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A GENERALIZATION OF FØLNER'S CONDITION

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Introduction.

Let G be a group acting on a set X and let A be a non-empty subset of X. We consider here a problem first posed by von Neumann [10]: when does there exist a finitely-additive G-invariant measure μ defined on all subsets of X and taking values in $[0,\infty]$ with $\mu(A)=1$? When the group G is amenable it is well-known that this is equivalent to whenever $a_1,\ldots,a_n$ are real numbers and $g_1,\ldots,g_n$ are in G then $\sum_{i=1}^n a_i\chi_{g_iA} \geqq 0$ implies $\sum_{i=1}^n a_i \geqq 0$ where χ_S is the characteristic function of S. Several equivalent versions of this translate property are derived. Let $\|\cdot\|$ be cardinality. The translate property is in particular equivalent to having a net of finite sequences $\{F_\alpha\}$ in X such that

$$\|F_\alpha \cap gA\|/\|F_\alpha \cap A\| \to 1 \quad \text{for all } g\in G .$$

For arbitrary groups we prove a theorem characterizing when the measure μ exists; this is the case if and only if there exists a net $\{F_\alpha\}$ of finite sets in X such that

$$\|(F_\alpha \,\Delta\, gF_\alpha)\cap A\|/\|F_\alpha\cap A\|$$

converges to 0 for all $g\in G$ where Δ denotes the symmetric difference. A corollary of this is that when G is amenable a measure μ as above exists if and only if there exists a net $\{F_\alpha\}$ of finite sets in X such that

$$\|F_\alpha\cap gA\|/\|F_\alpha\cap A\| \to 1 \quad \text{for all } g\in G .$$

These theorems can easily be generalized to other measures in X besides $\|\cdot\|$.

0. Preliminaries.

We will use Z, Q, R for the integers, the rational numbers, and the real numbers respectively. Let Z^+, Q^+, and R^+ be the positive elements of Z, Q, and R respectively. We will sometimes extend the real numbers

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by ∞ with $a+\infty=\infty$ for all $a \in \mathsf{R}\cup\{\infty\}$. We use the following set notation for subsets A and B of a set X:

$$A \cup B = \{x \in X : x \in A \text{ or } x \in B\}$$

$$A \cap B = \{x \in X : x \in A \text{ and } x \in B\}$$

$$A \setminus B = \{x \in X : x \in A \text{ and } x \notin B\}$$

$$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)\,.$$

χ_A is the characteristic function of A defined on X by $\chi_A(x)=1$ if $x \in A$ and $\chi_A(x)=0$ if $x \in X \setminus A$. $\|A\|$ will denote the number of elements of a finite set A. If $S=(s_1,\ldots,s_n)$ is a finite sequence in X then $\|S\|=n$. For a set $A \subset X$ and a sequence $S=(s_1,\ldots,s_n)$,

$$\|A \cap S\| = \textstyle\sum_{i=1}^n \chi_A(s_i)\,.$$

This is consistent with the case that S has no repetitions and is considered as a finite set.

Let E be a real linear space. A *cone* in E is a non-empty set P such that $P+P \subset P$ and $rP \subset P$ for all $r \in \mathsf{R}^+$. A *rational cone* in E is a non-empty set P such that $P+P\subset P$ and $qP \subset P$ for all $q \in \mathsf{Q}^+$. Given a set $X \subset E$ the cone (rational cone) generated by X is the smallest cone (rational cone) containing X. For instance, the rational cone generated by X is

$$\{\textstyle\sum_{i=1}^n q_i x_i : q_i \in \mathsf{Q}^+ \text{ and } x_i \in X\}\,.$$

The span of X is as usual the smallest linear subspace of E containing X.

Our use of topology and linear topological spaces follows the notation in Kelley [5] and Kelley, Namioka, et al. [6]. We will in particular need the following which we state here for convenience:

0.1 PROPOSITION. *Let A be a convex set in a locally convex space (E,τ). A point $x \in E$ is not in the τ-closure of A if and only if there exists $f \in E^*$ such that*

$$\sup_{a\in A}\langle f,a\rangle < \langle f,x\rangle\,.$$

0.2 COROLLARY. *Let A be a convex set in a locally convex space (E,τ). The τ-closure of A and the weak-closure of A are identical.*

0.3 PROPOSITION. *Let $\{(E_i,\tau_i): i \in I\}$ be a family of locally convex spaces. Let $F=\prod_{i\in I}E_i$ with the product topology $\prod_{i\in I}\tau_i$. Then the weak-topology of $(F, \prod_{i\in I}\tau_i)$ is the product of the weak-topologies of $\{(E_i,\tau_i)\}$.*

0.4 PROPOSITION. *Given a locally convex space* (E,τ), *a linear functional* φ *on* $(E,\tau)^*$ *is weak*-continuous if and only if there exists* $x \in E$ *such that* $\langle \varphi, f\rangle = \langle f, x\rangle$ *for all* $f \in (E,\tau)^*$.

Given a set X, by a finitely-additive measure on X we will always mean a function μ defined on all subsets of X taking values in $[0,\infty]$ such that when A and B are disjoint subsets of X, $\mu(A \cup B) = \mu(A) + \mu(B)$. For all notions in measure theory see Rudin [9] or Kelley, Namioka, et al. [6]. An excellent reference for information on amenable groups is Greenleaf [4]. This same source is good for a summary of von Neumann's work on the existence of invariant finitely-additive measures.

1. The translate property.

Let G be a group acting on a set X. A finitely-additive measure μ defined on all subsets of X is *G-invariant* if and only if when $g \in G$ and $S \subset X$ then $\mu(gS) = \mu(S)$. Let A be a non-empty subset of X. A problem first posed in [10] is to decide when there is a finitely-additive G-invariant measure μ defined on all subsets of X such that $\mu(A) = 1$? It is convenient to restate this question in terms of linear functionals.

A subset $B \subset X$ will be called *A-bounded* if and only if there exist $g_1, \ldots, g_n \in G$ such that $B \subset \bigcup_{i=1}^{n} g_i A$. Given a function $f: X \to \mathbb{R}$ let supp$f$ be $\{x \in X : f(x) \neq 0\}$. Let $B_A(X)$ consist of all bounded real-valued functions f such that supp$f$ is A-bounded. Under pointwise addition and the usual scalar multiplication, $B_A(X)$ is a real vector space. The group action of G on X induces an action of G on $B_A(X)$ defined by $(gf)(x) = f(g^{-1}x)$ for all $g \in G$, $f \in B_A(X)$, and $x \in X$. This represents G as a group of linear transformations of $B_A(X)$.

A function $f \in B_A(X)$ is positive, written $f \geqq 0$, if and only if $f(x) \geqq 0$ for all $x \in X$. A linear functional θ on $B_A(X)$ is positive if and only if $\langle \theta, f\rangle \geqq 0$ whenever $f \in B_A(X)$ and $f \geqq 0$. The linear functional θ is G-invariant if and only if $\langle \theta, gf\rangle = \langle \theta, f\rangle$ for all $g \in G$ and $f \in B_A(X)$.

