1, \ldots, \varphi_N)$. For given $\varphi \in (C_0^{\infty})^N$, we then define

$$\varphi^1 = E^+ * \varphi, \quad \varphi^{\nu+1} = E^+ * (B^+ \varphi^{\nu}) \quad \nu = 1, 2, \ldots . \tag{2.5}$$

where E^+* and B^+ are defined as adjoint operators:

$$\begin{aligned} \langle E * W, \varphi \rangle &= \langle W, E^+ * \varphi \rangle \\ \langle B \cdot W, \varphi \rangle &= \langle W, B^+ \varphi \rangle \end{aligned} \tag{2.6}$$

With these notations, we get for $\nu = 1, 2, \ldots$

$$\langle W^{\nu}, \varphi \rangle = \langle F, \varphi^{\nu} \rangle$$

and we see that the existence part of proposition 2.1 follows from

LEMMA 2.2. *Let φ, E and B be as above, and define φ^{ν} for $\nu = 1, 2, \ldots$ by (2.5). Then $\varphi^{\nu} \in (C^{\infty}(\mathsf{R}^{1+m}))^N$, and $\sum_{\nu=1}^{\infty} \varphi^{\nu}$ converges in $(C^{\infty}(\mathsf{R}^{1+m}))^N$.*

The proof of Lemma 2.2 is straight-forward, and follows closely the corresponding computations in [3]. Since it is rather tedious to write out, it will be omitted.

To prove the uniqueness of U, we let V be the difference between two solutions. V thus solves (2.1) with F replaced by 0, and we have $\operatorname{supp} V \subset \mathsf{R}_+^{1+m}$. For given $\varphi \in (C_0^{\infty})^N$, we choose $T \in \mathsf{R}$ such that $\operatorname{supp} \varphi \subset \{(t,x);\ t < T\}$ and write $V = V_1 + V_2$ where $\operatorname{supp} V_1$ is compact and $\operatorname{supp} V_2 \subset \{(t,x;\ t > T\}$. We define

$$V_1^1 = E * BV_1, \quad V_1^{\nu+1} = E * (BV_1^{\nu}), \quad \nu = 1, 2, \ldots .$$

and similarly for V_2. Since we have $V = E*(BV)$, we get

$$V = V_1^{\nu} + V_2^{\nu}$$

where

$$\operatorname{supp} V_2^\nu \subset \{(t,x)\ ;\ t>T\}\,.$$

Thus we have for $\nu=1,2,\ldots$

$$\langle V,\varphi\rangle = \langle V_1^\nu,\varphi\rangle = \langle V_1, B^+\varphi^{\nu-1}\rangle$$

and since $\sum\varphi^\nu$ converges by Lemma 2.2, we get $\varphi^\nu \to 0$, in $(C^\infty(\mathsf{R}^{1+m}))^N$, and $\langle V,\varphi\rangle=0$.

3. Distributions and bilinear functionals.

The usual identification of R^{2n} and C^n by

$$(3.1)\qquad (x,y)\to x+iy,\quad x\in \mathsf{R}^n,\ y\in\mathsf{R}^n\,,$$

makes the space $H=H(\mathsf{C}^n)$ of entire analytic functions on C^n a subspace of $C^\infty(\mathsf{R}^{2n})$. It follows from Cauchy's formulae for the derivatives of analytic functions that the topology induced on H from $C^\infty(\mathsf{R}^{2n})$ coincides with the usual topology of uniform convergence on compact sets (Trèves [5, p. 90]).

An element of the dual H' of H is called an analytic functional on C^n. By definition of the topology on H, a linear functional f on H is in H' if and only if there exist a constant C and a compact set $K\subset\mathsf{C}^n$ such that

$$|\langle f,\psi\rangle| \leqq C\sup_{z\in K}|\psi(z)|,\qquad \psi\in H\,.$$

An open set $U\subset\mathsf{C}^n$ is said to carry f if the compact K can be chosen in U.

It will be convenient to work with the space $B=B(C_0^\infty(\mathsf{R}),H)$ of separately continuous bilinear forms on $C_0^\infty(\mathsf{R})\times H(\mathsf{C}^n)$ instead of the space of distributions on R with values in H' (that is, the space of continuous linear maps from $C_0^\infty(\mathsf{R})$ to H'). It follows from Trèves [5, Proposition 42,2 (2)] that these two spaces are canonically isomorphic.

