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QUASI-DIRECT DECOMPOSITIONS OF TORSION-FREE ABELIAN GROUPS OF INFINITE RANK

L. FUCHS and G. VILJOEN

In memoriam Aleksandr Gennadievič Kuroš.

All groups under consideration are torsion-free abelian groups, written additively. For unexplained terminology and notation, the reader is referred to [4].

Jónsson [6] was the first to observe that the direct decompositions of torsion-free groups of finite rank into indecomposable summands need not be unique up to isomorphism. In another paper [7], however, he pointed out that if isomorphism is replaced by a weaker equivalence relation, called quasi-isomorphism, then uniqueness can be restored by establishing a Krull–Schmidt type theorem.

For groups of infinite rank, a new phenomenon arises. It was shown by Corner [2] that there exists a group A of countable rank such that $A = B \oplus C = \oplus_{n=-\infty}^{\infty} E_n$ where B, C are indecomposable of rank $\aleph_0$ and E_n are indecomposable of rank 2. Consequently, summands of direct sums of finite rank groups are not necessarily direct sums of finite rank groups. They need not even be quasi-isomorphic to such a direct sum, but Corner's example indicates that they are actually not far from being so. We wish to show, as a matter of fact, that by replacing quasi-isomorphism by a somewhat weaker relation, summands of arbitrary direct sums of finite rank groups will belong to an equivalence class containing direct sums of finite rank groups (Theorem 4). This result can also be interpreted as an analog of the celebrated Baer-Kulikov-Kaplansky theorem that direct summands of direct sums of rank 1 groups are again direct sums of rank 1 groups.

Some theorems on completely decomposable and separable groups can be generalized in our new setting (Theorems 1–3).

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1. Quasi-direct decompositions.

Let G and H be subgroups of a torsion-free divisible group D. Then G is said to be *quasi-contained* in H, in notation: $G \prec H$, if $nG \leqq H$ for some integer $n > 0$, and *quasi-equal* to H, $G \approx H$, if both $G \prec H$ and $H \prec G$ hold. A group K is *quasi-isomorphic* to H if $K \cong G$ with $G \approx H$; in this case, we write $K \sim H$.

Let G and H be as before. We shall say: G is *locally quasi-contained* in H, $G \prec^{\cdot} H$, if for every finite rank direct summand E of D we have

$$G \cap E \prec H \cap E . \tag{1}$$

The definitions of *local quasi-equality* $G \approx^{\cdot} H$ and *local quasi-isomorphism* $K \sim^{\cdot} H$ are analogous.

It is obvious that for groups of finite rank, the local and the global properties coincide. It is straightforward to check that the relation $\prec^{\cdot}$ is reflexive and transitive, while both $\approx^{\cdot}$ and $\sim^{\cdot}$ are equivalence relations. The following observations are elementary.

(a) To check (1) for some E, it suffices to show that $G \cap E' \prec H \cap E'$ for some summand E' of D containing E.

(b) If D' is a summand of D, then $G \prec^{\cdot} H$ [$G \approx^{\cdot} H$] implies $G \cap D' \prec^{\cdot} H \cap D'$ [$G \cap D' \approx^{\cdot} H \cap D'$].

(c) If $G_i \approx^{\cdot} H_i$ for $i \in I$, then

$$\oplus_{i \in I} G_i \approx^{\cdot} \oplus_{i \in I} H_i ;$$

and similarly, for $\sim^{\cdot}$ rather than $\approx^{\cdot}$. This fails to hold for infinite index sets I if $\approx^{\cdot}$ is replaced by $\approx$.

If A and B are quasi-contained in G such that $G \approx A \oplus B$, then we call $G \approx A \oplus B$ a *quasi-direct decomposition* of G, and A, B *quasi-direct summands* of G. The definition of *local quasi-direct decomposition* is analogous. Similar definitions apply to decompositions with infinitely many components. There is a third kind of direct decomposition we shall need: this is stronger than local quasi-direct decomposition, but weaker than quasi-direct decomposition.

Let A_i ($i \in I$) be subgroups of the divisible hull D of G. We write

$$G \approx^{\cdot} \oplus^{\wedge}{}_{i \in I} A_i$$

and call this an *admissible* [local quasi-direct] decomposition of G, if

(i) $G \approx^{\cdot} \oplus_{i \in I} A_i$,

(ii) for the ith coordinate-projection σ_i of D onto the divisible hull[1] $D(A_i)$ of A_i, we have $\sigma_i G \approx A_i$, for all $i \in I$.

