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ON THE SPECTRAL CATEGORY OF SOME RINGS

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Introduction.

In [1] Gabriel and Oberst showed that if $\boldsymbol{C}$ is an arbitrary Grothendieck-category then one can in a canonical way construct another Grothendieck-category $\operatorname{Spec}\boldsymbol{C}$ and a functor $P:\boldsymbol{C} \to \operatorname{Spec}\boldsymbol{C}$ which makes essential monomorphisms invertible. More explicitely: let $\operatorname{Spec}\boldsymbol{C}$ have the same objects as $\boldsymbol{C}$ and define

$$\operatorname{Hom}_{\operatorname{Spec}\boldsymbol{C}}(C,D) = \varinjlim \operatorname{Hom}_{\boldsymbol{C}}(C',D)$$

where $C' \to C$ is an essential monomorphism. P is the identity on objects and takes $f:C \to D$ to its image in $\varinjlim \operatorname{Hom}_{\boldsymbol{C}}(C',D)$. $\operatorname{Spec}C$ is indeed a Grothendieck category in which every morphism splits (so that all objects are injective and projective). They also showed that if $\boldsymbol{C}$ is a spectral category (that is, a Grothendieck category in which every morphism splits), then $\boldsymbol{C}$ can be represented as the full sub-category of $\operatorname{Mod}R.$ of direct summands of powers of R where R is a von Neumann-regular right self-injective ring, and conversely if R is such a ring then the full sub-category of $\operatorname{Mod}R.$ of direct summands of powers of R is a spectral category. It follows that the ring of endomorphisms of an arbitrary object in a spectral category is regular and right self-injective. It is also easy to show that if I is injective in $\boldsymbol{C}$ then $\operatorname{End}_{\boldsymbol{C}}I \to \operatorname{End}_{\operatorname{Spec}\boldsymbol{C}}I$ is surjective and its kernel is the Jacobson radical of $\operatorname{End}_{\boldsymbol{C}}I$.

In [8] Roos continued the study of spectral categories and introduced certain conditions on the lattice of sub-objects of an object to get a decomposition of the spectral category in a direct product of three different types of spectral categories. Call an object C in a spectral category $\boldsymbol{C}$ distributive if its lattice of sub-objects is distributive. Roos showed that this is equivalent to $\operatorname{End}_{\boldsymbol{C}}C$ being strongly regular ([8]). We thus get a decomposition

$$\boldsymbol{C} \simeq \boldsymbol{C}_{\mathrm{distr}} \times \boldsymbol{C}_{\mathrm{adistr}}$$

where the objects of $\boldsymbol{C}_{\mathrm{distr}}$ are direct sums of distributive objects and the objects of $\boldsymbol{C}_{\mathrm{adistr}}$ contain no non-zero distributive sub-objects. $\boldsymbol{C}_{\mathrm{adistr}}$

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can be further decomposed into types II and III where objects of type II are direct sums of AB5*-objects (see the definition in section 1.2 below) and objects of type III contain no non-zero AB5*-objects. Roos also gave examples of rings A such that SpecMod$A$ has a non-trivial anti-distributive part (its distributive part is always non-zero), but the problem if a commutative ring A with such a property could be found remained open and in fact inspired much of this work.

Summary.

In section 1 I study the ring of endomorphisms of a vector space of countably infinite dimension and show that its spectral category contains a non-zero type III spectral category. The question whether there exists a non-trivial type II part remains open but the problem is reduced to studying A/I where I is a left ideal which cannot be generated by a countable family.

The remaining part of the paper is devoted to the study of spectral categories of commutative rings and section 2 gives some general criteria for such a category to be of type I exclusively, and as an easy consequence we get a sufficient condition for the existence of a non-zero non-distributive part, which is used in section 3 in the study of the polynomial ring in countably many variables over a field. It is probable that the example given there is of type II, but this is far from proved. In section 4 I show that the spectral category of the ring of integer-valued functions on a compact topological space is of type I and in section 5 I show some results which might indicate the same result for the ring of continuous functions on the closed unit-interval.

On the whole the results are sadly incomplete and only give an idea of the difficulties which arise when trying to study the spectral category of even the most common rings. Finally I wish to thank Jan-Erik Roos for his interest in these problems, his suggestions and encouragement.

1.

