MATH. SCAND. 33 (1973), 145–152

THE THEOREMS OF F. AND M. RIESZ FOR CIRCULAR SETS

FRANK FORELLI

1. Introduction.

1.1. Let V be a vector space over $\mathbf{C}$ of complex dimension n with an inner product. We will denote (as is usual) by $\mathbf{T}$ the class of all z in $\mathbf{C}$ such that $z\bar{z}=1$.

Let $X \subset V$ be bounded, $\neq \emptyset$, and locally compact. (We define the topology of X by means of the metric that is associated with the inner product of V.) Thus $\bar{X}-X$ is closed. Furthermore let X be such that if $(z,x) \in \mathbf{T} \times X$, then $zx \in X$.

If A is a topological space, then we will denote (as is usual) by $C(A)$ the class of all continuous functions $f\colon A \to \mathbf{C}$ and we will denote by $C_0(A)$ the class of all functions in $C(A)$ that vanish at infinity. If A is a locally compact Hausdorff space, then we will denote by $M_+(A)$ the class of all Radon measures on A. Thus if $\mu \in M_+(A)$ and $E \subset A$, then $\mu(E) \geqq 0$. We will denote by $M(A)$ the complex linear span of those μ in $M_+(A)$ for which $\mu(A) < \infty$. (Thus if A is compact, then $M(A)$ is the complex linear span of $M_+(A)$.)

Let $\sigma \in M(X)$, $\sigma \neq 0$. Furthermore let σ be such that if $z \in \mathbf{T}$ and $E \subset X$, then $\sigma(zE)=\sigma(E)$.

We will denote by $H(\sigma)$ the $w(M(X),C_0(X))$ closure of the class of all measures in $M(X)$ of the form $g\sigma$ where g is in the polynomial ring $\mathbf{C}[\chi : \chi \in V^*]$. Thus if $\mu \in H(\sigma)$, if $F \subset C_0(X)$ is finite, and if $\varepsilon > 0$, then there is a polynomial g in $\mathbf{C}[\chi : \chi \in V^*]$ such that

$$|\textstyle\int f\,d\mu - \int fg\,d\sigma| < \varepsilon$$

for every f in F.

If k is a positive integer, then we will denote by H_k the class of all members of the polynomial ring $\mathbf{C}[\chi : \chi \in V^*]$ that are homogeneous of degree k. There is the following property which may or may not hold.

1.1.1. If $f \in \bigcup_{k=1}^{\infty} H_k$ and if $f\sigma \neq 0$, then $\sigma \ll f\sigma$.

The purpose of this paper is to prove the following two theorems.

Received January 15, 1973.

1.2. THEOREM. *If $\mu \in H(\sigma)$ and if $\varepsilon > 0$, then there is a polynomial g in $\mathsf{C}[\chi : \chi \in V^*]$ such that $\|\mu - g\sigma\| < \varepsilon$. Thus if $\mu \in H(\sigma)$, then $\mu \ll \sigma$.*

1.3. THEOREM. *Let the property* 1.1.1 *hold. If $\mu \in H(\sigma)$ and if $\mu \neq 0$, then $\sigma \ll \mu$.*

1.4. With regard to Theorem 1.2 we refer to [1], [4], and [6, Chapter 3].

1.5. If x and y are in V, then we will denote by $\langle x,y \rangle$ the inner product of x and y. If $\alpha \in [0,\infty)$, then we will denote by H^α the Hausdorff measure on X of dimension α. (We define H^α by means of the metric that is associated with the inner product of V. Thus if $z \in \mathsf{T}$ and $E \subset X$, then $H^\alpha(zE) = H^\alpha(E)$.) With regard to 1.1, 1.2, and 1.3 we cite the following example. Let

$$X = \{x : x \in V, \langle x,x \rangle = 1\}$$

and let $\sigma = H^{2n-1}$. Furthermore let

$$Y = \{y : y \in V, \langle y,y \rangle < 1\}.$$

We recall that the Poisson kernel of Y is the function $\beta : X \times Y \to (0,\infty)$ defined by

$$\beta(x,y) = [(1 - \langle y,y \rangle)/(1 - \langle x,y \rangle)(1 - \langle y,x \rangle)]^n .$$