[1] In general, $D(A)$ will denote the divisible hull of the group A.

Manifestly, for a finite index set I, an admissible decomposition of G is nothing else than a quasi-direct decomposition. But for infinite index sets I, we obtain something new, as is shown by the following examples.

EXAMPLE 1. Let $a_1,\ldots,a_n,\ldots$ be a basis of a vector space D over the rationals Q, and for a prime p, define

$$G = \langle a_1,\ldots,a_n,\ldots,p^{-n}(a_1+a_{n+1}) \text{ for all } n \geqq 1\rangle .$$

Then $G \approx^{\cdot} \oplus_{n=1}^{\infty} \langle a_n\rangle$, but this is not admissible, since the projection σ_1 to the divisible hull of $\langle a_1\rangle$ does not satisfy $\sigma_1 G \approx \langle a_1\rangle$.

EXAMPLE 2. Using the same notations, we now define

$$H = \langle a_1,\ldots,a_n,\ldots,p^{-n}(a_n+a_{n+1}) \text{ for all } n \geqq 1\rangle .$$

Then $H \approx^{\cdot} \oplus^{\wedge}{}_n \langle a_n\rangle$, but $H \approx \oplus_n \langle a_n\rangle$ fails to hold.

A group G is said to be *strongly indecomposable* if it has no non-trivial admissible decompositions.

Recall that an endomorphism η of the divisible hull D of G is said to be a *quasi-endomorphism* of G if $\eta G \prec G$. These η form a ring, called the *quasi-endomorphism ring* of G. [One should note, however, that the local quasi-endomorphisms fail to form a ring, in general!] Condition (ii) evidently guarantees that all the projections σ_i are quasi-endomorphisms of G.

By a result of Reid [9], the quasi-endomorphism ring of a group G of finite rank is local exactly if G is strongly indecomposable.

2. Elementary properties of admissible decompositions.

We are going to list a few elementary properties of admissible decompositions which will be required in the subsequent discussions.

(A) *If* $G \approx^{\cdot} \oplus^{\wedge} A_i$ *and if* $A_i \approx B_i$ *for every* i, *then* $G \approx^{\cdot} \oplus^{\wedge} B_i$, *too.*

(B) *If* $G \approx^{\cdot} \oplus^{\wedge}{}_{i\in I} A_i$ *and if* $A_i \approx^{\cdot} \oplus^{\wedge}{}_{j\in J} C_{ij}$, *then* $G \approx^{\cdot} \oplus^{\wedge}{}_{i,j} C_{ij}$ *too.* (*The converse implication fails unless* J *is finite.*)

(C) *If* $G \approx^{\cdot} \oplus^{\wedge} A_i$, *then* $G \cap D(A_i) \approx^{\cdot} A_i$.

(D) *If* $G \approx B \oplus C$ *and if* $B \prec X \prec G$, *then* $X \approx B \oplus (C \cap X)$.

The proofs of (A)–(D) are straightforward from the definitions, and therefore may be left to the reader.

(E) *If* $G \approx^{\cdot} \oplus^{\wedge}{}_{i \in I} A_i$ *and* $G \approx^{\cdot} \oplus^{\wedge}{}_{j \in J} C_j$, *and if* A_{i_1} *is contained in* $D(C_{j_1} \oplus \ldots \oplus C_{j_m})$ *for some* m, *then* A_{i_1} *is a quasi-direct summand of* $C_{j_1} \oplus \ldots \oplus C_{j_m}$.

Let $\varrho_i : D(G) \to D(A_i)$, $\sigma_j : D(G) \to D(C_j)$ be the obvious projections. Then

$$C_{j_1} \oplus \ldots \oplus C_{j_m} \approx (\sigma_{j_1} + \ldots + \sigma_{j_m})G$$

implies

$$\varrho_{i_1}(C_{j_1} \oplus \ldots \oplus C_{j_m}) \approx \varrho_{i_1}(\sigma_{j_1} + \ldots + \sigma_{j_m})G \approx A_{i_1}$$

whence the assertion is immediate.

(F) *If* $G \approx^{\cdot} \oplus^{\wedge}{}_{i \in I} A_i$ *and if* $I = I_1 \cup I_2$ *is a partition of* I *into two disjoint subsets, then*

$$\varphi G \approx^{\cdot} \oplus^{\wedge}{}_{i \in I_2} \varphi A_i$$

where φ *denotes the canonical map* $D(G) \to D(G)/D_1$ *with* $D_1 = D(\oplus_{i \in I_1} A_i)$.

