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SOME HYPERGEOMETRIC TRANSFORMATIONS AND GENERATING FUNCTIONS

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1. Introduction.

Transformations of infinite series the terms of which involve products of hypergeometric functions depending upon the summation index were investigated by Erdélyi [4; 5] and by Meixner [10], who considered Kummer's ${}_1F_1$ and Gauss's ${}_2F_1$; a typical result is [4, equation (16)]

$$(1.1)\qquad \sum_{r=0}^{\infty} \frac{(h;r)z^r}{r!}\, {}_1F_1\left[\begin{matrix}-r\\ c\end{matrix}\middle| x\right] {}_1F_1\left[\begin{matrix}-r\\ d\end{matrix}\middle| y\right]$$

$$= (1-z)^{-h} \sum_{r=0}^{\infty} \frac{(h;r)}{(c;r)(d;r)r!} \left[\frac{xyz}{(1-z)^2}\right]^r {}_1F_1\left[\begin{matrix}h+r\\ c+r\end{matrix}\middle| \frac{xz}{z-1}\right] {}_1F_1\left[\begin{matrix}h+r\\ d+r\end{matrix}\middle| \frac{yz}{z-1}\right]$$

where $(\alpha;r) \equiv \Gamma(\alpha+r)/\Gamma(\alpha)$ is the Pochhammer symbol.

More general transformations of this kind have been obtained: The hypergeometric functions may be of higher order, and the products, being special hypergeometric functions of two variables, may be replaced by certain general hypergeometric functions of two variables.

With an arbitrary number of variables, the mere notation of hypergeometric functions becomes difficult: to be of general use, the functional symbol must, in addition to the parameters and variables, in some manner indicate the "parametric structure", i.e., one must be able to infer with which variables any particular parameter is associated. While such an indication is trivially inherent in the current notations for hypergeometric functions of one and two variables, hypergeometric functions of three variables with general parametric structure require rather bulky symbols as suggested by Bhagchandani [2] and by Srivastava [15]; in the case of more than three variables, only special parametric structures have been considered at all.

A further generalization of these transformations might thus be considered virtually impossible because of the notational difficulties encountered. However, in transformations of this kind the parameters change

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in a simple and systematic manner. Therefore, it has been possible to introduce a contracted notation (section 2) which is satisfactory for the purpose of establishing generalized transformations involving multiple series the terms of which contain functions of twice as many variables (section 4). Some generating function relations are derived from the transformations (sections 6–8).

2. Preliminaries.

In order to obtain a suitably contracted notation, some conventions are introduced. An ordered set of P elements $(\alpha_1,\ldots,\alpha_P)$ is denoted by (α_p), and a similar rule applies to sets whose elements are of identical structure; for instance, $(\alpha_1/\beta_1,\ldots,\alpha_P/\beta_P)$ is written (α_p/β_p). (This symbol must, of course, not be interpreted as a set when an operation over the index is implied.) Multiple series are denoted by single summation signs and the summation limits $0,\infty$ are understood.

The functions in which we are primarily interested depend upon $2P$ complex variables (x_p), (y_p) and a number of complex parameters. The definition is

(2.1)
$$F\left[\begin{matrix}(a_p),\ (b_p)\\ (c_p),\ (d_p)\end{matrix}\Big|\mathscr{U}[(m_p),\ (n_p)]\Big|(x_p),\ (y_p)\right]$$
$$= \sum\nolimits_{(\mu_p),(\nu_p)} U((\mu_p+m_p),\ (\nu_p+n_p)) \prod\nolimits_{p=1}^{P} \frac{(a_p\,;\,\mu_p)x_p{}^{\mu_p}(b_p\,;\,\nu_p)y_p{}^{\nu_p}}{(c_p\,;\,\mu_p)\mu_p!(d_p\,;\,\nu_p)\nu_p!},$$

subject to convergence conditions (see below) unless this power series reduces to a polynomial. $(U((\mu_p),(\nu_p)))$ denotes an arbitrary multiple sequence of complex numbers; it may depend upon other parameters whose presence is indicated summarily by the set symbol $\mathscr{U}[(m_p),(n_p)]$ on the left-hand side of (2.1); (m_p) and (n_p) are sets of non-negative integers the significance of which will appear later. Thus F is not necessarily a hypergeometric function although the elements of the sets (a_p), (b_p), (c_p), (d_p) are of course hypergeometric numerator and denominator parameters, each associated with one variable. Any of these four sets may be empty; in such a case, the corresponding factors on the right-hand side of (2.1) are replaced by unity. No denominator parameter may, in general, be zero or a negative integer.

