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TOPICS IN MARKOV ADDITIVE PROCESSES

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The theory of bivariate Markov processes $\{(X_t,S_t) : t \in I\}$ in both discrete and continuous time which are additive in the second component (see section 1 for a precise definition of this concept) has been developed from a number of viewpoints recently, see Çinlar [10] and Cheong, de Smit and Teugels [8] for many references. In one sense such processes are slightly generalized Markov renewal processes, in another semi-Markov processes, but to give them one of these names is almost to prescribe a particular point of view, and in fact one different from the one taken below. Thus we have adopted the name *Markov additive process* (abbreviated MAP) used by Çinlar [10], [11] in the continuous time case. As we view them, MAP's are most like random walks defined on a Markov chain, see Miller [23], and this best describes the approach we take. Particularising in one direction gives random walks with Markov-dependent increments, and general state Markov chains arise as another case.

Our main concern is passage-time problems and related topics. For example, if we consider hitting times of the second component to half lines, we are also led quite naturally to study modified processes such as those obtained by adding an impenetrable barrier, and also the maximum process. The problem of first passage from a finite interval is also discussed and we obtain results extending those of Wald [33] and those of other writers e.g. Phatarfod [24]. Some examples to illustrate the formulae we obtain are given. It is hoped that the methods we describe below appear naturally suited to the problems discussed, and that many other special cases will be worked out in detail by others.

1. Preliminaries.

1.a. *Intuitive description of the process.*

The processes which we are considering can be roughly described as follows: We have an *underlying Markov process* with arbitrary state space $(E,\mathscr{E})$ beginning at $x \in E$ and proceeding $X_1,X_2,\ldots,X_n,\ldots$; we also have an R^m-valued process beginning $Y_0 \equiv 0$, and proceeding $Y_1,Y_2,\ldots,$

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$Y_n, \ldots$, where for $n \geqq 1$, the distribution of Y_n depends on X_{n-1} and X_n, and, given $(X_1, X_2, \ldots, X_n)$, the random variables $(Y_1, Y_2, \ldots, Y_n)$ are mutually independent. Our main interest is in the process $\{(X_n, S_n) : n \geqq 0\}$ where $S_n = \sum_0^n Y_k$. This process is sometimes called a *random walk defined on an underlying Markov process.*

Some special cases are:

(1.1) The chain has only one state. Here we are back to the case of a random walk which is the sum of a sequence of i.i.d. random variables, and in this case our treatment is most like that in Feller [14].

(1.2) The random variables $\{Y_n\}$ may in no way depend upon X_n or X_{n-1}; in this case we have case (1.1) with an independent Markov process alongside.

(1.3) The Markov process $\{X_n\}$ may be general, and $Y_n = f(X_n)$ where $f: E \to \mathsf{R}^m$ is a fixed $\mathscr{E}$-measurable function.

(1.4) The state space $(E, \mathscr{E})$ may be discrete and the random variables $\{Y_n\}$ non-negative. We are then dealing with a variant of the semi-Markov or Markov-renewal processes.

For further discussion and historical remarks, we direct the reader to our basic references Çinlar [10], [11].

1.b. *Semi-Markov transition functions.*

Our approach and notation will be based as far as possible upon Çinlar [10], [11] which in turn, is modelled upon Blumenthal and Getoor [6]. We recall some terminology. If $(G, \mathscr{G})$ and $(H, \mathscr{H})$ are measurable spaces and if $f: G \to H$ is measurable with respect to $\mathscr{G}$ and $\mathscr{H}$ then we write $f \in \mathscr{G}/\mathscr{H}$. If $H = \overline{\mathsf{R}}^1 = [-\infty, \infty]$ and $\mathscr{H} = \overline{\mathscr{R}}^1$, the Borel subsets of $\overline{\mathsf{R}}^1$, then we write $f \in \mathscr{G}$ instead of $f \in \mathscr{G}/\mathscr{H}$. Further

$$b\mathscr{G} = \{f \in \mathscr{G} : f \text{ is bounded}\}, \quad \mathscr{G}_+ = \{f \in \mathscr{G} : f \geqq 0\}$$

and $b\mathscr{G}_+ = b\mathscr{G} \cap \mathscr{G}_+$.

A mapping $N: F \times \mathscr{G} \to [0,1]$ is called *a transition function* from $(F, \mathscr{F})$ into $(G, \mathscr{G})$ if

a) $A \to N(x, A)$ is a measure on $\mathscr{G}$ for all fixed $x \in F$, and
b) $x \to N(x, A)$ is in $b\mathscr{F}$ for all fixed $A \in \mathscr{G}$.

Analogously, we define a mapping $Q: E \times (\mathscr{E} \times \overline{\mathscr{R}}^m) \to [0,1]$ to be a *semi-Markov transition function* (abbreviated SMTF) on $(E, \mathscr{E}, \overline{\mathscr{R}}^m)$ if

a) $x \to Q(x, A \times B)$ is in $b\mathscr{E}$ for every $A \in \mathscr{E}$, $B \in \overline{\mathscr{R}}^m$,
b) $A \times B \to Q(x, A \times B)$ is a measure on $\mathscr{E} \times \overline{\mathscr{R}}^m$ for every $x \in E$.

If Q, R are two SMTF's on $(E, \mathscr{E}, \bar{\mathscr{R}}^m)$ we may define the *convolution product* $Q * R$ as the function

$$(1.5)\qquad (x, A\times B) \to (Q*R)(x, A\times B) = \int_E \int_{\bar{\mathsf{R}}^m} Q(x, dx'\times ds)R(x', A\times(B-s)) .$$

$Q*R$ is easily checked to be an SMTF. For any SMTF Q we define $Q^0 = I$ where $I(x, A\times B) = \delta_x(A)\delta_0(B)$, and for $n \geqq 1$, $Q^n = Q^{n-1} * Q$.

There are many different ways of viewing a SMTF Q, and at various times we will be doing this. Thus Q may be viewed as a positive contraction valued measure defined on $(\bar{\mathsf{R}}^m, \bar{\mathscr{R}}^m)$ by the map $B \to Q(B)$, where

$$(Q(B)I_A)(x) = Q(x, A\times B) \qquad (I_A \text{ is the indicator function});$$

as a transition function on $(E\times \bar{\mathsf{R}}^m, \mathscr{E}\times\bar{\mathscr{R}}^m)$ which is homogenous in the second component by the map

$$((x,s), A\times B) \to Q(x, A\times(B-s));$$

as a transition function from $(E, \mathscr{E})$ to $(E\times \bar{\mathsf{R}}^m, \mathscr{E}\times\bar{\mathscr{R}}^m)$ by $(x, A\times B) \to Q(x, A\times B)$ (cf. Çinlar [10] (1.2)); and finally as giving a sequence $\{Q^n;\, n\geqq 0\}$ satisfying Definition (1.1) of Çinlar [11].

Any SMTF Q induces a family $\{\hat{Q}(\theta);\, \theta\in \mathsf{R}^m\}$ of contractions on the Banach space $b\mathscr{E}$ by writing

$$(\hat{Q}(\theta)f)(x) = \int_E \int_{\bar{\mathsf{R}}^m} Q(x, dx'\times dy) f(x') e^{i(\theta, y)} ,$$

where $(\cdot,\cdot)$ denotes the usual inner product in $\bar{\mathsf{R}}^m$. We call $\{\hat{Q}(\theta)\}$ the *Fourier transform* of Q.