To show that φG is locally quasi-equal to $\oplus \varphi A_i$, it suffices to show that $\varphi G \cap \varphi D(A_i) \approx \varphi A_i$ for all $i \in I_2$. This is an immediate consequence of

$$\varphi A_i \approx \varphi(D(A_i) \cap G) \leqq \varphi(D(A_i)) \cap \varphi G \leqq \varphi \sigma_i G \approx \varphi A_i$$

where σ_i stands for the ith projection $D(G) \to D(A_i)$. That the decomposition is admissible can be verified by using the obvious equality $\bar{\sigma}_i \varphi = \varphi \sigma_i$ where $\bar{\sigma}_i$ denotes the ith projection $\varphi D(G) \to \varphi D(A_i)$. We have, in fact, $\bar{\sigma}_i \varphi G = \varphi \sigma_i G \approx \varphi A_i$.

(G) (The exchange property.) *If* $G \approx X \oplus Y$ *with* X *of finite rank and if* $G \approx B_1 \oplus \ldots \oplus B_n$, *then there are quasi-direct decompositions* $B_i \approx B_i' \oplus B_i''$ $(i = 1, \ldots, n)$ *such that*

$$G \approx X \oplus B_1' \oplus \ldots \oplus B_n' \,. \tag{2}$$

If X is strongly indecomposable, then its quasi-endomorphism ring is local, and it follows that X has the exchange property (see e.g. Reid [9], [10]). Since every group of finite rank is a quasi-direct sum of a finite number of strongly indecomposable groups, the assertion follows by a straightforward induction (see Crawley and Jónsson [3]).

Note that $X \sim B_1'' \oplus \ldots \oplus B_n''$, so that if X is strongly indecomposable, then $X \sim B_i''$ for some i, and for all other indices j, $B_j' = B_j$ may be assumed in (2).

3. Completely quasi-decomposable groups.

Now we turn our attention to the analog of completely decomposable groups.

A torsion-free group G will be called *completely quasi-decomposable* if

$$G \approx^{\cdot} \oplus^{\wedge}{}_{i\in I} A_i \tag{3}$$

where each A_i is of finite rank. Since every group of finite rank is quasi-equal to a direct sum of strongly indecomposable groups, we may replace the A_i by these direct sums to obtain an admissible decomposition (3) with all A_i strongly indecomposable.

Analogously to completely decomposable groups, a uniqueness theorem can be established, though the proof is now more laborious.

THEOREM 1. *Let*

$$G \approx^{\cdot} \oplus^{\wedge}{}_{i\in I} A_i \quad \textit{and} \quad G \approx^{\cdot} \oplus^{\wedge}{}_{j\in J} C_j$$

be two admissible decompositions of G where all the groups A_i, C_j are strongly indecomposable of finite rank. Then there exists a bijection f between the index sets I and J such that

$$A_i \sim C_{f(i)} \quad \textit{for every } i \in I .$$

If I is finite, so is J, and the assertion reduces to Jónsson's theorem [7] on finite rank groups.

Next, let I be countable; then so is J, and there is no loss of generality in assuming $I=\{1,2,\ldots,n,\ldots\}=J$. Now, A_1 is quasi-contained in a direct sum $C_1\oplus\ldots\oplus C_m$ for some integer m. In view of (E), A_1 is a quasi-direct summand of $C_1\oplus\ldots\oplus C_m$. By the exchange property (G), one of these C_j, say C_1, is quasi-isomorphic to A_1 and can be replaced by A_1, that is,

$$C_1\oplus\ldots\oplus C_m \approx A_1\oplus C_2\oplus\ldots\oplus C_m .$$

We obtain

$$G \approx^{\cdot} A_1 \oplus^{\wedge} \oplus^{\wedge}{}_{n\geq 2} C_n .$$

If φ_1 denotes the natural projection $D(G) \to D(G)/D(A_1)$, then (F) implies

$$\varphi_1 G \approx^{\cdot} \oplus^{\wedge}{}_{n\geq 2} \varphi_1 A_n \quad \text{and} \quad \varphi_1 G \approx^{\cdot} \oplus^{\wedge}{}_{n\geq 2} \varphi_1 C_n .$$

We can now repeat the argument with $\varphi_1 C_2$ in the role of A_1, in order to obtain, say $\varphi_1 C_2 \sim \varphi_1 A_2$ [and hence $C_2 \sim A_2$] and

$$\varphi_2 G \approx^{\cdot} \oplus^{\wedge}{}_{n\geq 3} \varphi_2 A_n, \quad \varphi_2 G \approx^{\cdot} \oplus^{\wedge}{}_{n\geq 3} \varphi_2 C_n$$

where $\varphi_2\colon D(G) \to D(G)/D(A_1\oplus C_2)$ is the natural map. The existence of a correspondence f is now clear in the countable case.