A simplified F-function will appear in section 6, and a function involving hypergeometric parameters associated with two variables also proves to be of interest (section 8).

Convergence conditions for the series defined by (2.1), and in subsequent cases, will, for brevity, be sufficient conditions only, based upon Abel's lemma.

When the sets (a_p), (b_p), (c_p), (d_p) are all present, a sufficient condition for F to exist in a non-empty neighbourhood of the origin in $\mathbf{C}^{2P}$ is

$$|U((\mu_p),(\nu_p))| \leqq C_{\mathrm{I}}{}^{\Sigma_p(\mu_p+\nu_p)} \prod_p \{\mu_p!\,\nu_p!\}\,; \tag{2.2}$$

C_{I}, C_{II}, etc., denote positive constants. In fact, from this inequality, and from the elementary results

$$(\mu+m)!/(\mu!\,m!) \leqq 2^{\mu+m}\,, \tag{2.3}$$

and

$$(\alpha;\mu) \sim C_{\mathrm{II}}\mu^{\alpha-1}\mu!, \qquad \alpha \text{ fixed}\,, \tag{2.4}$$

we readily find that the absolute value of the general coefficient of the series in (2.1) is smaller than

$$C_{\mathrm{III}}{}^{\Sigma_p(\mu_p+\nu_p)}\,;$$

convergence then follows from Abel's lemma.

If the sets (a_p), (b_p), (c_p), (d_p) are not all present, the condition (2.2) must be accordingly amended. In particular,

$$|U((\mu_p),(\nu_p))| \leqq C_{\mathrm{IV}}{}^{\Sigma_p(\mu_p+\nu_p)} \tag{2.5}$$

is a sufficient condition when the sets (c_p), (d_p) are empty.

3. Hypergeometric functions as special cases.

By proper choice of the U-sequence the function defined by (2.1) will be expressible in terms of hypergeometric functions. Two examples will illustrate this fact.

For arbitrary P, let

$$U((\mu_p),(\nu_p)) = (\alpha;\textstyle\sum_{p=1}^{P}\mu_p)(\beta;\sum_{p=1}^{P}\nu_p)\,, \tag{3.1}$$

and take $(m_p)=(n_p)=(0)$; obviously, a product of two Lauricella functions

$$F_A[\alpha,(a_p);(c_p);(x_p)]F_A[\beta,(b_p),(d_p);(y_p)]$$

is then obtained by insertion into (2.1).

Next, take $P=1$, suppress unnecessary indices, let

$$U(\mu,\nu) = \sum_{\varrho=0}^{\infty} \frac{(\alpha;\nu+\varrho)(\beta;\mu+\varrho)z^{\varrho}}{(\gamma;\varrho)\varrho!}\,, \tag{3.2}$$

and take $m=n=0$; in this case, F becomes one of the functions considered by Saran [12], viz.,

$$F_K[a,\alpha,\alpha,\beta,b,\beta;\, c,d,\gamma;\, x,y,z];$$

the U-sequence may thus contain additional variables as well as additional parameters.

In order to elucidate the significance of the sets $(m_p,),(n_p)$, the first example is considered again. Using the elementary identity

$$(3.3) \qquad (\alpha;\, \mu+\nu) = (\alpha;\, \nu)(\alpha+\nu;\, \mu) = (\alpha;\, \mu)(\alpha+\mu;\, \nu)$$

we find that (3.1) implies

$$(3.4) \qquad U((\mu_p+m_p),(\nu_p+n_p))$$
$$= (\alpha;\, \textstyle\sum_{p=1}^P m_p)(\alpha+\sum_{p=1}^P m_p;\, \sum_{p=1}^P \mu_p)(\beta;\, \sum_{p=1}^P n_p)(\beta+\sum_{p=1}^P n_p;\, \sum_{p=1}^P \nu_p)\,.$$

It follows that the F-function now equals

$$(\alpha;\, \textstyle\sum_{p=1}^P m_p)F_A[\alpha+\sum_{p=1}^P m_p,(a_p);\,(c_p);\,(x_p)]\times$$
$$\times(\beta;\, \textstyle\sum_{p=1}^P n_p)F_A[\beta+\sum_{p=1}^P n_p,(b_p);\,(d_p);\,(y_p)]\,.$$