We identify R^{2n} with C^n by (3.1) and use the notations

$$\frac{\partial}{\partial z_j} = \tfrac12\left(\frac{\partial}{\partial x_j}-i\frac{\partial}{\partial y_j}\right),\quad \frac{\partial}{\partial \bar z_j} = (2i)^{-1}\left(\frac{\partial}{\partial x_j}+i\frac{\partial}{\partial y_j}\right).$$

Derivatives in B are defined in the obvious way by duality:

$$\langle \partial f/\partial t,\varphi,\psi\rangle = -\langle f,\varphi',\psi\rangle,\quad \langle \partial f/\partial z_j,\varphi,\psi\rangle = -\langle f,\varphi,\partial\psi/\partial z_j\rangle$$

etc. We note that the Cauchy-Riemann equations are valid:

$$(3.2)\qquad \partial f/\partial \bar z_j = \partial f/\partial x_j + i\,\partial f/\partial y_j = 0,\quad 1\leqq j\leqq n,\quad f\in B\,.$$

In this section we deduce some results about the relationship between elements of B and distributions on R^{1+2n}.

DEFINITION 3.1. Let $f \in B$ be given. f is said to be carried by the open set $C \subset \mathsf{R} \times \mathsf{C}^n$ if, for every compact interval I in R, and every $\varphi \in C_0^\infty(I)$, the analytic functional $f_\varphi : \psi \to \langle f, \varphi, \psi \rangle$ is carried by the projection on C^n of the set $(I \times \mathsf{C}^n) \cap C$.

LEMMA 3.2. *Let I be a compact subset of* R, *and let f be a separately continuous bilinear functional on $C_0^\infty(I) \times H(\mathsf{C}^n)$.*

Then there exists a compact set $K \subset \mathsf{C}^n$ such that for some seminorm $N(\cdot)$ on $C_0^\infty(I)$ we have

$$|\langle f, \varphi, \psi \rangle| \leqq N(\varphi) \sup_K |\psi| \tag{3.3}$$

for every $\varphi \in C_0^\infty(I)$, $\psi \in H'$, and there exists a distribution $F \in \mathscr{E}'(\mathsf{R}^{1+2n})$ with $\operatorname{supp} F \subset I \times K$, *such that*

$$\langle f, \varphi, \psi \rangle = \langle F, \varphi\psi \rangle$$

for $\varphi \in C_0^\infty(I)$ and $\psi \in H(\mathsf{C}^n) \subset C^\infty(\mathsf{R}^{2n})$.

PROOF. Since $C_0^\infty(I)$ and H are both Fréchet spaces, a separately continuous bilinear form on their product is simultaneously continuous (by [5, corollary to theorem 34,1]). This gives (3.3).

Further $B(C_0^\infty(I), H)$ is canonically isomorphic to the dual of the space $C_0^\infty(I) \otimes_\pi H$, which is a subspace of the Fréchet space $C^\infty(\mathsf{R}^{1+2n})$ (by [5, proposition 43,4 and theorem 51,6]). The last part of the lemma then follows from the Hahn–Banach theorem.

LEMMA 3.3. *Every $f \in B$ can be carried by some set C having the property that if $I \subset \mathsf{R}$ is a compact interval, then $(I \times \mathsf{C}^n) \cap C$ is relatively compact.*

Let C be such a carrier for f, and $\tilde{C}$ a neighbourhood of $\bar{C}$. Suppose that $(I \times \mathsf{C}^n) \cap \tilde{C}$ is also relatively compact for compact $I \subset \mathsf{R}$. Then there exists $F \in \mathscr{D}'(\mathsf{R}^{1+2n})$, with $\operatorname{supp} F \subset \tilde{C}$, *"extending" f in the sense that for $\varphi \in C_0^\infty(\mathsf{R})$, $\psi \in H(\mathsf{C}^n) \subset C^\infty(\mathsf{R}^{2n})$, we have*

$$\langle F, \varphi\psi \rangle = \langle f, \varphi, \psi \rangle .$$

(That F is defined for such functions, is clear since $\psi \to \langle F, \varphi\psi \rangle$ for $\psi \in C^\infty(\mathsf{R}^{2n})$, has compact support.)

PROOF. Let $\{I_\nu\}$ be a sequence of compact intervals in R, such that their interiors form a locally finite cover of R. Let $\{K_\nu\}$ be the corresponding sequence of compacts in C^n, according to Lemma 3.2. Then it is clear that any sufficiently small neighbourhood C of $\bigcup_\nu I_\nu \times K_\nu$ carries f and has the stated property.