1.1 PROPOSITION. *There exists a finitely-additive G-invariant measure* μ *defined on all subsets of* X *such that* $\mu(A) = 1$ *if and only if there exists a positive G-invariant linear functional* θ *on* $B_A(X)$ *such that* $\langle \theta, \chi_A\rangle = 1$.

PROOF. See Greenleaf [4].

We say that there is *an invariant for* (G, X, A) when the condition of 1.1 holds. An invariant for (G, X, A) will be a G-invariant positive linear functional θ on $B_A(X)$ such that $\langle \theta, \chi_A\rangle = 1$. It will be understood when

(G, X, A) is written that G is a group acting on a set X and A is a non-empty subset of X. Notice that G is amenable if and only if there is an invariant for (G, G, G) with the group action just group multiplication.

In Section 3 and 4 criteria for the existence of an invariant with G arbitrary will be given. For now, assume G is amenable. Let S_A be the span of $\{\chi_{gA} : g \in G\}$. Then S_A is a G-invariant subspace of $B_A(X)$ and in fact the smallest G-invariant subspace containing χ_A. A well-known result for amenable groups goes as follows:

1.2. PROPOSITION. *If G is amenable there exists an invariant for (G, X, A) if and only if there exists a positive G-invariant linear functional φ on S_A such that $\langle \varphi, \chi_A \rangle = 1$.*

PROOF. See Greenleaf [4].

From this proposition we get the following:

1.3 COROLLARY. *When G is amenable there exists an invariant for (G, X, A) if and only if whenever $g_1, \ldots, g_n \in G$ and $a_1, \ldots, a_n \in \mathsf{R}$, $\sum_{i=1}^n a_i \chi_{g_i A} \geqq 0$ implies $\sum_{i=1}^n a_i \geqq 0$.*

PROOF. Let us say (G, X, A) has the *translate property* when the above condition holds. If it holds define φ on S_A by

$$\langle \varphi, \textstyle\sum_{i=1}^n a_i \chi_{g_i A} \rangle = \textstyle\sum_{i=1}^n a_i .$$

φ is well-defined since if we have

$$\textstyle\sum_{i=1}^n a_i \chi_{g_i A} = \textstyle\sum_{j=1}^m b_j \chi_{g_j A}$$

then

$$H = \textstyle\sum_{i=1}^n a_i \chi_{g_i A} - \textstyle\sum_{j=1}^m b_j \chi_{g_j A}$$

is both positive and negative. Hence, $\sum_{i=1}^n a_i - \sum_{j=1}^m b_j$ is both positive and negative by the translate property; so $\sum_{i=1}^n a_i = \sum_{j=1}^m b_j$. It follows easily now that φ is a positive G-invariant linear functional on S_A such that $\langle \varphi, \chi_A \rangle = 1$.

Conversely, if such a φ exists then when $\sum_{i=1}^n a_i \chi_{g_i A} \geqq 0$,

$$0 \leqq \langle \varphi, \textstyle\sum_{i=1}^n a_i \chi_{g_i A} \rangle = \textstyle\sum_{i=1}^n a_i .$$

1.4 PROPOSITION. (*The translate property*). *Given (G, X, A) the following are equivalent:*

1) *For all* $a_1,\ldots,a_n \in \mathsf{R}$ *and* $g_1,\ldots,g_n \in G$,

$$\sum_{i=1}^{n} a_i\chi_{g_iA} \geqq 0 \quad \textit{implies} \quad \sum_{i=1}^{n} a_i \geqq 0 .$$

2) *For all* $g_1,\ldots,g_m,h_1,\ldots,h_n \in G$,

$$\sum_{j=1}^{m} \chi_{g_jA} - \sum_{i=1}^{n} \chi_{h_iA} \geqq 0 \quad \textit{implies}\ m \geqq n .$$

3) *For all* $g_1,\ldots,g_m,h_1,\ldots,h_{m+1} \in G$,

$$\sum_{i=1}^{m} \chi_{g_iA} - \sum_{j=1}^{m+1} \chi_{h_jA} \ngeqq 0 .$$

PROOF. Clearly 1) implies 2) implies 3). If 2) is not true, then there exist some $g_1,\ldots,g_m \in G$ and $h_1,\ldots,h_n \in G$ with

$$\sum_{i=1}^{m} \chi_{g_iA} - \sum_{j=1}^{n} \chi_{h_jA} \geqq 0$$

and $m<n$. Assume $m+n$ is minimal among all such possible sums. Then

$$\sum_{i=1}^{m} \chi_{g_iA} - \sum_{j=1}^{n-1} \chi_{h_jA} \geqq 0$$

also; so minimality implies $m \geqq n-1$. Thus $n=m+1$. So we have a contradiction of 3). Therefore 3) implies 2). For 2) implies 1), if 1) is not true then some $\sum_{i=1}^{n} a_i\chi_{g_iA} \geqq 0$ with $\sum_{i=1}^{n} a_i < 0$. Choose $r_1,\ldots,r_n \in \mathsf{Q}$ such that $r_i > a_i$ and $r_i - a_i < \varepsilon$ for all i. Then we have

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} (r_i - a_i) + \sum_{i=1}^{n} a_i < n\varepsilon + \sum_{i=1}^{n} a_i ,$$

and

$$\sum_{i=1}^{n} r_i\chi_{g_iA} = \sum_{i=1}^{n} (r_i - a_i)\chi_{g_iA} + \sum_{i=1}^{n} a_i\chi_{g_iA} \geqq \sum_{i=1}^{n} a_i\chi_{g_iA} \geqq 0 .$$

So if $\varepsilon < (-\sum_{i=1}^{n} a_i/n)$ then we get $\sum_{i=1}^{n} r_i < 0$. Let $r_i = p_i/q_i$ with $p_i \in \mathsf{Z}$, $q_i \in \mathsf{Z}^+$. Let $n_i = dr_i$ where d is any common multiple of $\{q_i : i=1,\ldots,n\}$. Then

$$\sum_{i=1}^{n} n_i\chi_{g_iA} = d\sum_{i=1}^{n} r_i\chi_{g_iA} \geqq 0$$

while $\sum_{i=1}^{n} n_i = d\sum_{i=1}^{n} r_i < 0$. Each $n_i \in \mathsf{Z}$, so repeating each g_i exactly $|n_i|$ times gives a form as in 2) with $n>m$. So 2) implies 1) and we have 1) if and only if 2) if and only if 3).

If we specialize Proposition 1.4 to $G=X$ and the G-action just group multiplication, then there is another form of the translate property. For a finite sequence $(s_1,\ldots,s_n)$ by $\|B \cap (s_1,\ldots,s_n)\|$ is meant $\sum_{i=1}^{n} \chi_B(s_i)$.