1.c. *Description of the basic process.*

We will consider a (discrete time, temporally homogeneous) *Markov process* (with translation operators) and with state space $(E, \mathscr{E})$ to be a sextuple

$$X = (\Omega, \mathscr{M}, \mathscr{M}_n, X_n, \theta_n, P^x) \qquad (x\in E) ,$$

and all such processes are assumed to be non-terminating. (For definitions see Blumenthal and Getoor [6]. Recall that, for all $k, l \geqq 0$, $X_l \circ \theta_k = X_{k+l}$.) Following Çinlar [11] we have:

(1.6) Definition. Let X be a Markov process with state space $(E, \mathscr{E})$, write $(F, \mathscr{F}) = (\bar{\mathsf{R}}^m, \bar{\mathscr{R}}^m)$, and let $S = \{S_n : n \geqq 0\}$ be a family of functions from $(\Omega, \mathscr{M})$ into $(F, \mathscr{F})$. Then $(X, S) = (\Omega, \mathscr{M}, \mathscr{M}_n, X_n, S_n, \theta_n, P^x)$ is called a (discrete time) *Markov additive process* (abbreviated MAP) provided the following hold:

a) $S_0 = 0$ a.s.;
b) for each $n \geqq 0$, $S_n \in \mathscr{M}_n/\mathscr{F}$;
c) for each $n \geqq 0$, $A \in \mathscr{E}$, $B \in \mathscr{F}$, the mapping

$$x \to P^x(X_n \in A, S_n \in B)$$

of E into $[0,1]$ is in $\mathscr{E}_+$;

d) for each $k,l \geqq 0$, $S_{k+l} = S_k + S_l \circ \theta_k$ a.s.;
e) for each $k,l \geqq 0$, $x \in E$, $A \in \mathscr{E}$, $B \in \mathscr{F}$:

$$P^x(X_l \circ \theta_k \in A,\ S_l \circ \theta_k \in B \mid \mathscr{M}_k) = P^{X_k}(X_l \in A,\ S_l \in B)\ .$$

We follow Çinlar [11] in our notation for objects associated with the definition, viz.

$$Q(x,C) = P^x\{(X_1,S_1) \in C\} \quad (C \in \mathscr{E} \times \mathscr{F})\ ,$$

$$P(x,A) = Q(x, A \times F) \quad (A \in \mathscr{E});$$

the completed σ-fields generated by the Markov process X (respectively MAP(X,S)) are denoted by $\{\mathscr{K}_n\}$ (respectively $\{\mathscr{L}_n\}$) and the limiting σ-fields by $\mathscr{K}$ (respectively $\mathscr{L}$). Further, we refer the reader to Çinlar [10], [11] for statements of all the fundamental results which amplify and illustrate the definitions.

2. Some general identities.

2.a. *Passage-time identities.*

Let $S_0 = 0$ and $S_n = \sum_1^n Y_k$ $(n \geqq 1)$ be a one-dimensional random walk with i.i.d. increments $\{Y_k\}$, and consider the following hitting time:

$$N = \inf\{n > 0 : S_n \notin B\} \quad B \in \mathscr{R}^1; \tag{2.1}$$

special cases include $B = (-\infty, 0]$, $(0, \infty)$. In determining the joint distribution of (N, S_N) in the case $B = (-\infty, 0]$ Feller [14, p. 600] and Spitzer [29, p. 177] make elegant use of the following identity, valid for real θ and $|\tau| < 1$:

$$E[\textstyle\sum_0^{N-1} \tau^n e^{i\theta S_n}][1 - \tau E[e^{i\theta Y_1}]] = 1 - E[\tau^N e^{i\theta S_N}]\ . \tag{2.2}$$

Spitzer points out that (2.2) is valid for a general stopping time N relative to the increasing sequence $\{\sigma(Y_1, \ldots, Y_n)\}$ of σ-fields, but that its main use appears to be in the special cases where N is the hitting time to an open or closed half-line. Also for these special cases Miller [23] derived a matrix analogue of (2.2) for a random walk defined on a Markov chain with a finite state space, and used the result to obtain a matrix Wiener-Hopf factorisation. More recently one of us, Arjas [1], has derived and applied a version of (2.2) valid for a general MAP. These applications show that despite the relative ease with which one obtains (2.2), it can be surprisingly effective in a variety of contexts.

Suppose now that X is a Markov process with state space $(E, \mathscr{E})$, and N a stopping time relative to $\{\mathscr{X}_n\}$. For any $f \in b\mathscr{E}_+$ and $|\tau| < 1$ we have:

$$E^x[\textstyle\sum_0^{N-1} \tau^n[f(X_n) - \tau E^{X_n}[f(X_1)]] = f(x) - E^x[\tau^N f(X_N)]\,, \tag{2.3}$$

where we adopt the convention that $\tau^N = 0$ on $\{N = \infty\}$. This identity is implicitly used in the theory of discrete-time Markov chains, see e.g. Feller [13], Kemeny, Snell and Knapp [19], and has been explicitly formulated by Ito [15] and by Port and Stone [25] in the continuous-time case. Equation (2.3) is called a passage-time identity and we note in passing that if X is a random walk with i.i.d. increments, and if $f(x) = e^{i\theta x}$ (here $E = \mathsf{R}^1$), then (2.3) reduces to (2.2). Kemperman [20], although he has not actually mentioned stopping times, makes an identity like (2.3) the key tool in his analysis of the passage problem for stationary Markov chains. The termination times (called "absorption") of Kemperman are not stopping times relative to $\{\mathscr{X}_n\}$, but can readily be shown to be so relative to an enlarged sequence of σ-fields.

2.b. *The passage-time identity.*

We now come to the basic tool in this paper. The identity we obtain includes all discrete time results mentioned in 2.a above.

Suppose (X, S) to be an MAP with SMTF Q on $(E, \mathscr{E}, \overline{\mathscr{R}}^m)$. If N is any stopping time relative to $\{\mathscr{M}_n\}$ we define the maps $\hat{G}_N = \hat{G}_N(T, \theta)$ and $\hat{H}_N = \hat{H}_N(T, \theta)$:

$$\begin{aligned}(\hat{G}_N f)(x) &= E^x[\textstyle\sum_0^{N-1} \exp(i(\theta, S_n))(T^n f)(X_n)]\,,\\ (\hat{H}_N f)(x) &= E^x[\exp(i(\theta, S_N))(T^N f)(X_N)]\,,\end{aligned} \tag{2.4}$$

where $x \in E$, $\theta \in \mathsf{R}^m$, $f \in b\mathscr{E}$ and T is a bounded linear operator on $b\mathscr{E}$ with $\|T\| < 1$. We continue to adopt the convention of 2.a that $T^N = 0$ on $\{N = \infty\}$. Recalling the definition of $\hat{Q} = \hat{Q}(\theta)$ in 1.b and denoting the identity operator by I we can state:

(2.5.) Theorem. (*Passage-time identity*). *If T commutes with $\hat{Q}(\theta)$ and $\|T\| < 1$, then:*

$$\hat{G}_N(T, \theta)[I - T\hat{Q}(\theta)] = I - \hat{H}_N(T, \theta)\,. \tag{2.6}$$