If C is a given carrier of f, $\tilde{C}$ a neighbourhood of its closure, a simple compacity argument shows that one can choose $\{I_\nu\}$ and $\{K_\nu\}$ such that

$$C \subset \bigcup_\nu I_\nu \times K_\nu \subset \tilde{C} .$$

For every ν, let f_ν be the restriction of f to $C_0^\infty(I_\nu) \times H(\mathsf{C}^n)$, and let $F_\nu \in \mathscr{E}'(\mathsf{R}^{1+2n})$ be the "extension" constructed in Lemma 3.2. If $\{\vartheta_\nu\}$ is a partition of unity on R, with $\operatorname{supp}\vartheta_\nu \subset I_\nu$, we have

$$\langle f,\varphi,\psi\rangle = \sum \langle f_\nu,\vartheta_\nu\varphi,\psi\rangle ,$$

and we can define $F \in \mathscr{D}'(\mathsf{R}^{1+2n})$ by

$$F = \sum_\nu \vartheta_\nu F_\nu$$

since the sum is locally finite.

It is then clear that $\operatorname{supp} F \subset \bigcup_\nu I_\nu \times K_\nu \subset \tilde{C}$ and that F extends f as stated.

LEMMA 3.4. *Let $F \in \mathscr{D}'(\mathsf{R}^{1+2n})$ be given, and suppose that for any compact set $I \subset \mathsf{R}$ the set*

$$K_I = (I \times \mathsf{R}^{2n}) \cap \operatorname{supp} F$$

is compact. Then the "restriction" f of F to $C_0^\infty(\mathsf{R}) \times H(\mathsf{C}^n)$ defined by

$$\langle f,\varphi,\psi\rangle = \langle F,\varphi\psi\rangle$$

is in B, and is carried by any open neighbourhood of $\operatorname{supp} F$.

PROOF. Let $I \subset \mathsf{R}$ be given, and $\varphi \in C_0^\infty(I)$. Then we have for suitable constants C,k,l:

$$\begin{aligned}|\langle f,\varphi,\psi\rangle| &\leqq C \textstyle\sum_{j=1}^k \sup_I |\varphi^{(j)}(t)| \cdot \left(\sum_{|\alpha+\beta|\leqq l} \sup_{K_I} |D_x^\alpha D_y^\beta \psi(x+iy)|\right)\\ &\leqq C \textstyle\sum_{j=1}^k \sup_I |\varphi^{(j)}(t)|\; C' \sup_{L_I} |\psi(x+iy)| ,\end{aligned}$$

where L_I is any neighbourhood of K_I, and C' is a constant (depending on L_I and l). The lemma follows.

4. The existence and uniqueness theorem.

We use the notations from the Introduction, and from Section 3. For brevity we say that $f \in B$ "vanishes for $t < t_0$" if

$$\langle f,\varphi,\psi\rangle = 0 \text{ whenever } \operatorname{supp}\varphi \subset \{t \in \mathsf{R}\,;\ t < t_0\}$$

regardless of $\psi \in H(\mathsf{C}^n)$. Our main result is the following

THEOREM. *For every* $f \in B^N$, *vanishing for* $t<0$, *there exists a unique* $u \in B^N$, *also vanishing for* $t<0$, *solving* (1.1).

Further, if f *is carried by the set* $C \subset \mathsf{R} \times \mathsf{C}^n$, *then* u *can be carried by any neighbourhood of* $\bar{C}+\Gamma^*$ *in* $\{(t,z);\, t \geqq 0\}$. *Here* Γ^* *is a proper convex cone in* $\mathsf{R}^+ \times \mathsf{C}^n$, (*see* (4.4) *below*).

PROOF. By the Cauchy–Riemann equations (3.2) we have, for any $u \in B^N$

$$\sum_{j=1}^n \bar{A}_j' \,\partial u/\partial \bar{z}_j = 0 \tag{4.1}$$

where $\bar{A}_j'$ is the complex conjugate and transpose of the matrix A_j. Adding this to (1.1), we obtain a symmetric hyperbolic system

$$\begin{aligned} I\,\partial u/\partial t + \textstyle\sum_{j=1}^n \left[\tfrac{1}{2}(A_j+\bar{A}_j')\partial u/\partial x_j + (2i)^{-1}(A_j-\bar{A}_j')\partial u/\partial y_j\right] \\ = B(t,z)u+f\,, \end{aligned} \tag{4.2}$$

which thus has exactly the same solutions as (1.1) in B^N.