1.5 COROLLARY. *Given* (G,G,A) *then the translate property holds if and only if when* $(s_1,\ldots,s_n)$ *and* $(t_1,\ldots,t_m)$ *are finite sequences in* G *with* $n>m$ *then there exists* $x \in G$ *with*

$$\|Ax \cap (s_1,\ldots,s_n)\| > \|Ax \cap (t_1,\ldots,t_m)\| .$$

PROOF. Let $H=\sum_{i=1}^m \chi_{t_iA} - \sum_{j=1}^n \chi_{s_jA}$. The translate property for (G,G,A) was equivalent to given any H as above with $n>m$ there exists $x \in G$ with $H(x)<0$. But

$$\begin{aligned} H(x) &= \textstyle\sum_{i=1}^m \chi_{t_iA}(x) - \sum_{j=1}^n \chi_{s_jA}(x) \\ &= \textstyle\sum_{i=1}^m \chi_{Ax^{-1}}(t_i^{-1}) - \sum_{j=1}^n \chi_{Ax^{-1}}(s_j^{-1}) \\ &= \|Ax^{-1} \cap (t_1^{-1},\ldots,t_m^{-1})\| - \|Ax^{-1} \cap (s_1^{-1},\ldots,s_n^{-1})\| . \end{aligned}$$

So the translate property is equivalent to given any $(t_1,\ldots,t_m)$ and $(s_1,\ldots,s_n)$ sequences in G with $n>m$ there exists $y \in G$ such that

$$\|Ay \cap (t_1^{-1},\ldots,t_m^{-1})\| < \|Ay \cap (s_1^{-1},\ldots,s_n^{-1})\| .$$

Take the sequences $(t_1^{-1},\ldots,t_m^{-1})$ and $(s_1^{-1},\ldots,s_n^{-1})$ to begin with to get the corollary.

EXAMPLES. The most immediate example of when there does not exist an invariant for (G,G,A) is when A is a subsemigroup of G generated freely by two elements x and y. Then $xA \cup yA \subset A$ but $xA \cap yA$ is empty; hence, there cannot exist an invariant for (G,G,A). It is in fact true that for some non-empty $A \subset G$ and some x and y in G we have $xA \cup yA \subset A$ with xA disjoint from yA if and only if x and y generate a free subsemigroup. See [8] for a proof.

As for examples of when there is an invariant for (G,X,A), we have the trivial cases of when G is finite or when A is finite. In the first case, let $x_0 \in A$ and let $\mu(S)=\|S \cap Gx_0\|/\|A \cap Gx_0\|$. In the second case let $\mu(S)=\|S\|/\|A\|$. Less trivial examples are available. If G is a nilpotent group then there is an invariant for any (G,X,A). We say a group G is supramenable if and only if there is an invariant for any (G,X,A). A solvable group is supramenable if and only if it contains no free subsemigroup on two generators. This is also true for connected Lie groups. For proofs of these facts and further exposition see [8].

2. The ratio property.

Given a group G acting on a set X and $A \subset X$ which is non-empty, we say (G,X,A) has the *ratio property* if and only if there exists a net $\{F_\gamma\}$ of finite sequences in X such that for all $g \in G$

$$\lim_\gamma \|F_\gamma \cap gA\|/\|F_\gamma \cap A\| = 1 .$$

We say (G,X,A) has the *ratio property without repetitions* if we can choose the net $\{F_\gamma\}$ as above with F_γ actually finite subsets of X. This section contains a proof that the translate property is equivalent to the ratio property.

Let E be a real vector space of functions on X under pointwise addition and the usual scalar multiplication. We say that $f \geqq 0$ if $f(x) \geqq 0$ for all $x \in X$.

2.1. LEMMA. *If $v \in E'$ is positive then there exists a net $\{F_\gamma\}$ of finite sequences in X and a net $\{D_\gamma\}$ of positive integers such that for all $f \in E$,*

$$D_\gamma^{-1} \sum_{x \in F_\gamma} f(x) \to \langle v, f \rangle .$$

PROOF. Let P be the rational cone generated by the point evaluations $\{e_x : x \in X\}$. Here $\langle e_x, f \rangle = f(x)$ and $e_x \in E'$ for all $x \in X$. Let $\bar{P}$ be the $w(E', E)$-closure of P in E'. We claim $v \in \bar{P}$. If not since $w(E', E)$ is a locally convex topology for E', 0.1 implies there exists a $w(E', E)$-continuous linear functional φ on E' such that

$$\sup_{p \in P} \langle \varphi, p \rangle < \langle \varphi, v \rangle .$$

But then 0.4 with τ = discrete topology implies there exists $f_0 \in E$ such that

$$\sup_{p \in P} \langle p, f_0 \rangle < \langle v, f_0 \rangle .$$

Hence each $\langle p, f_0 \rangle \leqq 0$ since $\langle v, f_0 \rangle \in \mathsf{R}$ and $qP \subset P$ for all $q \in \mathsf{Q}^+$. Therefore, $\sup_{p \in P} \langle p, f_0 \rangle = 0$. Since for any $x \in X$, $e_x \in P$, we have $f_0 \leqq 0$. But then $\langle v, f_0 \rangle \leqq 0$ which is a contradiction.

So there exists a net p_α in P such that $p_\alpha \to v$ in the topology $w(E', E)$. Each p_α is

$$p_\alpha = \sum_{i=1}^{N_\alpha} q_i^\alpha e_{x_i^\alpha} \quad \text{where } q_i^\alpha \in \mathsf{Q}^+ \text{ and } x_i^\alpha \in X.$$

Rewrite each p_α with $\{q_i^\alpha\}$ over a common denominator to get $p_\alpha = D_\alpha^{-1} \sum_{x \in F_\alpha} e_x$ where $D_\alpha \in \mathsf{Z}^+$ and F_α is a finite sequence in X.

Let G be a group acting on X. We can define for each $g \in G$ and $f: X \to \mathsf{R}$, $gf(x) = f(g^{-1}x)$. Then $gf: X \to \mathsf{R}$. We write $GE \subset E$ when $gf \in E$ if $g \in G$ and $f \in E$.