We recall that if A and B are sets, if f is a function defined on the Cartesian product $A \times B$, and if $(s,t) \in A \times B$, then f_s and f^t are the functions defined on B and A respectively by $f_s(b) = f(s,b)$ and $f^t(a) = f(a,t)$. If $\mu \in M(X)$, then for the purpose of this example we define $\mu^\# : Y \to \mathsf{C}$ by

$$\mu^\#(y) = \int \beta^y \, d\mu .$$

Thus $\mu^\# \in C^\infty(Y)$. We will denote (as is usual) by D the class of all z in C such that $z\bar{z} < 1$. We recall the following fact of the theory of the Poisson integral. If $(z,f,\mu) \in \mathsf{D} \times C(X) \times M(X)$ and if $z \to 1$, then

$$\int f(x)\mu^\#(zx)\, d\sigma(x) \to \int f\, d\mu ,$$

thus if $\mu^\#$ is holomorphic, then $\mu \in H(\sigma)$.

2. On the theory of flows.

2.1. If A, B, and N are sets, if $\varphi : A \to B$, and if $\mu : 2^A \to N$, then we define $\varphi^*(\mu) : 2^B \to N$ by

$$\varphi^*(\mu)(E) = \mu(\{a : a \in A, \varphi(a) \in E\}) .$$

With regard to this definition we recall the following fact of measure theory [2, page 72].

2.2. PROPOSITION. *If A and B are compact Hausdorff spaces, if $\varphi: A \to B$ is continuous, and if $\mu \in M_+(A)$, then $\varphi^*(\mu) \in M_+(B)$. Thus if $\mu \in M(A)$, then $\varphi^*(\mu) \in M(B)$.*

2.3. With regard to Proposition 2.2 we remark that if $f \in C(B)$, then

$$\int f d\varphi^*(\mu) = \int f \circ \varphi d\mu .$$

The following proposition (whose proof we omit) follows from Proposition 2.2.

2.4. PROPOSITION. *If A and B are locally compact Hausdorff spaces, if $\varphi: A \to B$ is continuous, if $\mu \in M_+(A)$, and if $\mu(A) < \infty$, then $\varphi^*(\mu) \in M_+(B)$. Thus if $\mu \in M(A)$, then $\varphi^*(\mu) \in M(B)$.*

2.5. We recall that $H^\infty(\mathsf{R})$ is the class of all functions f in $L^\infty(\mathsf{R})$ such that

$$\int \mathrm{Im}\,[1/(t-z)]f(t)\,dt$$

is holomorphic on $\{z: z \in \mathsf{C}, \mathrm{Im}(z) > 0\}$. Let (R, S, T) be a topological transformation group. (Thus by definition S is a locally compact Hausdorff space, $T: \mathsf{R} \times S \to S$, etc.) For the purpose of the proof of Theorem 1.2 we recall the following fact of the theory of flows [3, Theorem 4].

2.6. THEOREM. *Let $\mu \in M(S)$ and define $f: \mathsf{R} \to M(S)$ by $f(t) = (T_t)^*(\mu)$. If*

$$\int g \circ T^x d\mu(x) \in H^\infty(\mathsf{R})$$

for every g in $C_0(S)$, then f is continuous with respect to the norm topology of $M(S)$.

2.7. For the purpose of the proof of Theorem 1.3 we recall the following fact of the theory of flows [3, Theorem 3].

2.8. THEOREM. *Let $\mu \in M(S)$ and let $E \subset S$ be of $|\mu|$ measure 0. If*

$$\int g \circ T^x d\mu(x) \in H^\infty(\mathsf{R})$$

for every g in $C_0(S)$, then $\mu(T_t(E)) = 0$ for every t in R.

3. The proof of Theorem 1.2.

3.1. We define $Z\colon \mathsf{T}\times X\to X$ by $Z(z,x)=\bar{z}x$. Thus (T,X,Z) is a topological transformation group. We will denote by τ the Lebesgue measure on T such that $\tau(\mathsf{T})=1$. Thus if $f\in C(\mathsf{T})$, then

$$\int f\,d\tau = (2\pi)^{-1}\int_0^{2\pi} f(e^{it})\,dt\,.$$

We recall that if $f\in C(\mathsf{T})$, then $\hat{f}\colon \mathsf{Z}\to\mathsf{C}$ is defined by

$$\hat{f}(k) = \int \bar{z}^k f(z)\,d\tau(z)\,.$$

Furthermore we recall that $A(\mathsf{T})$ is the class of all f in $C(\mathsf{T})$ such that $\hat{f}(k)=0$ if $k<0$.