1.1. Let A be equal to $\mathrm{Hom}_k(V, V)$ where V is a vector space over k of countably infinite dimension, let I be the two-sided ideal in A consisting of endomorphisms of finite rank (that is $\dim_k \mathrm{Im}(f)$ is finite) and let $B = A/I$. Finally let P be the natural functor $\mathrm{Mod}A \to \mathrm{SpecMod}A$ (see [1]).

PROPOSITION 1. *$P(B)$ has no non-zero distributive sub-objects* (see [8, p. 160]).

PROOF. According to the proposition on page 161 in [8] it suffices to prove that for every non-zero sub-object C of $P(B)$ there exist sub-objects C' and C'' of C such that $C' \cap C'' = 0$ and

$$\mathrm{Hom}_{\mathrm{SpecMod}A}(C', C'') \neq 0 .$$

By definition $\mathrm{Hom}_{\mathrm{SpecMod}A}(C, P(B)) = \varinjlim \mathrm{Hom}_A(D, B)$ where D is essential in C (now regarded as an object in $\mathrm{Mod}A$). A monomorphism $i: C \to P(B)$ can therefore be represented by a homomorphism $j: D \to B$ which is necessarily a monomorphism since P is left-exact (see [1]). If we now can find non-zero sub-modules D' and D'' of D such that $D' \cap D'' = 0$ and a homomorphism $f: D' \to D''$ with non-essential kernel, then $P(D')$ and $P(D'')$ are non-zero sub-objects of $P(D) = C$ with $P(D') \cap P(D'') = P(D' \cap D'') = 0$ and

$$\mathrm{Hom}_{\mathrm{SpecMod}A}(P(D'), P(D'')) \neq 0 .$$

Identify D with its image in B under j. Now choose an element $\bar{f} \neq 0$ in D and let it be represented by the endomorphism f on V. $\mathrm{Im}(f) = U$ is infinite-dimensional since $\bar{f} \neq 0$. Let $U = V' \amalg W$ be a direct sum decomposition of U with V' and W infinite-dimensional and let g and h be the projections of V onto V' and W respectively. It is clear that $\bar{g}, \bar{h} \neq 0$ and that $\bar{g}A \amalg \bar{h}A \subset \bar{f}A$.

It now remains to show that there exists an A-homomorphism $\bar{g}A \to \bar{h}A$ with non-essential kernel (in fact the two right A-modules are isomorphic, cf. lemma 1 below). Let u' be a k-isomorphism of V' onto W with invers v' and extend u' and v' to endomorphisms u and v on V respectively. Consider the A-homomorphisms $s: gA \to hA$ and $t: hA \to gA$ defined by $s(ga) = uga$ and $t(ha) = vha$. Then we have

$$st(ha) = s(vha) = s(gvha) = ugvha = uvha = u'v'ha = ha$$

since $vha = gvha$ (g being an idempotent). Similarly $ts(ga) = ga$. Moreover uga has finite rank if and only if ga has so; hence s induces a monomorphism $\bar{s}: \bar{g}A \to \bar{h}A$ (which is even an isomorphism onto $\bar{h}A$ with invers $\bar{t}$).

1.2. LEMMA 1. *B is isomorphic to every non-zero cyclic sub-module.*

PROOF. Let $0 \neq \bar{f}A \subset B$ and let $\mathrm{Im}(f) = U$. Then U is infinite-dimensional and the same argument as in the preceding proof shows the existence of A-homomorphisms $\bar{s}: \bar{f}A \to A$ and $\bar{t}: A \to \bar{f}A$ which are inverses of each other.

Definition (see [8 p. 177]). An object C in a spectral category is an AB5*-object if for every sub-object D of C and for every decreasing filtered family $\{D_i\}$ of sub-objects of C the canonical map

$$\left(\bigcap D_i\right)+D \to \bigcap (D_i+D)$$

is an isomorphism.

Lemma 2. *If C is an* AB5**-object in a spectral category* **C** *then the ring* $R=\mathrm{Hom}_{\mathbf{C}}(C,C)$ *is finite, that is,* $xy=1$ *implies* $yx=1$ (see [3]).

Proof. The lattice of sub-objects of C is isomorphic to the lattice L of principal right ideals of R under the correspondence $D \to \mathrm{Hom}_{\mathbf{C}}(C,D)=eR$ where $e^2=e$ and $e(C)=D$ (note that $xR=yR$ if and only if $x(C)=y(C)$, and in that case they are both equal to $\mathrm{Hom}_{\mathbf{C}}(C,x(C))$). The hypotheses on C to be an AB5*-object then implies that R is a continuous regular ring and not only upper-continuous (see [5 p. 156]).