2.2 PROPOSITION. *Let G be a group acting on X and E as above with $GE \subset E$. Suppose there exists a G-invariant positive linear functional θ on E. Then there exists a net $\{F_\alpha\}$ of finite sequences in X and a net $\{D_\alpha\}$ of positive integers such that for all $f \in E$, $g \in G$*

$$\lim_\alpha D_\alpha^{-1} \left(\sum_{x \in F_\alpha} f(x) - \sum_{x \in F_\alpha} gf(x) \right) = 0 .$$

If $\langle \theta, f \rangle \neq 0$ then also we have

$$\lim_\alpha \sum_{x \in F_\alpha} gf(x) / \sum_{x \in F_\alpha} f(x) = 1 .$$

PROOF. Take $D_\alpha^{-1}\sum_{x\in F_\alpha} e_x \to \theta$ in the topology $w(E',E)$ as in Lemma 2.1. Then since $\langle\theta,gf\rangle=\langle\theta,f\rangle$, we get

$$\lim_\alpha D_\alpha^{-1}\Big(\sum_{x\in F_\alpha} f(x)-\sum_{x\in F_\alpha} gf(x)\Big) = 0 .$$

If $\langle\theta,f\rangle\neq 0$ then eventually, $D_\alpha^{-1}\sum_{x\in F_\alpha} f(x)$ is not zero. Dividing by this and taking the limit gives

$$\lim_\alpha\Big(1-\sum_{x\in F_\alpha} gf(x)\Big/\sum_{x\in F_\alpha} f(x)\Big) = (\langle\theta,f\rangle-\langle\theta,gf\rangle)/\langle\theta,f\rangle = 0 .$$

2.3 THEOREM. *(G,X,A) satisfies the translate property if and only if (G,X,A) satisfies the ratio property. If G is also amenable, there exists an invariant for (G,X,A) if and only if (G,X,A) satisfies the ratio property.*

PROOF. Apply Proposition 2.2 with

$$E = S_A = \operatorname{span}\{\chi_{gA} : g\in G\} .$$

The translate property implies there is a G-invariant positive linear functional θ on E such that $\langle\theta,\chi_A\rangle=1$ and so there exists a net $\{F_\alpha\}$ of finite sequences in X such that for all $g\in G$,

$$\lim_\alpha \sum_{x\in F_\alpha} g\chi_A(x)\Big/\sum_{x\in F_\alpha}\chi_A(x) = 1 .$$

This is the ratio property. For the converse, assume $\sum_{i=1}^n a_i\chi_{g_iA}\geqq 0$ with $a_1,\dots,a_n\in \mathsf{R}$ and $g_1,\dots,g_n\in G$. Choose $\{F_\alpha\}$ as in the definition of the ratio property. We have

$$0 \leqq \sum_{x\in F_\alpha}\sum_{i=1}^n a_i\chi_{g_iA}(x) = \sum_{i=1}^n a_i\|g_iA\cap F_\alpha\| .$$

Dividing by $\|A\cap F_\alpha\|$ and taking the limit as α gets large gives $\sum_{i=1}^n a_i\geqq 0$. Now 1.2 finishes the proof.

REMARK. We shall see in Section 4 that when we have an invariant for (G,X,A) we can get the ratio property without repetitions.

3. Weak*-Invariance.

The main purpose of this section and the next is to give necessary and sufficient conditions that there exist an invariant for (G,X,A) without making any restriction on the group G. Except for trivial cases, the major examples where an invariant exists are when G is an amenable group or a supramenable group.

3.1 Given a net $\{x_\gamma\}$ in a set X, we say $\{x_\gamma\}$ is a universal net if and only if for all $A\subset X$ either $\{x_\gamma\}$ is eventually in A or $\{x_\gamma\}$ is eventually in $X\setminus A$.

We will need a few facts about universal nets. First, every net has a universal subnet. Second, a universal net converges to each of its accumulation points. For further details see Kelley [5].

3.2 Lemma. *Given a net of linear functionals $\{l_\alpha\}$ on a linear space E such that for each $x \in E$, $\{\langle l_\alpha, x\rangle\}$ is eventually bounded, there exists a linear functional l on E and a subnet $\{l_\gamma\}$ of $\{l_\alpha\}$ such that $l_\gamma \to l$ pointwise on E.*

Proof. Let $\{l_\gamma\}$ be a universal subnet of $\{l_\alpha\}$. Then $\{\langle l_\gamma, x\rangle\}$ is a universal net of R for all $x \in E$. Since $\{\langle l_\alpha, x\rangle\}$ is eventually bounded, $\{\langle l_\gamma, x\rangle\}$ is also eventually bounded. But then $\{\langle l_\gamma, x\rangle\}$ has at least one accumulation point. Since $\{\langle l_\gamma, x\rangle\}$ is universal and converges to each of its accumulation points, $\{\langle l_\gamma, x\rangle\}$ must converge. Define

$$\langle l, x\rangle = \lim_\gamma \langle l_\gamma, x\rangle .$$

Because $\{\langle l_\gamma, x\rangle\}$ converges for each $x \in E$, l is a well-defined linear functional on E.

Recall that $l_1(X)$ and $l_\infty(X)$ are paired spaces where for $f \in l_1(X)$ and $H \in l_\infty(X)$,

$$\langle H, f\rangle = \textstyle\sum_{x \in X} H(x) f(x) .$$

3.3 Theorem. *The following are equivalent:*

1) *There exists an invariant for (G, X, A).*

2) *There exists a net $\{f_\alpha\} \subset l_1^+(X)$ such that for all $g \in G$ and for all $H \in B_A(X)$ we have*

$$(\langle f_\alpha, H\rangle - \langle f_\alpha, gH\rangle)/\langle f_\alpha, \chi_A\rangle \to 0 .$$

3) *There exists a net of finite sequences $\{F_\alpha\}$ in X such that for all $g \in G$ and $H \in B_A(X)$ we have*

$$\left(\textstyle\sum_{x \in F_\alpha} H(x) - \sum_{x \in F_\alpha} gH(x)\right)/\|F_\alpha \cap A\| \to 0 .$$

4) *There exists a net of finite sequences $\{F_\alpha\}$ in X such that for all $g \in G$ and A-bounded $M \subset X$*

$$(\|F_\alpha \cap M\| - \|F_\alpha \cap gM\|)/\|F_\alpha \cap A\| \to 0 .$$

Proof. For a finite sequence F in X let $\chi_F = \sum_{x \in F} e_x$, e_x evaluation at x. Given 1) choose an invariant φ for (G, X, A). By Proposition 2.2 there exists a net $\{D_\alpha\}$ in Z^+ and a net of finite sequences $\{F_\alpha\}$ in X such that $D_\alpha^{-1}\chi_{F_\alpha} \to \varphi$ pointwise on $B_A(X)$. Since $\langle \varphi, \chi_A\rangle = 1$,

$$\|F_\alpha \cap A\|/D_\alpha = D_\alpha^{-1}\langle \chi_{F_\alpha}, \chi_A\rangle \to 1 .$$

So $\|F_\alpha \cap A\|^{-1}\chi_{F_\alpha} \to \varphi$ pointwise on $B_A(X)$. Since φ is invariant, $\{F_\alpha\}$ satisfies 3). So 1) implies 3). If 3) is given for the net of sequences $\{F_\alpha\}$ then $\{\chi_{F_\alpha}\}$ satisfies 2) and $\{F_\alpha\}$ satisfies 4). So 3) implies 2) and 3) implies 4). Given a net $\{f_\alpha\}$ as in 2), let $H=\chi_A$. Then

$$1-\langle f_\alpha,\chi_{gA}\rangle/\langle f_\alpha,\chi_A\rangle \to 0 \quad \text{for all } g\in G\,.$$