To solve (4.2) we consider the corresponding equation in R^{1+2n}:

$$\begin{aligned} I\,\partial U/\partial t + \textstyle\sum_{j=1}^n \left[\tfrac{1}{2}(A_j+\bar{A}_j')\partial U/\partial x_j + (2i)^{-1}(A_j-\bar{A}_j')\partial U/\partial y_j\right] \\ = B(t,x+iy)U+F \end{aligned} \tag{4.3}$$

where $F \in (\mathscr{D}'(\mathsf{R}^{1+2n}))^N$ "extends" f according to Lemma 3.3. We note that if f can be carried by C, and $\tilde{C}$ is any neighbourhood of $\bar{C}$, we can choose F with $\operatorname{supp} F \subset \tilde{C}$. Now the conditions in Proposition 2.1 are fulfilled; except that F vanishes only for $t<t_0$ where $t_0<0$ can be chosen arbitrarily. Hence there exists a unique $U \in (\mathscr{D}'(\mathsf{R}^{1+2n}))^N$, solving (4.3), vanishing for $t<t_0$, and such that

$$\operatorname{supp} U \subset \Gamma^* + \operatorname{supp} F$$

where Γ^* is the dual cone to

$$\begin{aligned} \Gamma = \{(\tau,\xi,\eta) \in \mathsf{R}^{1+2n}\,;\ \tau \geqq 0 \text{ and} \\ \det\left[I(\tau+\gamma)+\textstyle\sum \tfrac{1}{2}(A_j+\bar{A}_j')\xi_j + (2i)^{-1}(A_j-\bar{A}_j')\eta_j\right] = 0 \Rightarrow \gamma<0\}\,. \end{aligned} \tag{4.4}$$

By Lemma 3.4 there exists a $u \in B'$ such that $\langle u,\varphi,\psi\rangle = \langle U,\varphi\psi\rangle$ for $\varphi \in (C_0^\infty(\mathsf{R}))^N$, $\psi \in H(\mathsf{C}^n)^N$, and which is carried by any neighbourhood of $\operatorname{supp} U$. It is then clear that u solves (4.2), and therefore also (1.1).

To prove that u is uniquely determined, we have to prove that if F vanishes on test functions of the form

$$\varphi(t,x,y) = \vartheta(t)\psi(x+iy), \quad \vartheta \in C_0^\infty(\mathsf{R}), \psi \in H(\mathsf{C}^n) \tag{4.5}$$

then so does U.

An argument similar to the beginning of the proof of Proposition 2.1 shows that we may suppose that $\operatorname{supp} F \subset \{(t,x,y)\,;\, t \leqq T\}$ for some finite $T \in \mathsf{R}$, and hence is compact. A simple Taylor expansion shows that if $\langle F,\varphi\rangle = 0$ whenever φ is of the form (4.5), then so is the case for all $\varphi \in C_0^\infty(\mathsf{R}^{1+2n})$ which depend analytically on $x+iy$ for (t,x,y) in a neighbourhood of $\operatorname{supp} F$.

Let φ, of the form (4.5), be given. We replace it by a $\tilde{\varphi} \in C_0^\infty(\mathsf{R}^{1+2n})$ in such a way that $\varphi = \tilde{\varphi}$ in a neighbourhood of $\operatorname{supp} F + \Gamma^*$, and such that

$$\left(\textstyle\bigcup_{j=1}^n \operatorname{supp}(\partial\tilde{\varphi}/\partial\bar{z}_j) + \Gamma^*\right) \cap \operatorname{supp} F \;=\; \emptyset\,.$$

This is possible since $\operatorname{supp} F$ is compact.

With E^+* and $B^+\cdot$ defined in (2.6) we have for $1 \leqq j \leqq n$

$$\partial(E^+*\tilde{\varphi})/\partial\bar{z}_j \;=\; E^+*\partial\tilde{\varphi}/\partial\bar{z}_j, \qquad \partial(B^+\tilde{\varphi})/\partial\bar{z}_j \;=\; B^+\partial\tilde{\varphi}/\partial\bar{z}_j$$

and hence, with $\tilde{\varphi}^\nu$ defined by (2.5) it is seen by induction that

$$\langle W^\nu,\tilde{\varphi}\rangle \;=\; \langle F,\tilde{\varphi}^\nu\rangle \;=\; 0, \qquad \nu = 1,2,\ldots\,.$$

On the other hand, we have

$$\langle U,\varphi\rangle \;=\; \langle U,\tilde{\varphi}\rangle \;=\; \textstyle\sum_{\nu=1}^\infty \langle W^\nu,\tilde{\varphi}\rangle \;=\; 0$$

and thus U vanishes on test functions of the form (4.5). The uniqueness is proved.

From the uniqueness it follows, since

$$\langle u,\varphi,\psi\rangle \;=\; 0$$

for $\operatorname{supp}\varphi \subset \{(t,z)\,;\, t < t_0\}$ for any $t_0 < 0$, that u vanishes for $t < 0$, and the proof is complete.

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