Suppose $xy=1$ in R. Then there is a factor-correspondence ([5 p. 149]) $t \to yt$ and $yxs \to xs$ between R and yxR, and Hilfssatz 1.4. of [5] shows that $R \sim yxR$, that is, R and yxR have a common complement in L which must necessarily be equal to zero. But then $yxR=R$, so $yxr=1$ for some $r \in R$; hence $xyxr=x$. Finally $xy=1$ so $x=xyxr=xr$ and $1=yxr=yx$.

Lemma 3. *There exist $\bar{f}$ and $\bar{g}$ in B with $\bar{f}\bar{g}=1$ and $\bar{g}\bar{f}\neq 1$.*

Proof. Let $V=U \amalg W$ with U and W infinite-dimensional and let g be a k-linear isomorphism of V onto W with invers f' and define an element f in A by $f(u+w)=f'(w)$, $u \in U$, $w \in W$. It is clear that $fg=1$ and $\mathrm{Im}(1-gf)=U$ which implies that $\bar{f}\bar{g}=1$ and $\bar{g}\bar{f}\neq 1$ in B since U is infinite-dimensional.

Proposition 2. *$P(B)$ has no non-zero* AB5**-subobjects.*

Proof. Note first that B is von Neumann regular since A is so; in particular B is right non-singular, so the injective hull $E(B)$ of B_B can be given an natural ring-structure which makes it into a von Neumann regular ring (see [9, section 8 and proposition 20.1. p. 113]). Since $P(B) \cong P(E(B))$ we have

$$\mathrm{Hom}_{\mathrm{SpecMod}A}(P(B),P(B)) \cong \mathrm{Hom}_A(E(B),E(B))/J$$

where J is the Jacobson radical of $\mathrm{Hom}_A(E(B),E(B))$ (see [8 p. 176]). But $J=0$ by [9 proposition 20.1] because $\mathrm{End}_A(E(B))=\mathrm{End}_B(E(B))$.

Hence B can be embedded in $\mathrm{End}_{\mathrm{SpecMod}A}(P(B))$ and lemma 3 shows that the latter is not finite, so according to lemma 2 $P(B)$ cannot be an AB5*-object. But by lemma 1 every non-zero sub-object C of $P(B)$ has a sub-object which is isomorphic to $P(B)$ so C cannot be an AB5*-object.

1.3. In order to look for objects of type II it is enough to study objects of the form $P(A/J)$ where J is a right ideal in A, for these objects constitute a set of generators for the category SpecMod$A$ (see [8])

PROPOSITION 3. *If J is a right-ideal in A with $J \supset I$ and if J can be generated by a countable set of elements then $P(A/J) \cong \coprod P_i$ where $P_i \cong P(B)$ for all i.*

PROOF. Let x_i, $i=1,2\ldots$ generate J and let $V_i = \mathrm{Im}(x_i)$. One can suppose that $V_i \subset V_j$ for $i<j$, for since A is regular every finitely generated right-ideal is generated by an idempotent and x_i can then be replaced by idempotents e_i such that

$$e_i A = x_1 A + \ldots + x_i A .$$

If W is not contained in any V_i then $W \cap V_i$ has infinite codimension in W for every i; for if $W \cap V_i + U = W$ with U finite-dimensional then if $\mathrm{Im}(y) = W \cap V_i$ and $\mathrm{Im}(z) = U$, both y and z belong to J which implies that $yA + zA = eA \subset J$ where $\mathrm{Im}(e) = W$, a contradiction ($z \in J$ iff $\mathrm{Im}(z) \subset V_i$ for some i). Let U_i be a complement of $W \cap V_i$ in W such that $U_i \supset U_j$ for $i<j$, and construct an infinite-dimensional sub-space W' of W such that $\dim_k W' \cap V_i$ is finite for all i, by choosing inductively $y_i \in U_i$ such that $y_1,\ldots,y_i$ are linearly independent (which is possible since the U_i are all infinite-dimensional), and let $W' = \coprod_{i=1}^{\infty} k y_i$. It is clear that W' has the required properties.