But for any $H\in B_A(X)$ there exists $K>0$ and $g_1,\ldots,g_n\in G$ such that $|H|\leqq K\sum_{i=1}^n \chi_{g_iA}$. Hence,

$$|\langle f_\alpha,\chi_A\rangle^{-1}\langle f_\alpha,H\rangle| \leqq K\langle f_\alpha,\chi_A\rangle^{-1}\sum_{i=1}^n \langle f_\alpha,\chi_{g_iA}\rangle \leqq 2nK$$

eventually. For all α let $\varphi_\alpha=f_\alpha/\langle f_\alpha,\chi_A\rangle$. Then φ_α is a positive linear functional on $B_A(X)$ and for all $H\in B_A(X)$ we have $\{\langle H,\varphi_\alpha\rangle\}$ is eventually bounded. By Lemma 3.2 there exist a subnet $\{\varphi_\gamma\}$ of $\{\varphi_\alpha\}$ and a linear functional φ on $B_A(X)$ such that $\varphi_\gamma\to\varphi$ pointwise on $B_A(X)$. But $\varphi_\alpha\geqq 0$ and $\langle\varphi_\alpha,\chi_A\rangle=1$ for all α implies $\varphi\geqq 0$ and $\langle\varphi,\chi_A\rangle=1$. Since

$$\lim_\alpha\langle\varphi_\alpha,gH-H\rangle = 0,$$

we have

$$\langle\varphi,gH-H\rangle = 0\,.$$

So φ is G-invariant. Therefore φ is an invariant for (G,X,A).

We need only show now 4) implies 1). As in 2) implies 1), 4) implies that there exists a subnet of $\{\|F_\alpha\cap A\|^{-1}\chi_{F_\alpha}\}$ which converges to a linear functional φ on

$$\operatorname{span}\{\chi_M:\ M \text{ is } A\text{-bounded}\}\,.$$

It follows that $\varphi\geqq 0$, φ is G-invariant, and $\langle\varphi,\chi_A\rangle=1$. Define a finitely-additive measure μ by $\mu(S)=\langle\varphi,\chi_S\rangle$ if S is A-bounded and $\mu(S)=\infty$ otherwise. Then μ is positive, G-invariant, and $\mu(A)=1$. Proposition 1.1 implies there exists an invariant for (G,X,A).

3.4 DEFINITION. *Let*

$$P_A = \{f\in l_1^+(X):\ \langle f,\chi_A\rangle=1\}\,.$$

We say a net $\{\varphi_\alpha\}$ in P_A converges to weak-invariance relative to A if and only if for all $g\in G$ and $H\in B_A(X)$,*

$$\lim_\alpha\langle\varphi_\alpha,H-gH\rangle = 0\,.$$

REMARK. Theorem 3.3 says there exists an invariant for (G,X,A) if and only if there exists a net φ_α which converges to weak*-invariance relative to A. If $A=X$ this weak*-invariance is Day's notion of weak*-invariance.

3.5. COROLLARY. *If there exists a net* $\{f_\alpha\}$ *in* P_A *which converges to weak*-invariance relative to* A *then there exists an invariant* φ *for* $\langle G,X,A\rangle$ *and a subnet* $\{f_\gamma\}$ *of* $\{f_\alpha\}$ *such that* $f_\gamma \to \varphi$ *pointwise on* $B_A(X)$.

PROOF. Given the net $\{f_\alpha\}$, the proof of 2) implies 1) in Theorem 3.3 together with Lemma 3.2 gives us some subnet $\{f_\gamma\}$ of $\{f_\alpha\}$ which converges to a linear functional on $B_A(X)$. Weak*-invariance implies φ is an invariant for (G,X,A).

4. The relative Følner condition.

We have a notion of weak*-invariance which generalizes Day's to the relative case. We need a similar generalization for strong invariance.

4.1 DEFINITION. *We say the Følner condition relative to* A *holds if there exists a net* $\{F_\gamma\}$ *of finite sets in* X *such that for all* $g \in G$,

$$\|(gF_\gamma \Delta F_\gamma) \cap A\|/\|F_\gamma \cap A\| \to 0 .$$

The net $\{F_\gamma\}$ *is called a Følner net relative to* A.

It should be understood that implicitly $\|F_\gamma \cap A\| > 0$ for all γ when $\{F_\gamma\}$ is a Følner net relative to A.

4.2 REMARK. By suitable indexing, the Følner condition relative to A holds if and only if for all $g_1,\ldots,g_n \in G$ and $\varepsilon > 0$ there exists a finite subset $F \subset X$ such that for all $i = 1,\ldots,n$

$$\|(g_iF\Delta F) \cap A\|/\|F \cap A\| < \varepsilon .$$

Notice also that for all $g \in G$

$$\begin{aligned}\big|1 - \|F \cap gA\|/\|F \cap A\|\big| &= \big|(\|F \cap A\| - \|g^{-1}F \cap A\|)/\|F \cap A\|\big| \\ &\leqq \|(F\Delta g^{-1}F) \cap A\|/\|F \cap A\| .\end{aligned}$$

So if there exists a Følner net relative to A, then (G,X,A) has the ratio property without repetitions.

4.3 PROPOSITION. *The Følner condition relative to* A *holds if and only if there exists a net* $\{F_\gamma\}$ *of finite sets in* X *such that for all* $L \subset G$ *with the identity* $e \in L$ *and* L *finite,*

$$\|LF_\gamma \cap A\|/\|F_\gamma \cap A\| \to 1 .$$

PROOF. Suppose $\{F_\gamma\}$ is given. Then let $e \in L \subset G, L$ finite and symmetric. For all $l \in L$,

$$lLF_\gamma \setminus LF_\gamma \subset L^2F_\gamma \setminus F_\gamma$$

and

$$LF_\gamma \setminus lLF_\gamma \subset L^2F_\gamma \setminus ll^{-1}F_\gamma = L^2F_\gamma \setminus F_\gamma\,.$$

So

$$\|(lLF_\gamma \Delta LF_\gamma) \cap A\| / \|LF_\gamma \cap A\| \leqq 2\, \|(L^2F_\gamma \setminus F_\gamma) \cap A\| / \|F_\gamma \cap A\|\,.$$

Since $\|L^2F_\gamma \cap A\|/\|F_\gamma \cap A\| \to 1$ and $L^2F_\gamma \supset F_\gamma$,

$$\|(L^2F_\gamma \setminus F_\gamma) \cap A\|/\|F_\gamma \cap A\| \to 0\,.$$

Thus, for all $\varepsilon > 0$ there exists γ such that

$$\|(lLF_\gamma \Delta LF_\gamma) \cap A\|/\|LF_\gamma \cap A\| < \varepsilon \quad \text{for all } l \in L\,.$$

To get such an estimate for an arbitrary finite $L_0 \subset G$ let $L = L_0 \cup \{e\} \cup L_0^{-1}$. This gives us the Følner condition relative to A. Conversely if $\{F_\gamma\}$ is a Følner net relative to A then by Remark 4.2

$$\|gF_\gamma \cap A\|/\|F_\gamma \cap A\| \to 1\,.$$

In addition, for all $g, h \in G$

$$(gF_\gamma \Delta hF_\gamma) \cap A \subset [(gF_\gamma \Delta F_\gamma) \cap A] \cup [(F_\gamma \Delta hF_\gamma) \cap A]\,.$$

An easy induction on $\|L\|$ for $e \in L \subset G$ with L finite implies $\|LF_\gamma \cap A\|/\|F_\gamma \cap A\| \to 1$.

