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A THEOREM OF BEURLING-HELSON TYPE

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Our main result is the following.

THEOREM. *Let* (ω_n), $n \in \mathsf{Z}$, *be a positive sequence such that for some* $\alpha > 0$, $\beta > 0$,

$$(1)\qquad \begin{aligned} 1+\alpha/n \ &\leqq\ \omega_n\omega_{n-1}^{-1}\ \leqq\ 1+\beta/n\ ,\\ 1+\alpha/n\ &\leqq\ \omega_{-n}\omega_{-n+1}^{-1}\ \leqq\ 1+\beta/n\ ,\end{aligned}$$

for every $n \in \mathsf{Z}_+$. *Let* f *be a continuous and real-valued function on* $[-\pi,\pi]$, *not a linear function. For every* $n \in \mathsf{Z}_+$, $(c_{m,n})$, $m \in \mathsf{Z}$, *is the sequence of Fourier coefficients of* e^{inf}. *Then*

$$\omega_n^{-1} \textstyle\sum_{-\infty}^{\infty} |c_{m,n}|\omega_m \to \infty, \qquad as\ \ n \to \infty\ .$$

If (ω_n), $n \in \mathsf{Z}$, is changed to the constant sequence with values 1, the theorem is transformed into the theorem of Beurling and Helson [1]. Hence our theorem can be considered as an analogue of theirs. The assumption (1) is in particular fulfilled when $\omega_n = (1+|n|)^\alpha$, $n \in \mathsf{Z}$, where $\alpha > 0$. In that case the theorem is due to N. Leblanc [3], who used it to determine the endomorphisms of the corresponding weighted l^1-algebras. In the cases when (ω_n) is submultiplicative, similar applications can be made of our theorem.

Our proof differs from Leblanc's primarily in the respect that we make use of P. J. Cohen's theory of idempotents in group algebras [2], and in that way we avoid a complicated analysis of certain Lebesgue measurable sets on T. The proof uses the lemma below, which might be of independent interest. The lemma can be considered as a substitute for Propositions 2.2–2.5 in Leblanc's paper. The deduction of the theorem from the lemma is fairly close to Leblanc's proof of his Proposition 2.1, but some modifications have been necessary in order to allow a more general weight sequence (ω_n).

LEMMA. *Let* f *and* $(c_{m,n})$, $m \in \mathsf{Z}$, $n \in \mathsf{Z}_+$, *be defined as in the theorem. Furthermore we assume that* f *has a finite derivative, except in a set of Lebesgue measure* 0. *Then*

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$$\sum_{a<m/n<b} |c_{m,n}| \to \infty, \quad as\ n \to \infty ,$$

for every $a, b \in \mathsf{R}$ *such that the set* $\{x \in [-\pi, \pi] \mid a < f'(x) < b\}$ *has positive Lebesgue measure.*

We shall first show how the theorem can be deduced from the lemma.

Initiating an indirect approach, we assume that f satisfies the assumptions of the theorem, while for some $C > 0$, and some subsequence $(n_k)_1^\infty$ of Z_+,

$$\omega_{n_k}^{-1} \sum_{-\infty}^{\infty} |c_{m,n_k}| \omega_m \leqq C , \tag{2}$$

for every $k \in \mathsf{Z}_+$.

(1) implies the existence of positive constants c and d such that

$$|m/n|^c \leqq \omega_m \omega_n^{-1} \leqq |m/n|^d, \quad \text{for } |m| \geqq |n|,\ n \neq 0 . \tag{3}$$

(2) and (3) imply that

$$\sum_{|m| \geqq |n_k|} |m/n_k|^c |c_{m,n_k}| \leqq C , \tag{4}$$

for every $k \in \mathsf{Z}_+$.

For every $k \in \mathsf{Z}_+$ we form the discrete measure μ_k on R, defined by pointmasses c_{m,n_k} at the points m/n_k, $m \in \mathsf{Z}$. By (4) μ_k is a bounded measure. We extend f by the period 2π and obtain

$$\hat{\mu}_k(x) = \exp(in_k f(x/n_k)) = \int_{-\infty}^{\infty} e^{itx}\, d\mu_k(t), \quad x \in \mathsf{R} .$$

Let us choose an arbitrary ε such that $0 < \varepsilon \leqq 1$. By (4)

$$\int_{|t| > \varepsilon^{-1}} |d\mu_k(t)| \leqq \varepsilon^c \int_{|t| > \varepsilon^{-1}} |t|^c |d\mu_k(t)| \leqq C\varepsilon^c . \tag{5}$$