3.2. Proposition. *If $\mu\in H(\sigma)$, if $f\in C_0(X)$, and if g in $C(\mathsf{T})$ is defined by*

$$g = \int f\circ Z^x\,d\mu(x)\,,$$

then $g\in A(\mathsf{T})$.

Proof. Let $k\in\mathsf{Z}$, let $f^{\#}$ in $C_0(X)$ be defined by

$$f^{\#} = \int f\circ Z_z z^k\,d\tau(z)\,,$$

let $\varepsilon>0$, and let h in $\mathsf{C}[\chi\colon\chi\in V^*]$ be such that

$$|\int f^{\#}d\mu-\int f^{\#}h\,d\sigma| < \varepsilon\,. \tag{3.1}$$

We have

$$\begin{aligned}\hat{g}(-k) &= \int z^k[\int f\circ Z_z\,d\mu]\,d\tau(z) = \int f^{\#}d\mu \\ &= (\int f^{\#}d\mu-\int f^{\#}h\,d\sigma)+\int f^{\#}h\,d\sigma\,.\end{aligned} \tag{3.2}$$

Furthermore if $z\in\mathsf{T}$, then

$$\int f\circ Z_z h\,d\sigma = \int f(\bar{z}x)h(x)\,d\sigma(x) = \int f(x)h(zx)\,d\sigma(x)\,,$$

hence

$$\int f^{\#}h\,d\sigma = \int[\int f(x)h(zx)\,d\sigma(x)]z^k\,d\tau(z)\,,$$

hence if $h=c+\sum_{j\geq1}h_j$ where $c\in\mathsf{C}$ and $h_j\in H_j$, then

$$\begin{aligned}\int f^{\#}h\,d\sigma &= \int f(x)[\int(c+\textstyle\sum_{j\geq1}h_j(x)z^j)z^k\,d\tau(z)]\,d\sigma(x) \\ &= \int f(x)[\int(cz^k+\textstyle\sum_{j\geq1}h_j(x)z^{j+k})\,d\tau(z)]\,d\sigma(x)\,.\end{aligned}$$

Thus if $k>0$, then

$$\int f^{\#}h\,d\sigma = 0\,, \tag{3.3}$$

hence by (3.1), (3.2), and (3.3) we have $|\hat{g}(-k)|<\varepsilon$ which completes the proof of Proposition 3.2.

3.3. PROPOSITION. *Let* $\mu \in M(X)$ *and define* $f\colon \mathsf{T} \to M(X)$ *by* $f(z) = (Z_z)^*(\mu)$. (*We refer to section* 2.1 *for the definition of* $(Z_z)^*$.) *If* $\mu \in H(\sigma)$, *then* f *is continuous with respect to the norm topology of* $M(X)$.

PROOF. If we define $T\colon \mathsf{R} \times X \to X$ by $T(t,x) = e^{-it}x$, then Proposition 3.3 follows from Proposition 3.2 and Theorem 2.6.

3.4. If A is a vector space, then we will denote (as is usual) by A' the class of all linear functionals on A.

3.5. PROPOSITION. *Let* A *be a vector space over* C, *let* N *be a subspace of* A, *and let* B *be a subspace of* A'. *If* N *is of finite dimension and if* B *distinguishes points of* A, *then* N *is* $w(A,B)$ *closed.*