4.4 THEOREM. *If the Følner condition relative to A holds, then there exists an invariant for* (G, X, A). *If* $\{F_\alpha\}$ *is a Følner net relative to A then there exists an invariant* φ *for* (G, X, A) *and some subnet* $\{F_\gamma\}$ *of* $\{F_\alpha\}$ *such that*

$$\|F_\gamma \cap A\|^{-1}\chi_{F_\gamma} \to \varphi$$

pointwise on $B_A(X)$.

PROOF. Let $\{F_\alpha\}$ be a Følner net relative to A. Let $f_\alpha = \|F_\alpha \cap A\|^{-1}\chi_{F_\alpha}$. For any $H \in B_A(X)$ there are $g_1, \ldots, g_n \in G$ with $\operatorname{supp} H \subset \bigcup_{i=1}^n g_iA$. Then $H = \sum_{i=1}^n H_i\chi_{g_iA}$ with $\|H_i\|_\infty \leqq \|H\|_\infty$ for all i. For $g \in G$,

$$\begin{aligned}
&|\langle f_\alpha, H\rangle - \langle f_\alpha, gH\rangle| \\
&\quad = \|F_\alpha \cap A\|^{-1} \left|\sum_{x \in F_\alpha} \sum_{i=1}^n (H_i(x)\chi_{g_iA}(x) - H_i(g^{-1}x)\chi_{g_iA}(g^{-1}x))\right| \\
&\quad \leqq \|F_\alpha \cap A\|^{-1} \left|\sum_{z \in F_\alpha \Delta g^{-1}F_\alpha} \sum_{i=1}^n H_i(z)\chi_{g_iA}(z)\right| \\
&\quad \leqq \|H\|_\infty \sum_{i=1}^n \|(F_\alpha \Delta g^{-1}F_\alpha) \cap g_iA\|/\|F_\alpha \cap A\|\,.
\end{aligned}$$

Since

$$\|(F_\alpha \Delta g^{-1}F_\alpha) \cap g_i A\| = \|(g_i^{-1}F_\alpha \Delta g_i^{-1}g^{-1}F_\alpha) \cap A\|$$
$$\leqq \|(g_i^{-1}F_\alpha \Delta F_\alpha) \cap A\| + \|(F_\alpha \Delta g_i^{-1}g^{-1}F_\alpha) \cap A\| ,$$

it follows that

$$\sum_{i=1}^{n} \|(F_\alpha \Delta g^{-1}F_\alpha) \cap g_i A\| / \|F_\alpha \cap A\| \to 0 .$$

Hence, $\{f_\alpha\}$ is weak*-invariant. By Corollary 3.5 some subnet of $\{f_\alpha\}$ converges to an invariant for (G, X, A).

We claim that the converse of this theorem is also true. The proof is an adaptation of Namioka's technique [7].

4.5 LEMMA. *If* $f \in l_1^+(X)$ *with* supp$f$ *finite and* $\|f\|_1 = 1$ *then there exists* $\lambda_1, \ldots, \lambda_m > 0$ *with* $\sum_{j=1}^{m} \lambda_j = 1$ *and there exists* $A_1 \subset \ldots \subset A_m$ *which are finite subsets of* X *such that*

$$f = \sum_{j=1}^{m} \lambda_j \chi_{A_j} / \|A_j\| .$$

PROOF. Let $0 < a_1 < \ldots < a_m$ be the distinct values of f. For all $m \geqq j \geqq 1$ define

$$A_j = \{x \in X : a_{m-j+1} \leqq f(x)\} .$$

Then $A_j \subset A_{j+1}$ for all $j = 1, \ldots, m-1$. Also,

$$f = a_1 \chi_{A_m} + (a_2 - a_1)\chi_{A_{m-1}} + \ldots + (a_m - a_{m-1})\chi_{A_1} .$$

Therefore

$$f = \sum_{j=1}^{m} \lambda_j \chi_{A_j} / \|A_j\|$$

where A_j are finite sets such that $A_j \subset A_{j+1}$ for all $j = 1, \ldots, m-1$. Since $\sum_{x \in X} f(x) = \|f\|_1 = 1$, $\sum_{j=1}^{m} \lambda_j = 1$.

4.6 DEFINITION. *Define the pseudo-norm* $\|\cdot\|_A$ *on* $l_1(X)$ *by* $\|f\|_A = \langle |f|, \chi_A \rangle$. *Let*

$$\mathscr{F}_A = \{f \in l_1^+(X) : \text{supp} f \textit{ is finite and } \|f\|_A = 1\} .$$

Note that

$$\mathscr{F}_A = P_A \cap \{f \in l_1(X) : \text{supp} f \text{ is finite}\} .$$

4.7. LEMMA. *If there exists a net* $\{f_\alpha\}$ *in* $\mathscr{F}_A$ *such that* $\|f_\alpha - g f_\alpha\|_A \to 0$ *for all* $g \in G$ *then the Følner condition relative to* A *holds.*

PROOF. We will show that for all $\varepsilon>0$ and $g_1,\ldots,g_n\in G$ there exists $K\subset X$ finite with

$$\|(g_iK\Delta K)\cap A\|/\|K\cap A\| < \varepsilon \quad \text{for all } i=1,\ldots,n\,.$$

Fix α. Let $D=\sum_{x\in X}f_\alpha(x)$. Since $f_\alpha\in\mathscr{F}$, $0<D<\infty$. By Lemma 4.5 there exist $\lambda_1,\ldots,\lambda_m>0$ with $\sum_{j=1}^m\lambda_j=1$ and finite subsets $A_1\subset\ldots\subset A_m$ of X such that $D^{-1}f_\alpha=\sum_{j=1}^m\lambda_j\chi_{A_j}/\|A_j\|$. Let s_0 be

$$s_0 = \max\{j=1,\ldots,m:\ A_j\cap A \text{ is empty}\}\,.$$

Since $\|f_\alpha\|_A=1$, we know $s_0<m$. Let

$$\gamma_j = D\lambda_j\|A_j\cap A\|/\|A_j\| \quad \text{for } j\geqq s_0+1\,.$$

Then

$$f_\alpha = D\sum_{i=1}^{s_0}\lambda_i\chi_{A_i}/\|A_i\|+\sum_{j=s_0+1}^m\gamma_j\chi_{A_j}/\|A_j\cap A\|\,.$$

We must have $1=\|f_\alpha\|_A=\sum_{j=s_0+1}^m\gamma_j$.