Proof. This proof is an extension of the one given by Spitzer [29] in the i.i.d. case. Take $f \in b\mathscr{E}$ and $x \in E$. Then

$$((([I-T\hat{Q}(\theta)]^{-1}-\hat{G}_N(T,\theta))f)(x)$$
$$= E^x[\textstyle\sum_{k=0}^{\infty} \exp(i(\theta, S_{N+k}))(T^{N+k}f)(X_{N+k})]$$
$$= E^x[\exp(i(\theta, S_N)) \textstyle\sum_{k=0}^{\infty} E^x[\exp(i(\theta, S_k\circ\theta_N))(T^{N+k}f)(X_k\circ\theta_N) \mid \mathscr{M}_N]]$$

by the general properties of conditional expectations, and this equals

$$E^x[\textstyle\sum_{n=0}^{\infty} \exp(i(\theta, S_n)) \sum_{k=0}^{\infty} E^{X_n}[\exp(i(\theta, S_k))(T^{n+k}f)(X_k)];\ N=n]$$

by the Markov property (cf. Çinlar [11] (1.4)). Now this equals

$$E^x[\textstyle\sum_{n=0}^{\infty} \exp(i(\theta, S_n)) \sum_{k=0}^{\infty} (\hat{Q}(\theta)^k T^{n+k}f)(X_n);\ N=n]$$
$$= E^x[\exp(i(\theta, S_N))(T^N[I-T\hat{Q}(\theta)]^{-1}f)(X_N)]$$

by the commutativity of T and $\hat{Q}(\theta)$, and finally this equals

$$(\hat{H}_N(T,\theta)[I-T\hat{Q}(\theta)]^{-1}f)(x)\ ,$$

and the proof follows by the inversion of $[I-T\hat{Q}(\theta)]^{-1}$.

(2.7) *Alternative proofs.* At least two other distinct proofs can be given of the preceding result and we will just mention these. Firstly, one can adapt Meyer's [22] martingale proof of the continuous-time analogue to our context. Alternatively, a proof can be given by integrating a suitable Chapman-Kolmogorov forward equation, cf. Arjas [1].

(2.8) Remark. In many situations there exists a P-excessive measure π on $(E, \mathscr{E})$ and then, under mild regularity assumptions, it is possible to extend the operators $\hat{Q}$, $\hat{G}$, $\hat{H}$ to act on the spaces $L^p(\pi)$. When this is done the identity just proved remains valid.

(2.9) Remark. If the definition (1.6) of an MAP is extended to allow a general starting point $S_0 = Y_0$ for the second component the passage-time identity becomes

$$\hat{G}_N(T,\theta)[I-T\hat{Q}(\theta)] = \hat{F}(\theta)-\hat{H}_N(T,\theta)\ , \tag{2.10}$$

where

$$(\hat{F}f)(x) = E^x[e^{i(\theta,\, S_0)}f(x)]\ . \tag{2.11}$$

The proof of (2.10) is almost identical to that given above.

2.c. *Lesser identities.*

In this section we specialise (2.6) and thus see more fully its relation to previous work.

One result which can be obtained immediately is (2.3) — simply put $T=\tau I$, $|\tau|<1$, and $\theta=0$. As remarked in 2.a this identity is due to Kemperman [20] for a special type of stopping time and is undoubtedly more widely known but we are unable to give a specific reference to it.

Let us turn to another kind of specialisation of (2.6). Suppose the identity to be valid for operators $\hat{G}$, $\hat{Q}$, $\hat{H}$ on a Banach space B which may be $b\mathscr{E}$ or $L^p(\pi)$ $(1 \leqq p \leqq \infty)$ where π is a P-excessive measure. Suppose further that for $\theta \in J \subseteqq \mathsf{R}^m$ there exists an eigenvalue $\varkappa(\theta)$ and eigenfunction $u_\theta(\cdot) \in \mathsf{B}$ for $\hat{Q}(\theta)$ that is

$$\hat{Q}(\theta)u_\theta = \varkappa(\theta)u_\theta, \quad \theta \in J . \tag{2.12}$$

For example, since $P1=\hat{Q}(0)1=1$, the function 1 and the number 1 form an eigenfunction-eigenvalue pair and so for small $|\theta|$ there will often be a pair u_θ and $\varkappa(\theta)$ satisfying $u_0=1$ and $\varkappa(0)=1$, see e.g. Kato [18]. The theory of positive operators (Karlin [17]) provides, at least for $i\theta$ real, another approach to the existence of such eigenvalues. We do not consider this problem here, but simply examine the consequences of assuming the existence of eigenvalues. Define the functions

$$\gamma = \hat{G}u_\theta, \quad \chi = \hat{H}u_\theta , \tag{2.13}$$

where $\hat{G}=\hat{G}_N(\tau,\theta)$ and $\hat{H}=\hat{H}_N(\tau,\theta)$ for suitable θ, τ, N and $T=\tau I$. Then also

$$\gamma = \gamma_N(\tau,\theta,\cdot), \quad \chi = \chi_N(\tau,\theta,\cdot)$$

and we have

(2.14) Proposition. *For each* $x \in E$,

$$\gamma(\tau,\theta,x)[1-\tau\varkappa(\theta)] = u_\theta(x)-\chi(\tau,\theta,x) .$$

The proof is immediate. In the special case where the chain has only one state, so that $\{S_n\}$ is a random walk with i.i.d. increments, one may readily check that $\varkappa(\theta)$, the common characteristic function of the increments, is an eigenvalue, and $u_\theta \equiv 1$ the corresponding eigenfunction, of $\hat{Q}(\theta)$. This provides another derivation of (2.2).

We close this section by mentioning the important special case in which the state space E of X is finite. Here all the operators may be considered as matrices relative to a fixed basis, and we obtain Miller's [23] result:

(2.15) Proposition. *Suppose the chain X to have a finite state space. Then the following matrix identity is valid: for $\|T\|<1$ and T commuting with $\hat{Q}$:*

$$\hat{G}[I-T\hat{Q}] = I-\hat{H} . \tag{2.16}$$

2.d *Wald identities.*

One of the uses to which Miller [23] and Kemperman [20] put their identities was to derive generalisations of Wald's fundamental identity of sequential analysis. Formally all that is required is that there be a θ such that for $\tau = 1/\varkappa(\theta)$, $\gamma(\tau, \theta, x) < \infty$ for all $x \in E$, and then in (2.14) this substitution leads to Wald's identity. Feller [14] also uses this method, and Arjas [1] in the general state space situation. Clearly we can adopt this approach here, but before we do so we make a few remarks. We suppose in this section that $m = 1$.

Recall that in (2.6) the free variable T was permitted to be an operator on B more general than τI which had been considered by previous authors (Miller, Arjas etc.). The reason for this can now be made clear. Suppose that the identity (2.6) holds not only for certain purely imaginary complex numbers $\lambda = i\theta$ but possibly more generally for numbers $\lambda = \sigma + i\theta \in \mathsf{C}$ where we use the natural definition of $\hat{Q}(\lambda)$ viz.

$$(2.17) \qquad (\hat{Q}(\lambda)f)(x) = E^x[e^{\lambda Y_1} f(X_1)] .$$

The reason we restricted the earlier discussion was simply to display the methods and avoid any doubts concerning the regions of validity of (2.6) as $\lambda \in \mathsf{C}$ varies. Of course on any wider domain than the imaginary axis the operators $\hat{Q}(\lambda)$ and $\hat{H}_N(T, \lambda)$ are not necessarily contractions.

(2.18) THEOREM. (Wald's identity) *Suppose that λ is a complex number such that*

(i) *the passage-time identity* (2.6) *is valid*;
(ii) $\hat{Q}(\lambda)^{-1}$ *exists as a bounded linear operator on* B;
(iii) $\|\hat{G}(\hat{Q}(\lambda)^{-1}, \lambda)\| < \infty$.