In the general case it can be proved, similarly, that the following rule holds:

Suppose that the general term of the sequence $(U((\mu_p),(\nu_p)))$ is a product of factors of the form

$$f = (\alpha;\, \textstyle\sum_{p=1}^P (j_p\mu_p+k_p\nu_p))^{\pm 1}\,,$$

where (j_p), (k_p) are prescribed sets whose elements are equal to 0 or 1. The function defined by (2.1) is then a hypergeometric function multiplied by a constant A; to each f corresponds a factor

$$(\alpha;\, \textstyle\sum_{p=1}^P (j_pm_p+k_pn_p))^{\pm 1}$$

in A and a numerator/denominator parameter

$$\alpha+\textstyle\sum_{p=1}^P (j_pm_p+k_pn_p)$$

associated with the variables corresponding to non-zero elements of (j_p) and (k_p).

4. The transformations.

In this section the transformations obtained for the F-functions introduced in section 2 are stated. Each transformation will hold for sufficiently small values of the variables if the U-sequence in question satisfies an inequality of the type considered in section 2 that implies the existence of the F-functions involved. One of the transformations is proved in section 5; the remaining ones are proved by similar methods. In addition we mention some known results (mainly recent ones) that are particular cases.

To Meixner's first transformation [10, equation (19a)] and to Erdélyi's transformation (1.1) corresponds

$$(4.1)\quad \sum_{(n_p)} F\left[{(-n_p),(-n_p) \atop ,}|\mathscr{U}[(0),(0)]|(x_p),(y_p)\right] \prod_{p=1}^{P} \frac{(h_p;\, n_p)t_p{}^{n_p}}{n_p!}$$
$$= \{\prod_{p=1}^{P}(1-t_p)^{-h_p}\} \times$$
$$\times \sum_{(n_p)} F\left[{(h_p+n_p),(h_p+n_p) \atop ,}|\mathscr{U}[(n_p),(n_p)]|\left(\frac{x_p t_p}{t_p-1}\right),\left(\frac{y_p t_p}{t_p-1}\right)\right] \times$$
$$\times \prod_{p=1}^{P}\left\{\frac{(h_p;\, n_p)}{n_p!}\left[\frac{x_p y_p t_p}{(1-t_p)^2}\right]^{n_p}\right\}.$$

This transformation contains as special cases recent results due to Carlitz [3, equation (1.6)], Sharma [14, equations (6), (17)], and to Srivastava [18, equation (2.5)].

We show how to obtain one of the special cases from (4.1). Take $P=1$, suppress unnecessary indices, and let

$$U(\mu,\nu) = (\alpha;\, \mu+\nu)/(\gamma;\, \mu+\nu)\,.$$

F then becomes an Appell function F_1 and the left-hand side of (4.1) reads

$$\sum_{n=0}^{\infty} F_1[\alpha;\, -n,\, -n;\, \gamma;\, x,y]\frac{(h;\, n)t^n}{n!}\,.$$

Next, by the rule in section 3 we find that the right-hand side of (4.1) now takes the form

$$(1-t)^{-h}\sum_{n=0}^{\infty} F_1\left[\alpha+2n;\, h+n, h+n;\, \gamma+2n;\, \frac{xt}{t-1},\frac{yt}{t-1}\right] \times$$
$$\times\, \frac{(h;\, n)(\alpha;\, 2n)(txy)^n}{n!(\gamma;\, 2n)(1-t)^{2n}};$$

we have thus (apart from renaming) found Sharma's result [14, equation (6)].

The second transformation is the confluent form of the first:

$$(4.2)\qquad \sum_{(n_p)} F\left[{(-n_p),(-n_p) \atop ,}|\mathscr{U}[(0),(0)]|(x_p),(y_p)\right]\prod_{p=1}^{P}\frac{t_p^{\,n_p}}{n_p!}$$

$$= \exp\,[\textstyle\sum_{p=1}^{P} t_p] \times$$

$$\times \sum_{(n_p)} F\left[{, \atop ,}|\mathscr{U}[(n_p),(n_p)]|(-x_pt_p),(-y_pt_p)\right]\prod_{p=1}^{P}\frac{(t_px_py_p)^{n_p}}{n_p!}.$$

It provides a generalization of transformations due to Erdélyi [5, equation (16)] and to Meixner [10, equation (32a)]; moreover, a result recently obtained by Carlitz [3, equation (1.8)] is a special case of (4.2).