Fix some $g\in G$. Let $B=\bigcup_{j=1}^m(A_j\setminus gA_j)$. Since $A_j\subset A_{j+1}$ for all $j=1,\ldots,m-1$, we have $(gA_j\setminus A_j)\subset X\setminus B$ for any $j=1,\ldots,m$. We know

$$\begin{aligned}&\|gf_\alpha-f_\alpha\|_A\\ &\geqq \sum_{x\in A\cap(X\setminus B)}|D\sum_{i=1}^{s_0}\gamma_i(\chi_{gA_i(x)}-\chi_{A_i(x)})/\|A_i\|+\\ &+\sum_{j=s_0+1}^m\gamma_j(\chi_{gA_j(x)-A_j(x)})/\|A_j\cap A\|\,|\,.\end{aligned}$$

Since each $\chi_{gA_i}-\chi_{A_i}\geqq 0$ on $X\setminus B$,

$$\|gf_\alpha-f_\alpha\|_A \geqq \sum_{x\in A\cap(X\setminus B)}\sum_{j+s_0+1}^m\gamma_j(\chi_{gA_j(x)}-\chi_{A_j(x)})/\|A_j\cap A\|\,.$$

Therefore

$$\|gf_\alpha-f_\alpha\|_A \geqq \sum_{j=s_0+1}^m\gamma_j\,\|(gA_j\setminus A_j)\cap A\|/\|A_j\cap A\|$$

because each $gA_j\setminus A_j\subset X\setminus B$. Similarly

$$\begin{aligned}\|g^{-1}f_\alpha-f_\alpha\|_A &= \sum_{x\in gA}|gf_\alpha-f_\alpha|(x)\\ &\geqq \sum_{x\in gA\cap(X\setminus B)}\sum_{j=s_0+1}^m\gamma_j(\chi_{gA_j(x)}-\chi_{A_j(x)})/\|A_j\cap A\|\\ &= \sum_{j=s_0+1}^m\gamma_j\|(gA_j\setminus A_j)\cap gA\|/\|A_j\cap A\|\\ &= \sum_{j=s_0+1}^m\gamma_j\|(A_j\setminus g^{-1}A_j)\cap A\|/\|A_j\cap A\|\,.\end{aligned}$$

Choose $g_1,\ldots,g_n\in G$ and $\varepsilon>0$. Let α be large enough so that

$$\varepsilon > 2\sum_{i=1}^n\|g_if_\alpha-f_\alpha\|_A+\|g_i^{-1}f_\alpha-f_\alpha\|_A\,.$$

Then by the estimates above,

$$\varepsilon > \sum_{j=s_0+1}^m\gamma_j\sum_{i=1}^n\left(\|(g_iA_j\Delta A_j)\cap A\|+\|(g_i^{-1}A_j\Delta A_j)\cap A\|\right)/\|A_j\cap A\|\,.$$

Since $\sum_{j=s_0+1}^{m} \gamma_j = 1$, there exists $K = A_j$ for some $j = s_0 + 1, \ldots, m$ such that

$$\varepsilon > \sum_{i=1}^{n} (\|(g_i K \Delta K) \cap A\| + \|(g_i^{-1} K \Delta K) \cap A\|)/\|K \cap A\| .$$

Certainly then for all $i = 1, \ldots, n$ we have

$$\|(g_i K \Delta K) \cap A\|/\|K \cap A\| < \varepsilon .$$

4.8 LEMMA. *Let (E, τ) be a locally convex linear topological space. Let G be a set of linear mappings of E. Let $C \subset E$ be a convex set. Then there exists a net $\{e_\alpha\} \subset C$ such that $e_\alpha - ge_\alpha \to 0$ weakly for all $g \in G$ if and only if there exists a net $\{d_\gamma\} \subset C$ such that $d_\gamma - gd_\gamma \to 0$ in the τ-topology.*

REMARK. No continuity assumption is made on the set G.

PROOF. Let $F = (E, \tau)^G = \prod_G (E, \tau)$ with the locally convex product topology. Define $T: E \to F$ by $T(e)(g) = e - ge$. Then T is a linear mapping and $T(C)$ is a convex set in F. There exists a net $\{e_\alpha\} \subset C$ such that $e_\alpha - ge_\alpha \to 0$ weakly for all $g \in G$ if and only if there exists a net $\{T(e_\alpha)\} \subset T(C)$ such that $T(e_\alpha) \to 0$ in the product of the weak topologies. Since F is given the product topology, the product of the weak topologies is the weak topology for F by 0.3. Thus if there exists a net $\{e_\alpha\} \subset C$ such that $e_\alpha - ge_\alpha \to 0$ weakly for all $g \in G$, then $T(e_\alpha) \to 0$ weakly. Since $T(C)$ is convex in the locally convex space F and 0 is in the weak-closure of $T(C)$, 0 is in the product topology closure by 0.2. But then there exists a net $\{d_\gamma\} \subset C$ such that $T(d_\gamma) \to 0$ in the product topology, that is $d_\gamma - gd_\gamma \to 0$ in the τ-topology for all $g \in G$. This is one direction of the lemma; the other is immediate since the τ-topology is stronger than the weak topology.

4.9 THEOREM. *If there exists an invariant for (G, X, A) then the Følner condition relative to A holds.*

PROOF. If there exists an invariant for (G, X, A) then by Theorem 3.3, part 3, there exists a net of finite sequences $\{F_\alpha\}$ in X such that for all $g \in G$ and $H \in B_A(X)$,

$$\left(\sum_{x \in F_\alpha} H(x) - \sum_{x \in F_\alpha} gH(x)\right)/\|F_\alpha \cap A\| \to 0 .$$

Let $f_\alpha = \|F_\alpha \cap A\|^{-1} \chi_{F_\alpha}$. Then $f_\alpha \in \mathscr{F}_A$ and $\langle f_\alpha, H - gH \rangle \to 0$ for all $g \in G$ and $H \in B_A(X)$. Since

$$\langle f_\alpha, H - gH \rangle = \langle f_\alpha - g^{-1} f_\alpha, H \rangle ,$$

$f_\alpha - kf_\alpha \to 0$ pointwise on $B_A(X)$ for all $k \in G$. Give $l_1(X)$ the topology τ induced by $\|\cdot\|_A$. It is easy to see that

$$(l_1(X), \|\cdot\|_A)^* \subset B_A(X) .$$

First, because $(l_1(X), \|\cdot\|_A) \cong (l_1(A), \|\cdot\|_1) \oplus N$ where N is the $\|\cdot\|_A$-closure of 0,

$$(l_1(X), \|\cdot\|_A)^* = (l_1(A), \|\cdot\|_1)^* .$$

Second, $(l_1(A), \|\cdot\|_1)^* = l_\infty(A) \subset B_A(X)$. Thus, we have a net $\{f_\alpha\}$ in the convex set $\mathscr{F}_A \subset l_1(X)$ such that when $l_1(X)$ is given the locally convex topology τ, $f_\alpha - gf_\alpha \to 0$ weakly for all $g \in G$. But then Lemma 4.8 implies that there is a net $\{h_\gamma\}$ in $\mathscr{F}_A$ such that $\|h_\gamma - gh_\gamma\|_A \to 0$ for all $g \in G$. Lemma 4.7 shows that the Følner condition relative to A holds.