Then:

$$(2.19) \qquad \hat{H}(\hat{Q}(\lambda)^{-1}, \lambda) = I .$$

Explicitly, for any $f \in \mathsf{B}$:

$$(2.19a) \qquad E^x[e^{\lambda S_N} \hat{Q}(\lambda)^{-N} f(X_N)] = f(x), \quad x \in E .$$

PROOF. The proof is immediate from (2.6) as described in the introduction to this section.

We can say a little more. It is not reasonable to think that a zero eigenvalue should, in the case $E = \{1, 2, \ldots, n\}$, prevent some form of the Wald identity from holding. Nor does it, for if we suppose that $\hat{Q}(\lambda)$

has a *commuting generalised inverse* $T(\lambda)$ i.e. that there exists a bounded linear operator $T(\lambda)$ on B such that

$$\hat{Q}(\lambda)T(\lambda)\hat{Q}(\lambda) = \hat{Q}(\lambda), \quad T(\lambda)\hat{Q}(\lambda)T(\lambda) = T(\lambda), \\ T(\lambda)\hat{Q}(\lambda) = \hat{Q}(\lambda)T(\lambda),$$

then we easily obtain

(2.20) THEOREM. *Suppose that λ is a complex number such that*:

(i) *the passage-time identity* (2.6) *is valid*;
(ii) $\hat{Q}(\lambda)$ *has a commuting generalised inverse* $T(\lambda)$;
(iii) $\|\hat{G}(T(\lambda),\lambda)\| < \infty$.

Then:

$$\hat{H}(T(\lambda),\lambda)T(\lambda) = T(\lambda). \tag{2.21}$$

PROOF. We put $T=T(\lambda)$ in (2.6) and multiply on the right by $T(\lambda)$.

In the case E is a finite set a simple sufficient condition for $\hat{Q}(\lambda)$ to have a commuting generalised inverse is that the zero eigenvalue of $\hat{Q}(\lambda)$, if such exists, is simple.

Next we suppose that $\hat{Q}(\lambda)$ is an operator on B and possesses an eigenvalue $\varkappa(\lambda)$ and eigenfunction $u_\lambda(\cdot)$ for λ in some $J\subseteq \mathsf{C}$, a slight extension of the situation described in (2.12) above. Then arguing as before we have, with the obvious extension of the notation (2.13):

(2.22) PROPOSITION. *Let $\lambda \in \mathsf{C}$ be such that*:

(i) *the identity* (2.6) *is valid*;
(ii) $\hat{Q}(\lambda)$ *has an eigenvalue* $\varkappa(\lambda)$ *and eigenfunction* $u_\lambda(\cdot)$;
(iii) $\gamma(\tau,\lambda,x)<\infty$ *at* $\tau=1/\varkappa(\lambda)$ *for all* $x\in E$.

Then $\chi(1/\varkappa(\lambda),\lambda,x)=u_\lambda(x)$. *Explicitly*:

$$E^x[e^{\lambda S_N}[1/\varkappa(\lambda)]^N u_\lambda(X_N)] = u_\lambda(x), \quad x\in E. \tag{2.23}$$

(2.24) REMARK. In the form (2.23) the Markovian Wald identity has a longer history. A result of this form was sketched by Bellman [5] and later proved with a fuller discussion by Tweedie [32], Miller [23] and Phatarfod [24]. See also § 2.23 of Barlett [3].

2.e *A moment identity.*

In the case of a random walk $\{S_n\}$ with i.i.d. increments one of the standard results relating to randomly stopped sums is the following identity of Wald [33]:

$$E[S_N] = E[Y_1]E[N], \tag{2.25}$$

where the existence of any two of the three expectations involved ensures the existence of the third and the validity of (2.25). Feller [14] gives an even better result for the case where $E[Y_1]$ does not exist but the characteristic function of the increment distribution has a derivative at zero. Since (2.25) was discovered, and a similar relation for $\mathrm{Var}[S_N]$, many moment identities relating the moments of N and S_N have been given e.g. by Brown [7].

We now consider the extension of (2.25) to stopped MAP's. To shorten the notation we write

$$E^\mu[f] = \langle \mu, f\rangle = \int f(x)\mu(dx)$$

for μ a measure on $(E, \mathscr{E})$ and $f \in L^1(\mu)$.

Suppose that the passage-time identity is valid on some Banach space of functions when $T = I$, $m = 1$ (the additive process is one-dimensional), and that we may expand $\hat{Q}(\theta)$, $\hat{G}_N(1,\theta)$ and $\hat{H}_N(1,\theta)$ as:

$$\begin{aligned} \hat{Q}(\theta) &= P + i\theta Q_1 + o(\theta) \quad (\hat{Q}(0) = P) \\ \hat{G}_N(1,\theta) &= G_0 + i\theta G_1 + o(\theta) \\ \hat{H}_N(1,\theta) &= H_0 + i\theta H_1 + o(\theta)\,. \end{aligned} \tag{2.26}$$

Then upon substituting the above into (2.6) and equating constant terms we obtain:

$$G_0(I-P) = I - H_0\,. \tag{2.27}$$

To interpret (2.27) we need to specialise drasticially. Suppose that

(i) X has an invariant probability measure π i.e. $\pi P = \pi$ and $\pi(E) = 1$;
(ii) the stopping time N is such that G_0 is invertible and H_0 is proper i.e. $H_0 1 = 1$ where $H_0 = \hat{H}_N(1,0)$.

These assumptions are fulfilled in a number of interesting special cases. Then it follows easily from (2.27) that πG_0^{-1} is H_0-invariant and so if $c_1^{-1} = \langle \pi, G_0^{-1} 1\rangle$ we have $\bar{\pi} \equiv c_1(\pi G_0^{-1})$ an invariant probability measure for the imbedded process $\bar{X} = \{\bar{X}_n : n \geqq 0\}$ which arises from "sampling" the original process X at times $n = N_1, N_1 + N_2, \ldots, N_1 + \ldots + N_k, \ldots$ where each N_i is a proper random time distributed as N. Conversely, if $\bar{\pi}$ is an invariant probability measure for $\bar{X}$ then $c_1^{-1}\bar{\pi} G_0 = \pi$. We collect these remarks in the following:

(2.28) PROPOSITION. *Suppose that the operator G_0 is invertible and $H_0 1 = 1$. Then the map $\pi \to c_1 \pi G_0^{-1}$ defines a bijection between finite P-*

invariant measures and finite H_0-invariant measures on (E,ξ). The inverse correspondence is $\bar{\pi} \to c_1^{-1}\bar{\pi}G_0$ where the constant c_1 is given by

$$c_1 = E^{\bar{\pi}}[N]\,. \tag{2.29}$$

PROOF. The majority of the proposition is clear from the remarks preceding. For the equation (2.29) we simply note that

$$c_1 = \langle \bar{\pi}, G_0 1\rangle = \langle \bar{\pi}, E^x[\textstyle\sum_{n=0}^{N-1} 1]\rangle = \langle \bar{\pi}, E^x[N]\rangle = E^{\bar{\pi}}[N]\,.$$

(2.30) REMARK. Apart from the fact that norming would be impossible, the correspondence above applies to not-necessarily-finite invariant measures and in fact to excessive measures as well.