If the index sets I and J are uncountable, then — using a famous Kaplansky argument — one can find well-ordered ascending chains of subsets,

$$\emptyset = I_0 \subset \ldots \subset I_\sigma \subset \ldots \subset I_\tau = I$$

and

$$\emptyset = J_0 \subset \ldots \subset J_\sigma \subset \ldots \subset J_\tau = J$$

for some ordinal τ, satisfying the following conditions:

1. $I_{\sigma+1} \setminus I_\sigma$ and $J_{\sigma+1} \setminus J_\sigma$ are countable, for every σ;
2. $I_\sigma = \bigcup_{\varrho<\sigma} I_\varrho$ and $J_\sigma = \bigcup_{\varrho<\sigma} J_\varrho$ for limit ordinals σ;
3. for every σ, the divisible hulls of $\oplus_{i\in I_\sigma} A_i$ and $\oplus_{j\in J_\sigma} C_j$ are the same; let D_σ denote this common divisible hull.

If we denote by φ_σ the canonical map $D \to D/D_\sigma$ (where $D = D(G)$), then (F) guarantees that

$$\varphi_\sigma G \approx^{\cdot} \oplus\hat{}_{i\in I\setminus I_\sigma} \varphi_\sigma A_i \quad \text{and} \quad \varphi_\sigma G \approx^{\cdot} \oplus\hat{}_{j\in I\setminus I_\sigma} \varphi_\sigma C_j \,.$$

Intersecting with $\varphi_\sigma D_{\sigma+1}$, we have the admissible decompositions

$$\oplus\hat{}_{i\in I_{\sigma+1}\setminus I_\sigma} \varphi_\sigma A_i \quad \text{and} \quad \oplus\hat{}_{j\in J_{\sigma+1}\setminus J_\sigma} \varphi_\sigma C_j$$

of the group $\varphi_\sigma G \cap \varphi_\sigma D_{\sigma+1}$. This reduces the general case to countably many summands settled above.

4. Quasi-separable groups.

A group G will be called *quasi-separable* if every finite subset $\{g_1, \ldots, g_n\}$ of G can be embedded in (the divisible hull of) a suitable quasi-direct summand A of G such that A is of finite rank; that is, $G \approx A \oplus B$ with $g_1, \ldots, g_n \in D(A)$ of finite rank and $B \prec G$. Naturally, A can be replaced by a direct sum $A_1 \oplus \ldots \oplus A_m$ of strongly indecomposable groups A_i. The property of being quasi-separable is visibly invariant under quasi-isomorphisms.

It is readily checked that the quasi-direct summand B has to be quasi-separable, too.

Completely quasi-decomposable groups are evidently quasi-separable. We turn out attention to a partial converse which corresponds to Baer's theorem that countable separable groups are completely decomposable [1].

THEOREM 2. *A countable quasi-separable group G is completely quasi-decomposable.*

Let $\{g_1, \ldots, g_n, \ldots\}$ be a generating system for G. There is a finite rank group A_1 such that $g_1 \in D(A_1)$ and $G \approx A_1 \oplus B_1$. In view of the quasi-separability of B_1, there is an A_2 of finite rank such that $B_1 \approx A_2 \oplus B_2$ for some $B_2 \prec B_1$ and $D(A_2)$ contains the $D(B_1)$-coordinate of g_2. Then $G \approx A_1 \oplus A_2 \oplus B_2$, and $g_1, g_2 \in D(A_1 \oplus A_2)$. Thus continuing, we obtain

$$G \approx^{\cdot} \oplus_{n=1}^{\infty} A_n .$$

This must be an admissible decomposition, for the projection $\sigma_n : D(G) \to D(A_n)$ does not change when B_m are split ($m \geqq n$).