4.10 THEOREM. *Given (G, X, A) the following are equivalent:*

1) *There exists an invariant for (G, X, A).*

2) *There exists a net $\{F_\alpha\}$ of finite sets in X such that for all $g \in G$ and $H \in B_A(X)$,*

$$\left(\textstyle\sum_{x \in F_\alpha} H(x) - \sum_{x \in F_\alpha} gH(x)\right)/\|F_\alpha \cap A\| \to 0 .$$

3) *There exists a net $\{f_\alpha\} \subset \mathscr{F}_A$ such that*

$$\|f_\alpha - gf_\alpha\|_A \to 0 \textit{ for all } g \in G .$$

4) *There exists a Følner net relative to A.*

PROOF. We saw in 3.3 and 4.9 that 2) gives 1) and 4). The proof of Theorem 4.9 showed that 2) implies 3). Lemma 4.7 was 3) implies 4). If $\{F_\alpha\}$ is a Følner net relative to A then it will satisfy 2).

4.11 COROLLARY. *If G is an amenable group then there exists an invariant for (G, X, A) if and only if the ratio property without repetitions holds.*

PROOF. Theorem 4.10 1) implies 4) and Remark 4.2 prove the only if part. The if part is Theorem 2.3.

5. The measurable invariant.

Many of the results of the preceding sections can be restated in the case that (X, β, v) is a measure space and G is a group of measurable transformations of X. We assume here that v is G-invariant in the sense that $v(gE) = v(E)$ for all $g \in G$ and $E \in \beta$. Given a subset of $A \in \beta$ with $v(A) > 0$ we say $M \in \beta$ is *A-bounded* if and only if there exists $g_1, \ldots, g_n \in G$

such that $M \subset \bigcup_{i=1}^{n} g_i A$ locally a.e. $[v]$. We let $B_A(X,v)$ denote the functions $f \in L_\infty(X,v)$ such that supp$f$ is A-bounded. This is well-defined even though supp$f$ is defined only locally a.e. $[v]$. G acts linearly on $B_A(X,v)$ by $gf(x) = f(g^{-1}x)$ for all $g \in G$, $f \in B_A(X,v)$, and $x \in X$. We say there exists an *invariant* for (G,X,A) in the above context if and only if there exists a positive G-invariant linear functional φ on $B_A(X,v)$ such that $\langle \varphi, \chi_A \rangle = 1$.

In the case that G is a group acting on a set X, β is all subsets of X, and v is counting measure, this is the problem we have been considering. The following propositions can be shown by techniques similar to the ones we have already used and are stated without proof.

5.1 PROPOSITION. *There exists an invariant for (G,X,A) if and only if there exists a positive G-invariant finitely-additive measure μ on all subsets of X such that $\mu(A) = 1$ and $\mu(M) = \mu(N)$ whenever $\chi_M = \chi_N$ locally* a.e. $[v]$.

Let S_A denote the span of $\{\chi_{gA} : g \in G\}$ in $B_A(X,v)$. Let $\mathscr{F}$ denote the functions $f \in L_1^+(X,v)$ such that $v(\text{supp}(f)) < \infty$ and f takes a finite number of real values locally a.e. $[v]$. Let

$$\mathscr{F}_A = \{f \in \mathscr{F} : \textstyle\int_A f\,dv = 1\}$$

and

$$\mathscr{F}(\mathsf{Z}^+) = \{f \in \mathscr{F} : \text{locally a.e. } [v]\ f \text{ takes values in } \mathsf{Z}^+\}\,.$$

5.2 PROPOSITION. *The following are equivalent*:

1) *There exists a positive G-invariant linear functional φ on S_A such that $\langle \varphi, \chi_A \rangle = 1$.*

2) *For all $g_1, \ldots, g_n \in G$ and $a_1, \ldots, a_n \in \mathsf{R}$, $\sum_{i=1}^{n} a_i \chi_{g_i A} \geqq 0$ locally* a.e. $[v]$ *implies $\sum_{i=1}^{n} a_i \geqq 0$.*

3) *There exists a net $\{f_\alpha\} \subset \mathscr{F}$ such that for all $g \in G$*

$$\textstyle\int_{gA} f_\alpha\,dv \big/ \int_A f_\alpha\,dv \to 1\,.$$

5.3 COROLLARY. *If G is amenable there exists an invariant for (G,X,A) if and only if there exists a net $\{f_\alpha\} \subset \mathscr{F}$ such that for all $g \in G$*

$$\textstyle\int_{gA} f_\alpha\,dv \big/ \int_A f_\alpha\,dv \to 1\,.$$

5.4 THEOREM. *There exists an invariant for (G,X,A) if and only if there exists a net $\{f_\alpha\} \subset \mathscr{F}$ such that for all $g \in G$ and $H \in B_A(X,v)$*

$$\langle f_\alpha, H - gH \rangle / \langle f_\alpha, \chi_A \rangle \to 0\,.$$

REMARK. In 5.2–5.4, $\mathscr{F}(\mathsf{Z}^+)$ can replace $\mathscr{F}$. $\mathscr{F}(\mathsf{Z}^+)$ is the measure-theoretic version of the set of finite sequences.

5.5 THEOREM. *Let* $\|f\|_A = \int_A |f| dv$ *for all* $f \in L_1(X, v)$. *Assume that* $(L_1(X), \|\cdot\|_A)^* \subset B_A(X, v)$. *Then the following are equivalent*:

1) *There exists an invariant for* (G, X, A).

2) *There exists a net* $\{f_\alpha\} \subset \mathscr{F}_A$ *such that for all* $g \in G$

$$\|f_\alpha - gf_\alpha\|_A \to 0 .$$

3) *There exists a net* $\{F_\alpha\}$ *of finite measure sets in* X *such that for all* $g \in G$

$$v((F_\alpha \Delta gF_\alpha) \cap A)/v(F_\alpha \cap A) \to 0 .$$

REMARK. 3) is equivalent to 2) and implies 1) with no restrictions on $(L_1(X), \|\cdot\|_A)$. It would also suffice to assume that $L_1(A, \beta|A, v|A)^* = L_\infty(A)$ where $\beta|A$ is $\{B \cap A : B \in \beta\}$ and $v|A(B) = v(B \cap A)$ for all $B \in \beta$. That is, it is enough to assume $(A, \beta|A, v|A)$ is localizable.

5.6 COROLLARY. *If* G *is amenable then there exists an invariant for* (G, X, A) *if and only if there exists a net* $\{F_\alpha\}$ *of finite measure sets such that for all* $g \in G$,

$$v(F_\alpha \cap gA)/v(F_\alpha \cap A) \to 1 .$$

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