We go on now to consider the terms in (2.6) expanded according to (2.26) which are linear in θ. Thus we have

$$G_1(I-P)-G_0Q_1 = -H_1\,. \tag{2.31}$$

From this equation we obtain a form of (2.25) valid for MAP's. For if π is P-invariant we have $\bar{\pi}=c_1\pi G_0^{-1}$ is H_0-invariant when H_0 is proper and G_0 invertible, and under these assumptions:

(2.32) PROPOSITION. $E^{\bar{\pi}}[S_N]=E^{\bar{\pi}}[N]E^{\pi}[Y_1]$.

PROOF. From (2.31) we have

$$\begin{aligned}
\langle \bar{\pi}, H_1 1\rangle &= \langle \bar{\pi}, [G_0Q_1-G_1(I-P)]1\rangle \\
&= \langle \bar{\pi}, G_0Q_1 1\rangle \qquad \text{since } P1=1\,, \\
&= c_1\langle \pi G_0^{-1}, G_0Q_1 1\rangle \quad \text{by definition of } \bar{\pi}\,, \\
&= c_1\langle \pi, Q_1 1\rangle \\
&= c_1E^{\pi}[Y_1] = E^{\bar{\pi}}[N]E^{\pi}[Y_1]\,,
\end{aligned}$$

as required.

2.f *An example.*

We close this chapter by considering briefly one situation where it is possible to determine certain expressions quite explicitly.

(2.33) *The one-step autoregressive process.* Let $\{\varepsilon_i\}$ be a sequence of i.i.d. random variables with distribution F and characterstic function ξ. For $|\varrho|<1$ write

$$\begin{cases} X_0 = 0\,, \\ X_{n+1} = \varrho X_n + \varepsilon_{n+1}, \quad n \geqq 0\,, \end{cases} \tag{2.34}$$

and consider the MAP (X, S) based upon the Markov chain X with $S_n = \sum_0^n X_k$. It is easy to see that the maximal eigenvalue $\varkappa(\theta)$ of the operator $\hat{Q}(\theta)$ on the space associated with the above autoregressive process is $\varkappa(\theta) = \xi(\theta/1-\varrho)$ with eigenfunction $u_\theta(x) = \exp(i\varrho\theta x/1-\varrho)$. Moreover, if $\{\varepsilon_i\}$ are normally distributed with mean 0 and variance 1, then X has an invariant measure π with

$$\pi(dx) = (2\pi)^{-\frac{1}{2}}(1-\varrho^2)^{\frac{1}{2}} \exp\{-\tfrac{1}{2}(1-\varrho^2)x^2\} dx , \tag{2.35}$$

where dx denotes Lebesgue measure on the state space $E = \mathbb{R}^1$.

3. Processes associated with a single boundary.

3.a *Orientation.*

In this final chapter we consider a number of applications. Most of the results proved can be obtained in other ways but we give only one way and leave the task of modelling alternative proofs to the reader.

All the results obtained share with the i.i.d. case the problem of finding, in particular cases, explicit formulae. We have a number of examples (relating to simple cases like negative exponential and gamma type increments) where this can be done but do not present them here.

3.b. *Maxima in Markov chains.*

In this section we will consider an ordered state space $(E; \leqq)$ for X; explicitly we assume that the binary relation $\leqq$ satisfies the following: for all $w, x, x' \in E$:

$$\begin{aligned} &\text{(i) } x \leqq x; \\ &\text{(ii) } x \leqq w \text{ and } w \leqq x' \text{ implies } x \leqq x'; \\ &\text{(iii) } x \leqq x' \text{ or } x' \leqq x . \end{aligned} \tag{3.1}$$

Let us define equivalence $\sim$ by $x \sim x'$ iff $x' \leqq x$ and $x \leqq x'$. Then we write $x' < x$ if $x' \leqq x$ and $x' \sim x$ is false. Then in terms of this we further suppose

$$\{(x', x) : x' < x\} \in \mathscr{E} \times \mathscr{E} . \tag{3.2}$$

Following these preliminaries cf. Dinges [12], we can define the (*first strict ascending*) *ladder index* N^+ by:

$$N^+ = \inf\{n : X_n > X_0\} . \tag{3.3}$$

Putting $N_0{}^+ = 0$, $N_1{}^+ = N^+$, subsequent ladder indices are defined in the obvious manner. Further, for any $n \geqq 0$ we define L_n, the *first position of the maximum up to time n* by

$$L_n = k \text{ iff } X_k > X_j,\ (0 \leqq j < k) \text{ and } X_k \geqq X_l,\ (k < l \leqq n) . \tag{3.4}$$

Finally the maximum state M_n at time n can be defined as

$$(3.5) \qquad M_n = X_{L_n}, \quad n \geqq 0 .$$

The aim of this section is to obtain an expression for the resolvent of the process $\{(M_n, S_n): n \geqq 0\}$ which we will refer to as (M, S). When X is a random walk with i.i.d. increments the corresponding process M has been extensively studied; in particular Spitzer's famous identity gives an exponential formula for the resolvent. We discuss this point briefly in 3.e below.

To obtain the result we desire we use an operator Π similar to the sweeping-up projection which Kingman [21] calls a Wendel projection. In our case Π is an idempotent linear transformation on the set of all operator-valued measures defined on $(\overline{\mathsf{R}}^m, \overline{\mathscr{R}}^m)$, and Π induces a transformation $\hat{\Pi}$ on the family of all Fourier transforms of these. We define Π as follows: Let $x \in E$ and define $\xi_x: E \to E$ by $\xi_x(x') \equiv x$. Then for T an operator-valued measure on $(\overline{\mathsf{R}}^m, \overline{\mathscr{R}}^m)$ ΠT is defined by

$$(3.6) \qquad (\Pi T)(x, A \times B) = T\big(x, (\xi_x{}^{-1}(A)) \times B\big),$$

$x \in E$, $A \in \mathscr{E}$, $B \in \overline{\mathscr{R}}^m$. From (3.6) follows immediately that

$$(3.7) \qquad ((\Pi T)f)(x) = (T1)(x) \cdot f(x), \quad f \in \mathsf{B} .$$

The definition of $\hat{\Pi}$ is also clear.

As stated above, our aim is to obtain an expression for the resolvent of (M, S); this is defined by: For $x \in E$, $0 \leqq \tau < 1$, $\theta \in \mathsf{R}^m$ and $f \in \mathsf{B}$

$$(3.8) \qquad (\hat{\Psi}(\tau, \theta)f)(x) = E^x[\textstyle\sum_{n=0}^\infty \exp(i(\theta, S_n))\tau^n f(M_n)] .$$

The main result is the following

(3.9) THEOREM. $\hat{\Psi}(\tau, \theta) = [I - \hat{H}_{N+}(\tau, \theta)]^{-1} \hat{\Pi} \hat{G}_{N+}(\tau, \theta)$.