Hence, decomposing $\hat{\mu}_k$ by the relation

$$\begin{aligned} \hat{\mu}_k(x) &= \int_{|t| \leqq \varepsilon^{-1}} + \int_{|t| > \varepsilon^{-1}} e^{itx}\, d\mu_k(t) \\ &= \hat{\tau}_k(x) + \hat{\varrho}_k(x), \quad x \in \mathsf{R} , \end{aligned}$$

we obtain from (5)

$$\|\hat{\varrho}_k\|_\infty \leqq C\varepsilon^c . \tag{6}$$

Since $\|\hat{\mu}_k\|_\infty = 1$, this implies that

$$\|\hat{\tau}_k\|_\infty \leqq 1 + C\varepsilon^c . \tag{7}$$

Now, varying k, we see that the functions $\hat{\tau}_k$ are entire functions of uniformly bounded exponential type and hence (7) and Bernstein's theorem show that the functions $\{\hat{\tau}_k\}$, $k \in \mathsf{Z}_+$, are uniformly equicontinuous. Since $\hat{\mu}_k = \hat{\tau}_k + \hat{\varrho}_k$, and since ε can be made arbitrarily small, (6) shows that the functions $\{\hat{\mu}_k\}$, $k \in \mathsf{Z}_+$, are uniformly equicontinuous as well. Hence the functions on R with values $n_k f(x/n_k)$ are uniformly equicontinuous and this implies that f is Lipschitz continuous. Hence f is absolutely continuous with almost everywhere existing bounded derivative.

By the assumptions of the theorem, f is not identically constant. Hence there exists an interval $[a,b]$, not containing 0, such that

$$\{x \in [-\pi,\pi] \mid a < f'(x) < b\}$$

has positive Lebesgue measure. By (2), (3) and (4)

$$\sum_{a<m/n_k<b} |c_{m,n_k}|$$

is uniformly bounded in k. Since this contradicts the assertion of the lemma, our theorem is proved.

We shall now prove the lemma. Here, too, we give an indirect proof, assuming that there exist a $C>0$ and a subsequence $(n_k)_1^\infty$ of Z_+ such that

$$\sum_{a<m/n_k<b} |c_{m,n_k}| \leqq C , \tag{8}$$

for every $k \in \mathsf{Z}_+$. We shall show that this implies a contradiction, by proving that it leads to the existence of an idempotent measure on a certain compact group, this idempotent having properties not consistent with Cohen's description of the family of idempotents.

In the proof we use the standard auxiliary function $\varphi\colon \mathsf{R} \to \mathsf{R}$, defined by

$$\varphi(t) = \begin{cases} 1-|t|, & |t| \leqq 1 , \\ 0 \quad , & |t| > 1 . \end{cases}$$

The values of its Fourier transform $\hat{\varphi}$ are given by

$$\hat{\varphi}(x) = \int_{\mathsf{R}} e^{itx}\varphi(t)\,dt, \qquad x \in \mathsf{R} .$$

Obviously $\hat{\varphi} \in L^1(\mathsf{R})$, and the Fourier inversion formula holds for φ.

For every $d \in \mathsf{R}$, $\varepsilon > 0$, $k \in \mathsf{Z}_+$, we introduce the function $\hat{F}_{d,\varepsilon,k}$ on T, defined by

$$\hat{F}_{d,\varepsilon,k}(x) = \sum_{m\in\mathsf{Z}} c_{m,n_k}\varphi(\varepsilon^{-1}(mn_k{}^{-1}-d))e^{imx} . \tag{9}$$

Extending f periodically, we obtain from Parseval's relation

$$\begin{aligned}\hat{F}_{d,\varepsilon,k}(x) &= (2\pi)^{-1}\int_{\mathsf{R}} \exp\big(in_k f(x-t)\big)\,\varepsilon n_k\hat{\varphi}(\varepsilon n_k t)\,\exp(idn_k t)\,dt \\ &= (2\pi)^{-1}\int_{\mathsf{R}} \exp i\big(n_k f(x-s\varepsilon^{-1}n_k{}^{-1})+sd\varepsilon^{-1}\big)\hat{\varphi}(s)\,ds ,\end{aligned}$$

for $x \in \mathsf{R}$.

If $f'(x)$ exists finitely, we have by dominated convergence,

$$\text{(10)} \quad \lim_{k\to\infty} \hat{F}_{d,\varepsilon,k}(x)\exp(-in_k f(x)) = (2\pi^{-1}\int_{\mathsf{R}} \exp\big(i(-f'(x)+d)s\varepsilon^{-1}\big)\hat{\varphi}(s)\,ds = \varphi\big((d-f'(x))\varepsilon^{-1}\big)\ .$$

In particular (10) implies, if $f'(x)=d$,

$$\text{(11)} \quad \lim_{k\to\infty} \hat{F}_{d,\varepsilon,k}(x)\exp(-in_k f(x)) = 1\ .$$

Let $d_1,\ldots,d_m$ be different values in $]a,b[$, all attained by $f'(x)$, when $x\in\,]-\pi,\pi[$. We choose $\varepsilon>0$ so small that all intervals $]d_\nu-\varepsilon,d_\nu+\varepsilon[$ are contained in $]a,b[$ and are disjoint, $\nu=1,2,\ldots,m$. Then, by (8) and (9), for every $x_1,x_2,\ldots,x_m\in\,]-\pi,\pi[$,

$$\sum_{\nu=1}^{m} |\hat{F}_{d_\nu,\varepsilon,k}(x_\nu)| \leqq \sum_{a<m/n_k<b} |c_{m,n_k}| \leqq C\ ,$$