Let L be a non-zero sub-module of A/J and let $\bar{f} \neq 0$ in L be represented by f in A. $f \notin J$ so by the above argument there is an infinite-dimensional sub-space W' of $\mathrm{Im}(f)$ such that $W' \cap V_i$ is finite-dimensional for every i. Choose g in A with $\mathrm{Im}(g) = W'$. Then $gA \subset fA$ and if $h \in gA \cap J$ then $\mathrm{Im}(h) \subset W' \cap V_i$ so h has finite rank, that is $h \in I$ so $gA \cap J = gA \cap I$. Therefore

$$(gA+J)/I \cong (gA+I)/I .$$

But $g \notin I$ and by lemma 1, $(gA+I)/I \cong A/I = B$, so L has a sub-module isomorphic to B.

Consider now the family F of direct sums of cyclic sub-modules of A/J isomorphic to B. F is not empty according to the preceding paragraph and is inductively ordered by inclusion on the indexing sets, hence has a maximal element $\coprod (f_iA+J)/J$ by Zorn's lemma. Suppose that $\coprod (f_iA+J)/J$ is not essential in A/J. Then there exists $0 \neq L \subset A/J$ such that

$$(\coprod (f_iA+J)/J) \cap L = 0 .$$

But L has a non-zero sub-module isomorphic to B contradicting the maximality of $\coprod (f_iA+J)/J$. Therefore $\coprod (f_iA+J)/J$ is essential in A/J which implies that

$$P(A/J) \cong P(\coprod (f_iA+J)/J) \cong \coprod P((f_iA+J)/J) = \coprod P_i$$

where $P_i \cong P(B)$ for all i.

2.

Let now A be an arbitrary commutative ring. According to [8 proposition 2 p. 161] an object C in a spectral category $\boldsymbol{C}$ is distributive if and only if $\mathrm{Hom}_{\boldsymbol{C}}(C_1,C_2)=0$ for all sub-objects C_1,C_2 of C with $C_1 \cap C_2 = 0$. An element $f \neq 0$ in $\mathrm{Hom}_{\mathrm{Spec\,Mod}\,A}(C,D)$ can be represented by an A-homomorphism $g\colon C' \to D$ where C' is essential in C and $\mathrm{Ker}(g)$ is not essential in C'. But then there is a cyclic sub-module $aA \neq 0$ of C' such that $aA \cap \mathrm{Ker}(g)=0$ and the restriction of g to aA is a monomorphism into D. If moreover $b=g(a)$ then g induces an isomorphism $aA \cong bA$. A is commutative implies that $(0:a)=(0:b)$.

If now $P(C)$ is not distributive then there exist sub-objects $P(C_1)$ and $P(C_2)$ of $P(C)$ with $P(C_1) \cap P(C_2)=0$ and a non-zero morhism $f\colon P(C_1) \to P(C_2)$. One can suppose that C_1 and C_2 are submodules of C (cf. the proof of proposition 1) so by the argument above one can find non-zero a and b in C with $(0:a)=(0:b)$. Moreover $C_1 \cap C_2 = 0$ implies that $aA \cap bA = 0$.

Conversely suppose there exist non-zero a,b in C with $(0:a)=(0:b)$ and $aA \cap bA = 0$. Then $aA \cong bA$ and $P(aA) \cong P(bA)$ are non-zero sub-objects of $P(C)$ with intersection $=0$, so $P(C)$ cannot be distributive. We have therefore proved the following result:

Proposition 4. *Let A be a commutative ring and C a non-zero A-module. Then $P(C)$ is a distributive object in* Spec Mod A *if and only if there are no non-zero a,b in C with $aA \cap bA = 0$ and* $(0:a)=(0:b)$.

Corollary. *Let I be an ideal in A and $a \in A - I$. If there is no $b \in A$ such that $ab \notin I$ and $a^2b^2 \in I$ then $P((aA+I)/I)$ is distributive. In particular $P(A/I)$ is distributive if $I=\sqrt{I}$.*

PROOF. If $P((aA+I)/I)$ is not distributive then there exist, according to proposition 4, b and c in A such that $(I:ab)=(I:ac)\neq A$ and $(abA+I)\cap(acA+I)=I$. But then $abac \in abA \cap acA \subset I$ so $ab \in (I:ac)=(I:ab)$.