PROOF.

$$\begin{aligned}
(\hat{\Psi}(\tau, \theta)f)(x) &= E^x[\textstyle\sum_{n=0}^\infty \exp(i(\theta, S_n))\tau^n f(M_n)] \\
&= E^x[\textstyle\sum_{k=0}^\infty \sum_{n=0}^{N_{k+1}^+ - N_k^+ - 1} \exp(i(\theta, S_{N_k^+ + n}))\tau^{N_k^+ + n} f(X_{N_k^+})] \\
&= E^x[\textstyle\sum_{k=0}^\infty \exp(i(\theta, S_{N_k^+}))\tau^{N_k^+} \cdot \\
&\quad \cdot E^x[\textstyle\sum_{n=0}^{(N^+ - 1)\circ\theta_{N_k^+}} \exp(i(\theta, S_n \circ \theta_{N_k^+}))\tau^n \mid \mathscr{M}_{N_k^+}] f(X_{N_k^+})] \\
&= E^x[\textstyle\sum_{k=0}^\infty \exp(i(\theta, S_{N_k^+}))\tau^{N_k^+} (\hat{G}_{N+}(\tau, \theta)1)(X_{N_k^+}) f(X_{N_k^+})]
\end{aligned}$$

by the (strong) Markov property, and by (3.7) this equals

$$\begin{aligned}
&E^x[\textstyle\sum_{k=0}^{\infty} \exp(i(\theta, S_{N_k^+}))\tau^{N_k^+}((\hat{\Pi}\hat{G}_{N^+}(\tau,\theta))f(X_{N_k^+})] \\
&\quad = (\textstyle\sum_{k=0}^{\infty} \hat{H}_{N^+}(\tau,\theta)^k(\hat{\Pi}\hat{G}_{N^+}(\tau,\theta))f)(x) \\
&\quad = ([I-\hat{H}_{N^+}(\tau,\theta)]^{-1}(\hat{\Pi}\hat{G}_{N^+}(\tau,\theta))f)(x) .
\end{aligned}$$

The probabilistic interpretation of (3.9) is as follows: the maximum M_n at time n is achieved by following a sequence of ascending ladder indices and then keeping below or equal to the position achieved. Notice that we are simply recording S at time n together with the position of the maximum up to that time. If, however, we desire to record S at the time L_n at which the maximum M_n up to time n is achieved, then a modification of the above proof gives with

$$(\hat{\Psi}_1(\tau,\theta)f)(x) = E^x[\textstyle\sum_{n=0}^{\infty} \exp(i(\theta, S_{L_n}))\tau^n f(M_n)] \tag{3.10}$$

(3.11) THEOREM. $\hat{\Psi}_1(\tau,\theta) = [I-\hat{H}_{N^+}(\tau,\theta)]^{-1}\hat{\Pi}\hat{G}_{N^+}(\tau,0)$.

The formulation we have just given makes it possible for us to enquire after the transform of the *ultimate maximum* M_∞ which X achieves (if such exists), and the position S_{L_∞} of S when this ultimate maximum is achieved. We have the following:

(3.12) THEOREM. *If $M = \lim_{n\to\infty} X_{L_n}$ exists and*

$$(\tilde{\Psi}(\theta)f)(x) = E^x[\exp(i(\theta, S_{L_\infty}))f(M_\infty)] \tag{3.13}$$

then

$$\tilde{\Psi}(\theta) = [I-\hat{H}_{N^+}(1,\theta)]^{-1}[I-\hat{\Pi}\hat{H}_{N^+}(1,0)] . \tag{3.14}$$

PROOF. We use an Abelian argument on (3.11) giving

$$\begin{aligned}
(\tilde{\Psi}(\theta)f)(x) &= \lim_{\tau\uparrow 1}(1-\tau)([I-\hat{H}_{N^+}(\tau,\theta)]^{-1}(\hat{\Pi}\hat{G}_{N^+}(\tau,0))f)(x) \\
&= \lim_{\tau\uparrow 1}(1-\tau)([I-\hat{H}_{N^+}(\tau,\theta)]^{-1}(\hat{G}_{N^+}(\tau,0)1\cdot f))(x) \\
&\qquad\qquad \text{by (3.7)}, \\
&= \lim_{\tau\uparrow 1}(1-\tau)([I-\hat{H}_{N^+}(\tau,\theta)]^{-1}([I-\hat{H}_{N^+}(\tau,0)]\cdot \\
&\qquad \cdot [I-\tau\hat{Q}(0)]^{-1}1\cdot f))(x) \qquad \text{by (2.6)}, \\
&= \lim_{\tau\uparrow 1}([I-\hat{H}_{N^+}(\tau,\theta)]^{-1}([I-\hat{H}_{N^+}(\tau,0)]1\cdot f))(x) \\
&\qquad\qquad \text{by } \hat{Q}(0)1 = 1, \\
&= ([I-\hat{H}_{N^+}(1,\theta)]^{-1}[I-\hat{\Pi}\hat{H}_{N^+}(1,0)]f)(x) \qquad \text{by (3.7)}.
\end{aligned}$$

The above theorem is a Markovian generalisation of a result due originally to Täcklind [31] and later derived by Spitzer [28]. (See, however, the remarks on exponential formulae in 3.e.) It is also closely related to work of Baxter [4] and Stone [30] although these authors do not consider the limiting properties.

3.c *Maxima in the additive processes.*

We turn now to a topic closely related to that of the last section, but with the emphasis on the second component of the MAP (X, S). Assume throughout this section that S is one-dimensional i.e. that $m=1$, and that $S_0=0$ a.s.; it is then clear that the following are well defined:

(3.15) $$N^+ = \inf\{n : S_n > 0\};$$

(3.16) $$L_n = k \text{ iff } S_k > S_j,\ (0 \leqq j < k) \text{ and } S_k \geqq S_l,\ (k < l \leqq n).$$

The ladder indices $N_0^+=0$, $N_1^+=N^+, N_2^+, \ldots$ and the times of first attaining the maximum are interpreted exactly as in the previous section, the only difference being the fact that they refer to S rather than X. There seems no danger in using the same notation as in 3.b for in any particular case it will be clear which is intended. Thus we define the maxima of the additive process S by

(3.17) $$M_n = S_{L_n}, \quad n \geqq 0.$$

As one would expect, our aim is to obtain the resolvent of the process

$$(X, M) = \{(X_n, M_n) : n \geqq 0\}$$

and again this generalises, in a different way, the work relating to maxima of partial sums of i.i.d. random variables. For $x \in E$, $0 \leqq \tau < 1$, $\theta \in \mathsf{R}^1$ and $f \in \mathsf{B}$ define

(3.18) $$(\hat{\Phi}(\tau, \theta) f)(x) = E^x[\textstyle\sum_{n=0}^{\infty} \exp(i\theta M_n)\, \tau^n f(X_n)].$$

(3.19) THEOREM. $\hat{\Phi}(\tau, \theta) = [I - \hat{H}_{N^+}(\tau, \theta)]^{-1} \hat{G}_{N^+}(\tau, 0)$.

PROOF. The proof is similar to that of Theorem (3.9).

The probabilistic interpretation of (3.19) is as that of (3.9) in the last section, but with emphasis on the second component of the MAP (X, S).

Another possibility is to consider the process $\{(X_{L_n}, M_n) : n \geqq 0\}$, that is to record the values of X at the times a maximal value is obtained in S. If

(3.20) $$(\hat{\Phi}_1(\tau, \theta) f)(x) = E^x[\textstyle\sum_{n=0}^{\infty} \exp(i\theta M_n)\, \tau^n f(X_{L_n})],$$

then a minor modification in the proof of (3.9) gives, with Π as in (3.6), (3.21) THEOREM. $\hat{\Phi}_1(\tau,\theta)=[I-\hat{H}_{N+}(\tau,\theta)]^{-1}\hat{\Pi}\hat{G}_{N+}(\tau,0)$.

Observe that the right hand sides of (3.21) and (3.11) are formally identical. We can also state the corresponding limiting result here, the proof being exactly the same as for (3.14).