PROOF. Since B distinguishes points of A, the subspace of A of dimension 0 is $w(A,B)$ closed. We assume that Proposition 3.5 holds for every subspace of A of dimension m, and we let N be of dimension $m+1$. Let $\{x_1,\ldots,x_{m+1}\}$ be a basis of N. If $1 \leqq k \leqq m+1$, then by the induction hypothesis and [5, Corollary 14.4] there is an α^k in B such that $\alpha^k(x_j) = \delta_j{}^k$. Let $y \in A$ and let

$$x = \sum_{k=1}^{m+1} \alpha^k(y) x_k .$$

If $y \neq x$, then since B distinguishes points of A there is a β in B such that $\beta(y) \neq \beta(x)$. If

$$\gamma = \beta - \sum_{k=1}^{m+1} \beta(x_k)\alpha^k ,$$

then

$$\gamma(x_j) = \beta(x_j) - \sum_{k=1}^{m+1} \beta(x_k)\delta_j{}^k = 0 ,$$

hence $\gamma = 0$ on N. Furthermore

$$\gamma(y) = \beta(y) - \sum_{k=1}^{m+1} \beta(x_k)\alpha^k(y) = \beta(y) - \beta(x) \neq 0$$

which completes the proof of Proposition 3.5.

3.6. PROPOSITION. *Let* $\mu \in M(X)$ *and let* f *in* $C(\mathsf{T})$ *be a trigonometric polynomial. If* $\mu \in H(\sigma)$, *then*

$$Z^*(f\tau \times \mu) = \left(\hat{f}(0)c + \sum_{k \geqq 1} \hat{f}(-k) g_k\right)\sigma$$

where $c \in \mathsf{C}$ *and* $g_k \in H_k$.

PROOF. Let $g \in \mathsf{C}[\chi : \chi \in V^*]$. If $(z,h) \in \mathsf{T} \times C_0(X)$, then (as before)

$$\int h \circ Z_z g \, d\sigma = \int h(\bar{z}x) g(x) \, d\sigma(x) = \int h(x) g(zx) \, d\sigma(x) ,$$

hence

$$\int h\,d(Z^*(f\tau\times g\sigma)) = \int[\int h(x)g(zx)\,d\sigma(x)]f(z)\,d\tau(z)\,,$$

hence if $g=c+\sum_{k\geqq 1}g_k$ where $c\in\mathbf{C}$ and $g_k\in H_k$, then

$$\begin{aligned}\int h\,d(Z^*(f\tau\times g\sigma)) &= \int h(x)[\int(c+\textstyle\sum_{k\geqq 1}g_k(x)z^k)f(z)\,d\tau(z)]\,d\sigma(x)\\ &= \int h(\hat{f}(0)c+\textstyle\sum_{k\geqq 1}\hat{f}(-k)g_k)\,d\sigma\,,\end{aligned}$$

hence

$$Z^*(f\tau\times g\sigma) = (\hat{f}(0)c+\textstyle\sum_{k\geqq 1}\hat{f}(-k)g_k)\sigma\,. \tag{3.4}$$

For the purpose of the proof of Proposition 3.6 we will denote by N the class of all measures in $M(X)$ of the form $Z^*(f\tau\times g\sigma)$ where $g\in\mathbf{C}[\chi:\chi\in V^*]$. Since the vector space H_k is of finite dimension $(=\binom{n+k-1}{n-1})$, it follows from the identity (3.4) that the vector space N is of finite dimension.

If $h\in C_0(X)$, then for the purpose of the proof of Proposition 3.6 we define $h^\#$ in $C_0(X)$ by

$$h^\# = \int h\circ Z_z f(z)\,d\tau(z)\,.$$

If $\varepsilon>0$ and if $F\subset C_0(X)$ is finite, then there is a g in $\mathbf{C}[\chi:\chi\in V^*]$ such that

$$|\int h^\#\,d\mu-\int h^\# g\,d\sigma| < \varepsilon \tag{3.5}$$

for every h in F. If $h\in C_0(X)$, then

$$\begin{aligned}&\int h\,d(Z^*(f\tau\times\mu))-\int h\,d(Z^*(f\tau\times g\sigma))\\ &\quad= \int[\int h\circ Z_z f(z)\,d\tau(z)]\,d\mu-\int[\int h\circ Z_z f(z)\,d\tau(z)]g\,d\sigma\\ &\quad= \int h^\#\,d\mu-\int h^\# g\,d\sigma\,,\end{aligned}$$

hence if $h\in F$, then by (3.5)

$$|\int h\,d(Z^*(f\tau\times\mu))-\int h\,d(Z^*(f\tau\times g\sigma))| < \varepsilon\,.$$

Thus $Z^*(f\tau\times\mu)$ is in the $w(M(X),C_0(X))$ closure of N, hence by Proposition 3.5 $Z^*(f\tau\times\mu)\in N$. Thus there is a polynomial g in $\mathbf{C}[\chi:\chi\in V^*]$ such that

$$Z^*(f\tau\times\mu) = Z^*(f\tau\times g\sigma)$$

which by means of (3.4) completes the proof of Proposition 3.6.