The third transformation,

$$(4.3)\qquad \sum_{(n_p)} F\left[{(-n_p), \quad (-n_p) \atop ,\;(1-h_p-n_p)}|\mathscr{U}[(0),(0)]|(x_p),(y_p)\right]\prod_{p=1}^{P}\frac{(h_p;\,n_p)t_p^{\,n_p}}{n_p!}$$

$$= \{\textstyle\prod_{p=1}^{P}(1-t_p)^{-h_p}\}\times$$

$$\times \sum_{(n_p)} F\left[{(h_p), \atop ,}|\mathscr{U}[(n_p),(n_p)]|\left(\frac{x_pt_p}{t_p-1}\right),(y_pt_p)\right]\prod_{p=1}^{P}\frac{(-x_py_pt_p)^{n_p}}{n_p!},$$

generalizes a result due to Srivastava [19, equation (2.2)].

The generalization of Meixner's second transformation [10, equation (19b)] is

$$(4.4)\qquad \sum_{(n_p)} F\left[{(-n_p),(h_p+n_p) \atop ,}|\mathscr{U}[(0),(0)]|(x_p),(y_p)\right]\prod_{p=1}^{P}\frac{(h_p;\,n_p)t_p^{\,n_p}}{n_p!}$$

$$= \{\textstyle\prod_{p=1}^{P}(1-t_p)^{-h_p}\}\times$$

$$\times \sum_{(n_p)} F\left[{(h_p+n_p),(h_p+n_p) \atop ,}|\mathscr{U}[(n_p),(n_p)]|\left(\frac{x_pt_p}{t_p-1}\right),\left(\frac{y_p}{1-t_p}\right)\right]\times$$

$$\times \prod_{p=1}^{P}\left\{\frac{(h_p;\,n_p)}{n_p!}\left[-\frac{x_py_pt_p}{(1-t_p)^2}\right]^{n_p}\right\}.$$

With $P=3$, $t_2=t_3=0$, and suitable U-sequences, recent results due to Sharma [13, equations (19), (20)], and to Srivastava [16, equation (3.2)] could be obtained from (4.4).

Finally, we have the generalization of Meixner's third transformation [10, equation (19c)]:

$$(4.5)\quad \sum_{(n_p)} F\left[\begin{matrix}(h_p+n_p),(h_p+n_p)\\ ,\end{matrix}\middle|\mathscr{U}[(0),(0)]\middle|(x_p),(y_p)\right] \prod_{p=1}^{P} \frac{(h_p;n_p)t_p^{\,n_p}}{n_p!}$$

$$= \{\prod_{p=1}^{P} (1-t_p)^{-h_p}\}\times$$

$$\times \sum_{(n_p)} F\left[\begin{matrix}(h_p+n_p),(h_p+n_p)\\ ,\end{matrix}\middle|\mathscr{U}[(n_p),(n_p)]\middle|\left(\frac{x_p}{1-t_p}\right),\left(\frac{y_p}{1-t_p}\right)\right]\times$$

$$\times \prod_{p=1}^{P} \left\{\frac{(h_p;n_p)}{n_p!}\left[\frac{x_p y_p t_p}{(1-t_p)^2}\right]^{n_p}\right\},$$

together with the related result

$$(4.6)\quad \sum_{(n_p)} F\left[\begin{matrix}(h_p+n_p),(h_p+n_p)\\ ,\end{matrix}\middle|\mathscr{U}[(0),(0)]\middle|(x_p),(y_p)\right] \prod_{p=1}^{P} \frac{(h_p;n_p)t_p^{\,n_p}}{n_p!}$$

$$= \{\prod_{p=1}^{P} (1-t_p)^{-h_p}\}\times$$

$$\times \sum_{(n_p)} F\left[\begin{matrix}(h_p+n_p),(h_p+n_p)\\ ,\end{matrix}\middle|\mathscr{U}[(n_p),(0)]\middle|(x_p),\left(\frac{y_p}{1-t_p}\right)\right]\times$$

$$\times \prod_{p=1}^{P} \left\{\frac{(h_p;n_p)}{n_p!}\left[\frac{x_p t_p}{1-t_p}\right]^{n_p}\right\}.$$

The transformation (4.5) contains as particular cases ($P=2, t_2=0$) recent results due to Sharma [13, equations (17), (18)], and to Srivastava [16, equation (3.3)].