To show that $\operatorname{Spec}\operatorname{Mod} A$ is locally distributive (of type I) it suffices to show that $P(A/I)$ is a direct sum of distributive objects for every ideal I in A, for these objects constitute a system of generators for $\operatorname{Spec}\operatorname{Mod} A$ (see [8]). In order to show this, it is enough to show that every cyclic non-zero module $aA+I/I$ contains a non-zero cyclic sub-module $abA+I/I$ such that $P(abA+I/I)$ is distributive. For if $\coprod I_i/I$ is a maximal element in the family $\mathscr{F}$ of direct sums of modules I_i/I such that $P(I_i/I)$ is distributive and if $\coprod I_i/I$ is not essential in A/I it is possible to find a strictly bigger direct sum which belongs to $\mathscr{F}$. Therefore $\coprod I_i/I$ is essential in A/I and

$$P(A/I) \cong P(\coprod I_i/I) \cong \coprod P(I_i/I)\,.$$

According to the corollary to proposition 4 one can suppose that $a^2 \in I$; for either there is no b such that $ab \notin I$ and $a^2b^2 \in I$ and then $P(aA+I/I)$ is distributive (by the corollary) or there exists such a b and then a can be replaced by ab (if $abA+I/I$) contains a cyclic non-zero sub-module which is distributive in $\operatorname{Spec}\operatorname{Mod} A$ then so does $aA+I/I$). Thus we have:

PROPOSITION 5. *Let A be a commutative ring. If for every $a\in A$ and ideal I in A satisfying $a\notin I$, $a^2\in I$ there exists $b\in A$ such that $P(abA+I/I)$ is non-zero and distributive, then* $\operatorname{Spec}\operatorname{Mod} A$ *is locally distributive.*

The following result is also an easy corollary of proposition 4 and will be used in the next paragraph to show the existence of a commutative ring with a spectral category that is not locally distributive.

PROPOSITION 6. *Let A be a commutative ring and I an ideal in A. Suppose there are elements a_k, b_k in $A-I$ such that*

(1) $(a_kA+I)\cap(b_kA+I)=I$,
(2) $(I:a_k)=(I:b_k)$,
(3) *for every $c\notin I$ there is a k such that $a_kc\notin I$.*

Then $P(A/I)$ contains no non-zero distributive sub-objects (is anti-distributive).

PROOF. Let $cA+I/I$ be an arbitrary non-zero cyclic sub-module of A/I. It is clearly sufficient to show that $P(cA+I/I)$ is not distributive. Take a_k such that $a_kc\notin I$. Then we have

$$(a_k cA + I)/I \cap (b_k cA + I)/I = 0$$

and

$$(I : a_k c) = ((I : a_k) : c) = ((I : b_k) : c) = (I : b_k c),$$

so according to proposition 4, $P(cA + I/I)$ is not distributive.

3.

Let now $A = k[X_1, Y_1, X_2, Y_2, \ldots]$ be the polynomial ring in countably many variables over a field k. Let I be the ideal generated by X_i^2, Y_i^2 and $X_i Y_i$, $i = 1, 2, \ldots$. We have:

PROPOSITION 7. SpecMod$A$ *is not locally distributive. More explicitely we have that* $P(A/I)$ *is anti-distributive* (*that is contains no non-zero distributive sub-objects*).

PROOF. All we have to do is to check properties (1)–(3) in proposition 6 for the pair of elements X_k, Y_k, which is quite trivial. For example (1): Suppose the contrary and take

$$P \in (X_k A + I) \cap (Y_k A + I) - I .$$

Since all the ideals are generated by monomials one can suppose that P is a monomial which must contain both X_k and Y_k (since A is a unique factorisation domain) and thus belong to I, a contradiction. For (2) note that if $P \in (I : X_k)$ then every monomial of P must belong to $(I : X_k)$ and therefore contains a X_k or Y_k if it is not already in I. (3) is immediate since an element in A is a finite sum of monomials.

CONJECTURE. $P(A/I)$ is of type II, that is more precisely: it is an AB5*-object or its ring of endomorphisms in SpecMod$A$ is continuous. This seems however hard to prove.

4.