(3.22) THEOREM. *If* $M_\infty=\lim_{n\to\infty}S_{L_n}<\infty$ a.s. *and*

$$(\tilde{\Phi}(\theta)f)(x) = E^x[\exp(i\theta M_\infty)f(X_{L_\infty})]$$

then

$$\tilde{\Phi}(\theta) = [I-\hat{H}_{N+}(1,\theta)]^{-1}[I-\hat{\Pi}\hat{H}_{N+}(1,0)]\,. \tag{3.23}$$

3.d. MAP'*s with one impenetrable barrier.*

One of the more important processes associated with the one-dimensional random walk $\{S_n\}$ with i.i.d. increments $\{Y_k\}$ is the process modified by the placing of an impenetrable barrier at zero i.e. the process $\{W_n\}$ defined recursively by

$$\begin{aligned} W_0 &= Y_0{}^+; \\ W_{n+1} &= (W_n+Y_{n+1})^+, \quad n\geqq 0\,. \end{aligned} \tag{3.24}$$

A classical result which goes back to Wald [34] is that when $Y_0=0$ a.s. $\{W_n\}$ and $\{M_n\}$ (see (3.17)) have the same distribution. As has been mentioned this distribution was later elegantly expressed by Spitzer [28]; see also Kingman [21] for an alternative derivation based on (3.24). Let us define the (*first weak descending*) *ladder index* $\bar{N}$ of $\{S_n\}$ by

$$\bar{N} = \inf\{n>0 : S_n\leqq 0\} \tag{3.25}$$

and $\bar{N}_2,\bar{N}_3,\ldots$ have the obvious definitions following $\bar{N}_0=0$ and $\bar{N}_1=\bar{N}$. Further we define *the position of the last minimum up to time* n:

$$\bar{L}_n = k \text{ iff } S_k \leqq S_j,\ (0\leqq j<k) \text{ and } S_k < S_l,\ (k<l\leqq n)\,. \tag{3.26}$$

In terms of the above we can easily check that

$$W_n = S_n-S_{L_n} = \textstyle\sum_k (S_n-S_k)I_{\{L_n=k\}}\,. \tag{3.27}$$

Turning now to our MAP (X,S) (with $m=1$) we define a related bivariate process called (X,W) as follows:

$$\begin{cases}(X_0,W_0) = (X_0,Y_0{}^+); \\ (X_{n+1},W_{n+1}) = (X_{n+1},(W_n+Y_{n+1})^+), \quad n\geqq 0\,.\end{cases} \tag{3.28}$$

The resolvent of (X,W) is denoted $\hat{A}(\tau,\theta)$, that is for $x \in E$, $0 \leqq \tau < 1$, $\theta \in \mathsf{R}^1$ and $f \in \mathsf{B}$

$$(\hat{A}(\tau,\theta)f)(x) = E^x[\textstyle\sum_{n=0}^{\infty} \exp(i\theta W_n)\,\tau^n f(X_n)]\,. \tag{3.29}$$

With $\bar{N}$, $\bar{L}_n$ defined as in (3.25), (3.26) we have the following result where for simplicity we suppose $S_0 = Y_0 = 0$ a.s. (see 3.e for a more general result).

(3.30) THEOREM. *If* $S_0 = 0$ a.s. *then*

$$\hat{A}(\tau,\theta) = [I - H_{\bar{N}}(\tau,0)]^{-1}\hat{G}_{\bar{N}}(\tau,\theta)\,. \tag{3.31}$$

PROOF.

$$\begin{aligned}(\hat{A}(\tau,\theta)f)(x) &= E^x[\textstyle\sum_{n=0}^{\infty} \exp(i\theta W_n)\,\tau^n f(X_n)] \\ &= E^x[\textstyle\sum_{k=0}^{\infty} \sum_{n=0}^{\bar{N}_{k+1}\bar{N}_k - 1} \exp(i\theta(S_{\bar{N}_k+n} - S_{\bar{N}_k}))\,\tau^{\bar{N}_k+n} f(X_{\bar{N}_k+n})] \\ &= E^x[\textstyle\sum_{k=0}^{\infty} \tau^{\bar{N}_k} E^x[\sum_{n=0}^{(\bar{N}-1)\circ\theta_{\bar{N}_k}} \exp(i\theta(S_n\circ\theta_{\bar{N}_k}))\cdot \\ &\quad \cdot \tau^n f(X_n\circ\theta_{\bar{N}_k}) \mid \mathscr{M}_{\bar{N}_k}]] \\ &= E^x[\textstyle\sum_{k=0}^{\infty} \tau^{\bar{N}_k}(\hat{G}_{\bar{N}}(\tau,\theta)f)(X_{\bar{N}_k})]\end{aligned}$$

by the (strong) Markov property, and finally this equals

$$(\textstyle\sum_{k=0}^{\infty} \hat{H}_{\bar{N}}(\tau,0)^k \hat{G}_{\bar{N}}(\tau,\theta)f)(x) = ([I - \hat{H}_{\bar{N}}(\tau,0)]^{-1}\hat{G}_{\bar{N}}(\tau,\theta)f)(x)\,.$$

As is ususally done we go on to obtain an expression for the limiting distribution of W_n when it exists. In the i.i.d. case one has the result that W_n and M_n have the same distribution and so when $\lim_{n\to\infty} M_n < \infty$ a.s. a limiting distribution for W_n also exists. The situation here is more subtle — see the next section — so we simply postulate the existence of the distribution.

For the limiting distribution in (X,W) to exist some regularity condition must be assumed on X. Referring to Šidak [27] for the necessary definitions we state the following:

(3.32) ASSUMPTION. The chain X is irreducible and positive recurrent with a unique P-invariant measure π; further we require that the condition CD be valid with $d = 1$ (aperiodicity) and PS (positivity for some n).

Under this assumption, for any $A \in \mathscr{E}$ with $\pi(A) > 0$ we have

$$\lim_{n\to\infty} n^{-1} \textstyle\sum_0^{n-1} P^{(r)}(x,A) = \pi(A) \quad \text{for a.e. } x(\pi)\,. \tag{3.33}$$

(3.34) LEMMA. *Under Assumption* (3.32), *if* $f \in L^1(\pi)$,

(i) $\lim_{n\to\infty}(n^{-1}\sum_{k=0}^{n-1}P^k f)(x) = E^\pi[f]$ *for* a.e. $x(\pi)$;

(ii) $\lim_{\tau\uparrow 1}(1-\tau)([I-\tau P]^{-1}f)(x) = E^\pi[f]$ *for* a.e. $x(\pi)$.

PROOF. (i) is a consequence of the ergodic theorem and results in Šidak [27] and (ii) follows from the fact that (pointwise)

$$\lim_{\tau\uparrow 1}(1-\tau)([I-\tau P]^{-1}f)(x) = \lim_{n\to\infty}(n^{-1}\textstyle\sum_0^{n-1} P^k f)(x)$$

which follows from a familiar Abelian argument (see Feller [14]).