5. Proof of the first transformation.

To prove formally the transformation (4.1) denote its left-hand side by L and replace F by its power series. This yields

$$L = \sum_{(n_p),(\mu_p),(\nu_p)} U((\mu_p),(\nu_p)) \prod_{p=1}^{P} \frac{(h_p;n_p)t_p^{\,n_p}(-n_p;\mu_p)x_p^{\,\mu_p}(-n_p;\nu_p)y_p^{\,\nu_p}}{n_p!\,\mu_p!\,\nu_p!}.$$

Next, we apply the elementary identity

$$(5.1)\qquad (1+\alpha;\ -\mu)(-\alpha;\mu) = (-1)^{\mu}, \quad \mu \text{ integral},$$

and obtain, after changing the order of summation,

$$(5.2)\quad L = \sum_{(\mu_p),(\nu_p)} U((\mu_p),(\nu_p)) \prod_{p=1}^{P} \left\{\frac{(-x_p)^{\mu_p}(-y_p)^{\nu_p}}{\mu_p!\,\nu_p!}\Phi(\mu_p,\nu_p,h_p,t_p)\right\},$$

where we have defined

$$(5.3) \qquad \Phi(\mu,\nu,h,t) \equiv \sum_{n=0}^{\infty} \frac{n!(h;\, n)t^n}{(n-\mu)!\,(n-\nu)!}.$$

In order to transform this series, take $n-\mu$ as summation index and utilize (3.3). As a result,

$$\begin{aligned}(\mu-\nu)!\,\Phi(\mu,\nu,h,t) &= t^{\mu}(h;\,\mu)\mu!\;{}_2F_1[1+\mu,h+\mu;\;1+\mu-\nu;\;t]\\ &= t^{\mu}(h;\,\mu)\mu!\,(1-t)^{-h-\mu-\nu}{}_2F_1[-\nu,1-\nu-h;\;1+\mu-\nu;\;t]\,,\end{aligned}$$

by one of Euler's transformations. This hypergeometric polynomial is written in reverse order, and (3.3) and (5.1) are utilized again. We then find that

$$(5.4) \qquad \Phi(\mu,\nu,h,t)$$
$$= t^{\mu+\nu}(1-t)^{-h-\mu-\nu}\mu!\,\nu!(h;\,\mu)(h;\,\nu)\sum_{n=0}^{\infty}\big((\mu-n)!\,(\nu-n)!\,n!(h;\,n)t^n\big)^{-1}$$

We now insert this expression into (5.2), change the order of summation, and replace the inner summation indices μ_p,ν_p by μ_p+n_p, ν_p+n_p; the result is

$$(5.5) \quad L = \Big\{\prod_{p=1}^{P}(1-t_p)^{-h_p}\Big\}\sum_{(n_p),(\mu_p),(\nu_p)} U\big((\mu_p+n_p),(\nu_p+n_p)\big)\times$$
$$\times \prod_{p=1}^{P}\left\{\left[\frac{x_p y_p t_p}{(1-t_p)^2}\right]^{n_p}\left[\frac{x_p t_p}{t_p-1}\right]^{\mu_p}\left[\frac{y_p t_p}{t_p-1}\right]^{\nu_p}\frac{(h_p;\,\mu_p+n_p)(h_p;\,\nu_p+n_p)}{(h_p;\,n_p)\mu_p!\,\nu_p!\,n_p!}\right\}.$$

The right-hand side of (5.5) is readily transformed to that of (4.1); the formal proof is thus complete.

Next, we utilize the inequality (2.5), which implies the existence of the F-functions in (4.1), together with (2.4), to estimate the general coefficient B in the power series S on the right-hand side of (5.5) in the following way:

$$B \leqq C_{\mathrm{V}}{}^{\Sigma_p(\mu_p+\nu_p+2n_p)}\prod_{p=1}^{P}\frac{(\mu_p+n_p)^{k_p}(\mu_p+n_p)!\,(\nu_p+n_p)^{k_p}(\nu_p+n_p)!}{n_p{}^{k_p}n_p!\,\mu_p!\,\nu_p!\,n_p!},$$

where k_p denotes the real part of h_p-1. Next, by (2.3) we find that

$$B \leqq C_{\mathrm{VI}}{}^{\Sigma_p(\mu_p+\nu_p+n_p)}.$$

Hence, by Abel's lemma, S is absolutely convergent for sufficiently small values of its variables. It is readily seen that, for sufficiently small values of the original variables, substitution in S leads to an absolutely convergent power series which upon multiplication by P binomial series yields another absolutely convergent power series $T((t_p),(x_p),(y_p))$, say. Since the two sides of (4.1) are derangements of T and S, the proof is now complete.