It is easy to establish the following theorem (the corresponding result on separability seems to be considerably deeper, cf. Fuchs [5]).

THEOREM 3. *If G is quasi-separable and if $G \approx B \oplus C$, then both B and C are quasi-separable.*

Let $b_1, \ldots, b_n \in B$. There exists a finite rank quasi-direct summand X of G, $G \approx X \oplus Y$, such that $b_1, \ldots, b_n \in X$. In view of (G), there are quasi-direct summands B', C' of B, C, respectively, such that $G \approx X \oplus B' \oplus C'$. From (D) we infer that

$$B \approx B' \oplus [(X \oplus C') \cap B] .$$

Here the group in the square brackets contains $b_1, \ldots, b_n$, and — as is easily verified — is of finite rank.

Our last result is an analog of the important Baer–Kulikov–Kaplansky theorem [8]:

THEOREM 4. *A quasi-direct summand of a direct sum of groups of finite rank is completely quasi-decomposable.*

Let $G = \oplus_{i \in I} A_i \approx B \oplus C$ where A_i are of finite rank. Using again a standard argument, one can find a well-ordered ascending chain of subsets,

$$\emptyset = I_0 \subset \ldots \subset I_\sigma \subset \ldots \subset I_\tau = I$$

for some ordinal τ such that

(α) $I_{\sigma+1} \setminus I_\sigma$ is countable for every σ,

(β) $I_\sigma = \bigcup_{\varrho < \sigma} I_\varrho$ for limit ordinals σ,

(γ) $G_\sigma = [G_\sigma \cap D(B)] \oplus [G_\sigma \cap D(C)]$ for all σ, where $G_\sigma = \oplus_{i \in I_\sigma} A_i$.

We obtain

$$G_{\sigma+1} \cap D(B) = [G_\sigma \cap D(B)] \oplus B_\sigma$$

for some B_σ. Since B_σ is countable and since $G_{\sigma+1}$ can be viewed as a quasi-separable group, Theorems 2 and 3 imply that $B_\sigma \approx \cdot \oplus^\wedge{}_n B_{\sigma n}$ with all $B_{\sigma n}$ of finite rank. Now $B \approx \cdot \oplus_{\sigma, n} B_{\sigma n}$ is obvious. To see that this is admissible, notice that $B_{\sigma n} \prec B$ is clear, thus the projection $\pi: D(B) \to D(B_{\sigma n})$ satisfies $B_{\sigma n} \prec \pi B$. Since π can be factored as

$$D(B) \to D(G_{\sigma+1}) \to D(B_\sigma) \to D(B_{\sigma n})$$

with the obvious projections throughout, we have $\pi B \prec B_{\sigma n}$, too. Hence $B \approx \cdot \oplus^\wedge B_{\sigma n}$, in fact.

BIBLIOGRAPHY

1. R. Baer, *Abelian groups without elements of finite order*, Duke Math. J. 3 (1937), 68–122.
2. A. L. S. Corner, *A note on rank and direct decompositions of torsion-free abelian groups* II, Cambridge Philos. Soc. 66 (1969), 239–240.
3. P. Crawley and B. Jónsson, *Refinements for infinite direct decompositions of algebraic systems*, Pacific J. Math. 14 (1964), 797–855.
4. L. Fuchs, *Infinite Abelian Groups*, vol. I, Academic Press, New York and London, 1970.
5. L. Fuchs, *Summands of separable abelian groups*, Bull. London Math. Soc. 2 (1970), 205–208.
6. B. Jónsson, *On direct decompositions of torsion-free abelian groups*, Math. Scand. 5 (1957), 230–235.
7. B. Jónsson, *On direct decomposition of torsion free abelian groups*, Math. Scand. 7 (1959), 361–371.
8. I. Kaplansky, *Projective modules*, Ann. of Math. 68 (1958), 372–377.
9. J. D. Reid, *On quasi-decompositions of torsion-free abelian groups*, Proc. Amer. Math. Soc. 13 (1962), 550–554.
10. J. D. Reid, *On the ring of quasi-endomorphisms of a torsion-free group*, *Topics in Abelian Groups*, pp. 57–68. Edited by J. M. Irwin and E. A. Walker, Scott, Foresman and Co., Chicago, 1963.

TULANE UNIVERSITY, NEW ORLEANS, LA. U.S.A.

AND

UNIVERSITY OF SOUTH AFRICA, PRETORIA, REPUBLIC OF SOUTH AFRICA