3.7. We will now prove Theorem 1.2. If $(\lambda,g)\in M(\mathbf{T})\times C_0(X)$ and if $\lambda(\mathbf{T})=1$, then

$$\int g\,d(Z^*(\lambda\times\mu)-\mu) = \int[\int g\,d((Z_z)^*(\mu)-\mu)]\,d\lambda(z)\,,$$

hence

$$\|Z^*(\lambda\times\mu)-\mu\| \leqq \int\|(Z_z)^*(\mu)-\mu\|\,d|\lambda|(z)\,.$$

Thus by Proposition 3.3 there is a trigonometric polynomial f such that

$$\|Z^*(f\tau \times \mu) - \mu\| < \varepsilon$$

which by means of Proposition 3.6 completes the proof of Theorem 1.2.

4. The proof of Theorem 1.3.

4.1. Proposition. *Let $\mu \in M(X)$ and let $E \subset X$ be of $|\mu|$ measure 0. If $\mu \in H(\sigma)$, then $\mu(zE) = 0$ for every z in T.*

Proof. If (as before) we define $T: \mathsf{R} \times X \to X$ by $T(t,x) = e^{-it}x$, then Proposition 4.1 follows from Proposition 3.2 and Theorem 2.8.

4.2. We will now prove Theorem 1.3. If $(\lambda, f) \in M(\mathsf{T}) \times C_0(X)$ and if g in $C(\mathsf{T})$ is defined by

$$g = \int f \circ Z^x \, d\mu(x) ,$$

then

$$\int f d(Z^*(\lambda \times \mu)) = \int g d\lambda .$$

If $\int f d\mu \neq 0$, then $g \neq 0$ and hence $\hat{g} \neq 0$. Thus since $\mu \neq 0$ there is a j in Z such that if e is the trigonometric polynomial defined by $e(z) = \bar{z}^j$, then $Z^*(e\tau \times \mu) \neq 0$. By Proposition 3.6 we have

$$Z^*(e\tau \times \mu) = g\sigma \tag{4.1}$$

where $g \in \mathsf{C} \cup \bigcup_{k=1}^{\infty} H_k$. Let $E \subset X$ be a Borel set of $|\mu|$ measure 0. If $F \subset E$, then by Proposition 4.1 $\mu(zF) = 0$ for every z in T, hence if F is a Borel set and if f is the characteristic function of F, then by (4.1)

$$\begin{aligned} \int_F g\, d\sigma &= \int f d(Z^*(e\tau \times \mu)) \\ &= \int \left[\int f(\bar{z}x)\, d\mu(x)\right] e(z)\, d\tau(z) = 0 , \end{aligned}$$

hence

$$\int_E |g|\, d|\sigma| = 0 ,$$

hence by property 1.1.1 $|\sigma|(E) = 0$ which completes the proof of Theorem 1.3.

REFERENCES

1. S. Bochner, *Classes of holomorphic functions of several variables in circular domains*, Proc. Nat. Acad. Sci. U.S.A. 46 (1960), 721–723.
2. H. Federer, *Geometric Measure Theory* (Grundlehren Math. Wissensch. 153), Springer-Verlag, Berlin, Heidelberg, New York, 1969.
3. F. Forelli, *Analytic and quasi-invariant measures*, Acta Math. 118 (1967), 33–59.

4. L. Gårding and L. Hörmander, *Strongly subharmonic functions*, Math. Scand. 15 (1964), 93–96.
5. J. Kelley and I. Namioka, *Linear Topological Spaces*, D. Van Nostrand Company Inc., Princeton, 1963.
6. W. Rudin, *Function Theory in Polydiscs*, W. A. Benjamin Inc., New York, 1969.

THE UNIVERSITY OF WISCONSIN, MADISON, U. S. A.