and by (11) we can conclude that $m\leqq C$. Hence only finitely many values in $]a,b[$ are attained by $f'(x)$, and for that reason there exists a value d, taken in a subset of $]-\pi,\pi[$ of positive Lebesgue measure. We choose $\varepsilon_0>0$ so small that $]d-\varepsilon_0,d+\varepsilon_0[\,\subset\,]a,b[$, and consider the family $\mathscr{F}$ of functions $\hat{F}_{d,\varepsilon,k}$ on T with $0<\varepsilon<\varepsilon_0$, $k\in\mathsf{Z}_+$. By (8) the functions in the family have uniformly bounded norm in $A(\mathsf{T})$, and so have the functions in the family $\mathscr{G}$ of all $|\hat{F}_{d,\varepsilon,k}|^2$, too. By (11),

$$\lim_{k\to\infty} |\hat{F}_{d,\varepsilon,k}(x)|^2 = 1$$

for every x in a set $E_1\subset\mathsf{T}$ of positive Lebesgue measure. By (10) and the assumption that $f'(x)$ exists almost everywhere and that f is not linear, we have

$$\lim_{\varepsilon\to 0}\lim_{k\to 0} |\hat{F}_{d,\varepsilon,k}(x)|^2 = 0$$

in a subset $E_2\subset\mathsf{T}$ of positive Lebesgue measure, and such that $E_3=\mathsf{T}\setminus(E_1\cup E_2)$ has zero Lebesgue measure.

The family $\mathscr{G}$ can be considered as a norm-bounded family of functions in $B(\mathsf{T}_d)$, the space of Fourier–Stieltjes transforms on the discrete circle group. By the compactness of the dual group of T_d we can conclude that $B(\mathsf{T}_d)$ contains a function φ taking the values 1 on E_1, 0 on E_2.

Let G denote the subgroup of T_d of all $x\in\mathsf{T}_d$ with $x=2\pi p\cdot 2^{-q}$, where $p\in\mathsf{Z}$, $q\in\mathsf{N}$. Since G is denumerable, we have that E_3+G has Lebesgue measure zero. If F_i denotes the set of all points x in E_i such that $(x+G)\cap E_i$ is finite, $i=1,2$, it is easy to see that F_1 and F_2 are Lebesgue nullsets as well. From this we can conclude the existence of a $x_0\in\mathsf{T}$ such that x_0+G only contains points where $\varphi(x)=0$ or 1, infinitely many of each kind. Putting $\psi(x)=\varphi(x_0+x)$, for each $x\in G$, we obtain a Fourier–

Stieltjes transform on G, only taking the values 0 and 1, each of these values taken on an infinite set of points.

Each proper subgroup of G is finite. Hence the coset-ring of G consists of all finite subsets of G and their complements. By Theorem 3 in Cohen [2], ψ is the characteristic function of a member of the coset-ring. But both $\psi^{-1}(0)$ and $\psi^{-1}(1)$ are infinite which gives a contradiction. Hence the lemma is proved.

If (ω_n), $n \in \mathbf{Z}$, is positive, even and satisfies

$$1+\alpha/n \leqq \omega_n \omega_{n-1}^{-1} \leqq 1+\varepsilon_n n^{-1} \log n ,$$

for sufficiently large positive n, where $\alpha > 2$, $\varepsilon_n \to 0$, as $n \to \infty$, then the conclusion of the theorem still holds. For the left hand inequality shows that every f, giving a counter-example to the theorem for (ω_n), has to be twice continuously differentiable. Since it is not linear, it is a well-known consequence of van der Corput's lemma that for some a,b,

$$\liminf_{n\to\infty} n^{-\frac{1}{2}} \sum_{a<m/n<b} |c_{m,n}| > 0 .$$

From this the desired contradiction follows, using the right-hand inequality. More general results of this type have been announced in Leibenzon [4], see also Levina [5].

On the other hand, if (ω_n), $n \in \mathbf{Z}$, is positive, even and of growth at infinity slower than exponential, and satisfying

$$1+an^{-1}\log n \leqq \omega_n \omega_{n-1}^{-1}, \quad a > 0 ,$$

for large n, then $f(x) = \varepsilon \cos x$, $x \in [-\pi,\pi]$, $\varepsilon > 0$, gives for small ε a function f which does not fulfil the conclusion of the theorem. For using the inequalities

$$\begin{aligned} \sum |c_{m,n}|^2 |r|^{2m} &= 2\pi^{-1} \textstyle\int_{-\pi}^{\pi} \exp\big(n\varepsilon(r+r^{-1})\cos\theta\big)\, d\theta \\ &\leqq \exp\big(n\varepsilon(r+r^{-1})\big), \quad 0 < r \leqq \infty , \end{aligned}$$

it is easy to conclude that $\omega_n{}^{-1} \sum_{-\infty}^{\infty} |c_{m,n}| \omega_n$ is bounded, as $n \to \infty$, if ε is small in comparison with a. A larger class of counter-examples is given in Leibenzon [4].

It is an open problem to investigate what happens, when $\omega_n \omega_{n-1}^{-1} = 1+o(1/n)$ at infinity. The reason why our method does not work in that case is that we can not then prove the Lipschitz' continuity for the function f in the indirect proof.

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