6. Generating functions (i).

The transformations in section 4 assume a particularly simple form when (x_p) or (y_p) equals the zero set: only one term in the multiple series of the right-hand side in question is non-zero and a generating function of P variables is obtained. To state these results conveniently, the notation will be simplified by writing $\mathscr{U}$ in place of $\mathscr{U}[(0),(0)]$ and by omission of variables equal to zero and of parameters associated with these variables. It should be noted that, in the equations of this section. the two members contain the same set $\mathscr{U}$, i.e., the same additional parameters and variables.

The first generating function relation, obtained by taking $(y_p)=(0)$ in (4.1), (4.3), or in (4.4), is

$$\sum_{(n_p)} F\left[{}^{(-n_p)}|\mathscr{U}|(x_p)\right] \prod_{p=1}^{P} \frac{(h_p;n_p)t_p^{\,n_p}}{n_p!} = \{\prod_{p=1}^{P} (1-t_p)^{-h_p}\} F\left[{}^{(h_p)}|\mathscr{U}|\left(\frac{x_pt_p}{t_p-1}\right)\right]. \tag{6.1}$$

A recent result due to Manocha [7, equation (5)] is a particular case of (6.1).

Next, with $(y_p)=(0)$ in (4.2) we find the relation

$$\sum_{(n_p)} F\left[{}^{(-n_p)}|\mathscr{U}|(x_p)\right] \prod_{p=1}^{P} \frac{t_p^{\,n_p}}{n_p!} = \exp\,[\sum_{p=1}^{P} t_p]\, F[|\mathscr{U}|(-x_pt_p)]\,, \tag{6.2}$$

which contains formulae given by Munot [11, equation (4.1)], and by Jain & Sharma [6, equation (4)] as special cases.

When $(x_p)=(0)$ in (4.3) the relation

$$\sum_{(n_p)} F\left[{}^{(-n_p)}_{(1-h_p-n_p)}|\mathscr{U}|(y_p)\right] \prod_{p=1}^{P} \frac{(h_p;n_p)t_p^{\,n_p}}{n_p!} = \{\prod_{p=1}^{P} (1-t_p)^{-h_p}\}F[|\mathscr{U}|(y_pt_p)] \tag{6.3}$$

is obtained; finally, $(x_p)=(0)$ in (4.4), (4.5), or in (4.6) yields

$$(6.4)\qquad \sum_{(n_p)} F\left[{}^{(h_p+n_p)}|\mathscr{U}|(y_p)\right] \prod_{p=1}^{P} \frac{(h_p;\, n_p)t_p^{\,n_p}}{n_p!}$$

$$= \{\prod_{p=1}^{P} (1-t_p)^{-h_p}\} F\left[{}^{(h_p)}|\mathscr{U}|\left(\frac{y_p}{1-t_p}\right)\right].$$

Some results due to Anandani [1, equations (2.1), (2.3), (2.4)] are special cases of (6.4).

7. Generating functions (ii).

The multiple sums in section 4 are expressible in terms of known functions for special U-sequences; in such cases the transformations reduce to generating function relations. A non-trivial result of this kind, involving two Lauricella functions, is

$$(7.1)\quad \sum_{(n_p)} F_A[a,(-n_p);\,(c_p);\,(x_p)]F_A[b,(-n_p);\,(c_p);(y_p)] \prod_{p=1}^{P} \frac{(c_p;\, n_p)t_p^{\,n_p}}{n_p!}$$

$$= [\prod_{p=1}^{P} (1-t_p)^{-c_p}]\left[1-\sum_{p=1}^{P} \frac{x_p t_p}{t_p-1}\right]^{-a}\left[1-\sum_{p=1}^{P} \frac{y_p t_p}{t_p-1}\right]^{-b} \times$$

$$\times F_C[a,b;\,(c_p);\,(X_p)]\,,$$

$$X_p \equiv x_p y_p t_p (1-t_p)^{-2}\left[1-\sum_{r=1}^{P} \frac{x_r t_r}{t_r-1}\right]^{-1}\left[1-\sum_{r=1}^{P} \frac{y_r t_r}{t_r-1}\right]^{-1}.$$

When $P=1$, the Lauricella functions F_A and F_C both reduce to Gaussian hypergeometric functions and (7.1) specializes to a result due to Meixner [10, equation (34)], also known as Weisner's formula.