Let now

$$(\tilde{A}(\theta)f)(x) = E^x[e^{i\theta W_\infty}f(X_\infty)] \tag{3.35}$$

whenever $W_\infty = \lim_{n\to\infty}W_n < \infty$ a.s. and $X_\infty = \lim_{n\to\infty}X_n$ exists, and define P_1 by

$$(P_1 f)(x) = E^\pi[f] \quad \text{(constant)}\,. \tag{3.36}$$

Then the following limiting result holds:

(3.37) THEOREM. *If* $W_\infty = \lim_{n\to\infty}W_n < \infty$ a.s., *then under Assumption* (3.32)

$$\tilde{A}(\theta) = P_1\hat{G}_N{}^{-1}(1,0)\hat{G}_N(1,\theta)\,. \tag{3.38}$$

PROOF. Arguing as before,

$$\begin{aligned}\lim_{\tau\uparrow 1}(1-\tau)\hat{A}(\tau,\theta) &= \lim_{\tau\uparrow 1}(1-\tau)[I-\hat{H}_N(\tau,0)]^{-1}\hat{G}_N(\tau,\theta)\\ &= \lim_{\tau\uparrow 1}(1-\tau)[I-\tau\hat{Q}(0)]^{-1}\hat{G}_N{}^{-1}(\tau,0)\hat{G}_N(\tau,\theta) \qquad \text{by (2.6)}\\ &= P_1\hat{G}_N{}^{-1}(1,0)\hat{G}_N(1,\theta)\,.\end{aligned}$$

As before the limit is to be taken as a pointwise limit when operating on functions $f \in L^1(\pi)$.

Ignoring the chain completely gives us an expression for the transform of the limiting random variable. Here $(\bar{P}_1 f)(x) = E^{\bar{\pi}}[f(X_{\bar{N}})]$ where $\bar{\pi}$ and c_1 are defined as in 2.e with $N = \bar{N}$.

(3.39) COROLLARY. $E^x[e^{i\theta W_\infty}] = c_1{}^{-1}(\bar{P}_1\hat{G}_{\bar{N}}(1,\theta)1)(x)$.

PROOF. $E^x[e^{i\theta W_\infty}] = (\hat{A}(\theta)1)(x)$ and (3.39) arises after we note (with the notation as in 2.e) that

$$c_1{}^{-1}\bar{\pi} = \pi\hat{G}_N{}^{-1}(1,0) \quad \text{implies} \quad c_1{}^{-1}\bar{P}_1 = P_1\hat{G}_N{}^{-1}(1,0)\,.$$

This completes the proof.

3.e *Remarks on the preceding sections.*

The reader can hardly fail to note a number of points which have not been explained after reading the two preceding sections. Firstly, the striking similarity between (3.19) and (3.31), and also that between (3.23) and (3.38); secondly we have used a number of times the fact that for certain stopping times N the operators $\hat{G}_N(\tau, \theta)$ are invertible. Finally we have not fully explained why the random variables W_n of the preceding section should have a limiting distribution. It turns out that all of these points are aspects of the same phenomenon — *duality.*

Recall that at the beginning of 3.d we remarked that Wald had proved (for $Y_0 = 0$ a.s.) $\{M_n\}$ and $\{W_n\}$ have the same distribution; it is apparent when the proof of this fact is examined that the only requirement on the increments is reversibility: that $\{Y_1, \ldots, Y_n\}$ has the same distribution as $\{Y_n, \ldots, Y_1\}$. Armed with this knowledge it is natural to ask whether a duality exists between (X, W) and a process (X, M) where the reversing of the increments is done in some suitable manner. This is indeed the case as we have shown elsewhere [2], and so for a suitable dual (reversed) process $\{(\tilde{X}_n, \tilde{Y}_n) : n \geqq 0\}$ we can prove that the resolvents of the processes

$$(3.40) \qquad \{(\tilde{X}_n, \tilde{W}_n) : n \geqq 0\} \quad \text{and} \quad \{(X_n, M_n) : n \geqq 0\}, \quad \text{or}$$
$$\{(\tilde{X}_n, \tilde{M}_n) : n \geqq 0\} \quad \text{and} \quad \{(X_n, W_n) : n \geqq 0\}$$

are adjoints.

This duality completely explains the similarities noted above; in fact (3.19) can be obtained from (3.31) (or vice-versa) by taking adjoints, and so only one of these results needs to be proved. Similarly we can show that

$$\hat{G}_N^{-1} = [I - \hat{\tilde{H}}^*{}_{\tilde{N}+}]$$

where

$$(3.41) \qquad \tilde{N}^+ = \inf\{n : \tilde{S}_n > 0\}$$

and * denotes adjoint, and so the problem concerning the inverses of the $\hat{G}$-operators is solved. Finally whenever the monotone increasing random variables $\{\tilde{M}_n\}$ have a proper limiting distribution the relation (3.40) shows that the same is true for the random variables $\{W_n\}$. All these points are fully explained in Arjas and Speed [2].

Another point which suggests itself after studying the previous two paragraphs is the absence of any exponential formulae like that of Spitzer's for the resolvents. Let us see why this should be so. In the i.i.d. case with the notation of Feller [14] XVIII we have

$$\sum_0^\infty \tau^n E[e^{i\theta M_n}] = [1-\chi(\tau,\theta)]^{-1}\gamma(\tau,0) \tag{3.42}$$

which of course is very like (3.19). To obtain explicit expressions for $\chi(\tau,\theta)$ and $\gamma(\tau,\theta)$ Feller writes the passage-time identity (2.2) in the form

$$(1-\tau E[e^{i\theta Y_1}])^{-1} = \exp(\log(1-\chi(\tau,\theta))^{-1}+\log\gamma(\tau,\theta)), \quad 0 \leqq \tau < 1 ,$$

and expands the logarithms. This shows just where such an approach breaks down in our situation — we are dealing with operators and it is just *not* true that $\exp(\log U + \log V) = UV$ in that case. When all the relevant operators commute the above result holds and we obtain the following extension of a result of Baxter [4].

(3.43) THEOREM. *For N^+ as in* (3.15) *let us suppose that $\hat{G}_{N+}$ and $\hat{H}_{N+}$ commute. Then the following formulae are valid for $0 \leqq \tau < 1$, $\theta \in \mathsf{R}^1$:*

$$I-\hat{H}_{N+}(\tau,\theta) = \exp[-\textstyle\sum_{n=1}^\infty n^{-1}\tau^n\Gamma(\hat{Q}(\theta)^n)] \tag{3.44}$$

$$\hat{G}_{N+}(\tau,\theta) = \exp[\textstyle\sum_{n=1}^\infty n^{-1}\tau^n[I-\Gamma](\hat{Q}(\theta)^n)] \tag{3.45}$$

where Γ is a suitable projection operator.

PROOF. We omit the straightforward proof. It is very similar to the i.i.d. case in Feller [14] where in our case the action of Γ can be described as follows: for $f \in \mathsf{B}$

$$(\Gamma(\hat{Q}(\theta)^n)f)(x) = E^x[e^{i\theta S_n}f(X_n)\,;\, S_n > 0]\,. \tag{3.46}$$

We close these remarks by stating without proof an analogue of (3.30) which is valid for an arbitrary initial random variable Y_0 rather than $Y_0 = 0$ a.s. To do this we need another projection operator Π which "sweeps up" analogously to the one defined in 3.b above. To avoid lengthy preliminaries we simply state that if for some transform operator U we have

$$(Uf)(x) = E^x[\alpha(X_0,\ldots,X_n,\ldots\,;\,S_0,\ldots,S_n,\ldots)]$$

then

$$(\Pi Uf)(x) = E^x[\alpha(X_0,\ldots,X_n,\ldots\,;\,S_0^+,\ldots,S_n^+,\ldots)]\,.$$

(3.47) THEOREM. *With a general starting distribution defining the operator $\hat{F}(\theta)$ as in Chapter* 2, *we have*

$$\hat{A}(\tau,\theta) = \Pi(\hat{F}(\theta)[I-\hat{H}_N(\tau,\theta)]^{-1})\hat{G}_N(\tau,\theta)\,. \tag{3.48}$$

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