Equation (7.1) follows from (4.1) by taking

$$U((\mu_p),(\nu_p)) = (a;\, \sum_{p=1}^{P} \mu_p)(b;\, \sum_{p=1}^{P} \nu_p)/\prod_{p=1}^{P} \{(c_p;\, \mu_p)(c_p;\, \nu_p)\}\,.$$

The F-functions in (4.1) then become F_A's multiplied by constants (cf. the rule mentioned in section 3) and the left-hand side of (7.1) is readily obtained from that of (4.1). The right-hand side of (4.1), with $n \equiv n_1 + \ldots + n_P$, becomes

$$\{\prod_{p=1}^{P} (1-t_p)^{-c_p}\} \sum_{(n_p)} (a;\, n)F_A\left[a+n,(c_p+n_p);\,(c_p+n_p);\,\left(\frac{x_p t_p}{t_p-1}\right)\right] \times$$

$$\times (b;\, n)F_A\left[b+n,(c_p+n_p);\,(c_p+n_p);\,\left(\frac{y_p t_p}{t_p-1}\right)\right] \times$$

$$\times \prod_{p=1}^{P}\left\{\frac{1}{(c_p;\, n_p)n_p!}\left[\frac{x_p y_p t_p}{(1-t_p)^2}\right]^{n_p}\right\}.$$

This expression simplifies since

$$F_A[\alpha,(\beta_p);(\beta_p);(\xi_p)] = [1-\sum_{p=1}^{P}\xi_p]^{-\alpha},$$

and the right-hand side of (7.1) is readily obtained.

Results similar to (7.1) follow from (4.4) and (4.5); they can also be obtained from (7.1) by linear transformations of F_A.

8. Generating functions (iii).

In this section we briefly mention two further generating functions related to the transformations in section 4. Both belong to a class of functions involving hypergeometric parameters associated with two variables and defined by

$$(8.1)\qquad G\left[\begin{matrix}(a_p)\\(c_p)\end{matrix}\Big|\mathscr{U}[(m_p),(n_p)]\Big|(x_p),(y_p)\right]$$
$$= \sum_{(\mu_p),(\nu_p)} U((\mu_p+m_p),(\nu_p+n_p))\prod_{p=1}^{P}\frac{(a_p;\mu_p+\nu_p)x_p^{\mu_p}y_p^{\nu_p}}{(c_p;\mu_p+\nu_p)\mu_p!\,\nu_p!}.$$

The introduction of this class is suggested by the proof of the transformation (4.4). In fact, the two sides of (4.4) are expressible in terms of the G-function:

$$(8.2)\quad \sum_{(n_p)} F\left[\begin{matrix}(-n_p),(h_p+n_p)\\ ,\end{matrix}\Big|\mathscr{U}[(0),(0)]\Big|(x_p),(y_p)\right]\prod_{p=1}^{P}\frac{(h_p;n_p)t_p^{n_p}}{n_p!}$$
$$= \{\prod_{p=1}^{P}(1-t_p)^{-h_p}\}\,G\left[\begin{matrix}(h_p)\\ \end{matrix}\Big|\mathscr{U}[(0),(0)]\Big|\left(\frac{x_pt_p}{t_p-1}\right),\left(\frac{y_p}{1-t_p}\right)\right],$$

and

$$(8.3)\qquad \sum_{(n_p)} F\left[\begin{matrix}(h_p+n_p),(h_p+n_p)\\ ,\end{matrix}\Big|\mathscr{U}[(n_p),(n_p)]\Big|(u_p),(v_p)\right]\times$$
$$\times\prod_{p=1}^{P}\frac{(h_p;n_p)(u_pv_p)^{n_p}}{n_p!}$$
$$= G\left[\begin{matrix}(h_p)\\ \end{matrix}\Big|\mathscr{U}[(0),(0)]\Big|(u_p),(v_p)\right].$$

Some generating function relations recently obtained are particular cases of these results. Equation (8.2) contains results due to Manocha [7, equation (9)], Manocha & Sharma [8, equation (35); 9, equations (2.2), (2.3)], Srivastava [18, equation (2.2); 20, equation (6)], and to Srivastava & Singhal [21, equations (6.7), (6.9)]. Equation (8.3) contains a formula given by Srivastava [17, equation (11)] as a special case.

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