Let now $A = C(X, \mathsf{Z})$ be the ring of continuous integer-valued functions on a compact topological space (see [7]). I shall show that SpecMod$A$ is locally distributive. Let I be an ideal in A and f an element in $A - I$ but such that $f^2 \in I$. X is compact and f is continuous so the range of f is finite and X can be decomposed as $X = \bigcup_1^n X_i$, where the X_i are open and closed in X and pair-wise disjoint and f is constant on every X_i. If one lets $g_i = 1$ on X_i and $= 0$ otherwise then g_i is continuous and $f = \sum_1^n g_i f$. $f \notin I$ implies the existence of an i such that $g_i f \notin I$. By substituting this $g_i f$ for f one can suppose from the beginning that $X=X_1 \cup X_2$, $X_1 \cap X_2=\varnothing$ and X_1 is open and closed in X and that $f=0$ on X_1 and $f=a\neq 0$ on X_2 (also that $X_2\neq\varnothing$ for $f\neq 0$). If $a=\prod_1^n p_j$, p_j primes, and if one sets $f_j(X_2)=p_j$, $f_j(X_1)=0$, then f_j is continuous and $f=\prod_1^n f_j$. Let $f'=f\prod_1^{m-1} f_j$ where m is the first integer such that $f\prod_1^m f_j \in I$ (this exists and is $\geqq 1$ since $f \notin I$ and $f^2 \in I$), so $f' \in fA-I$, and if $g(X_2)=p_m$ and $g(X_1)=1$ then $gf' \in I$.

If E is open and closed in X, define the element h_E in A by $h_E(X-E)=1$ and $h_E(E)=p_m$. Let $\mathscr{E}$ be the family of such E such that $h_E f' \in I$. $\mathscr{E}$ is not empty as was shown above.

Lemma 4. *$\mathscr{E}$ is a filter, that is,*

(1) *if $E \in \mathscr{E}$ and $F \supset E$ where F is open and closed in X then $F \in \mathscr{E}$,*
(2) *if E and $F \in \mathscr{E}$ then $E \cap F \in \mathscr{E}$.*

Proof. (1) Suppose that F is open and closed in X and contains E in $\mathscr{E}$. Define $k(E)=1$, $k(F-E)=p_m$ and $k(X-F)=1$. Then $k \in A$, $kh_E=h_F$ and $kh_E f' \in I$, so $F \in \mathscr{E}$.

(2) Define $k(E-E\cap F)=0$ and $=1$ otherwise and define $g(E-E\cap F)=1$ and $=0$ otherwise. Then we have $k,g \in A$,

$$kh_E+gh_F = h_{E\cap F} \quad \text{and} \quad (kh_E+gh_F)f' \in I$$

by the assumption on E and F so $E\cap F \in \mathscr{E}$.

Define $c(\mathscr{E})=\bigcap E$ where E varies over $\mathscr{E}$. $c(\mathscr{E})$ is non-empty for X is compact, the E's are closed in X and every finite intersection is non-empty.

Lemma 5. *If E is open and closed in A and $E \supset c(\mathscr{E})$ then $E \in \mathscr{E}$.*

Proof. $X-E$ is compact and $(X-E)\cap E_i$ are closed in $X-E$ and

$$\bigcap_{E_i\in\mathscr{E}} \left((X-E)\cap E_i\right) = \varnothing .$$

But then some finite intersection $\bigcap_1^k \left((X-E)\cap E_i\right)=\varnothing$, that is, $E\supset \bigcap_1^k E_i$, so $E \in \mathscr{E}$ by lemma 4.

Lemma 6. *If $g(x) \in p_m \mathbf{Z}$ for all x in $c(\mathscr{E})$ then $gf' \in I$.*

Proof. $g^{-1}(p_m \mathbf{Z})=U$ is open and closed in X and $U \supset c(\mathscr{E})$ so $U \in \mathscr{E}$ and if g' is defined by $g'=g$ on $X-U$ and $g'(x)=g(x)/p_m$ on U then $g' \in A$ and $g=g'h_U$. But $h_U f' \in I$ so $gf'=g'h_U f' \in I$.

LEMMA 7. *If* $(I:gf')=(I:hf')$, $x \in c(\mathscr{E})$ *and* $g(x) \in p_m \mathsf{Z}$ *then* $h(x) \in p_m \mathsf{Z}$.

PROOF. Suppose the contrary. Since g and h are continuous and Z discrete one can find an open and closed neighbourhood U of x such that g and h take a constant value on U. Let $k(U)=1$, $k(X-U)=p_m$. Then $kg(x)=k'h_X(x)$ for all x in X and some $k' \in A$. Now $h_X f' \in I$ so $kgf' \in I$ that is $k \in (I:gf')$. Hence it is now sufficient to show that $k \notin (I:hf')$. Choose $a,b \in \mathsf{Z}$ such that

$$akh(x)+bp_m = 1, \quad x \in U ,$$

and define elements α and β in A by $\alpha(U)=a$ and $\alpha=0$ otherwise, $\beta(U)=b$ and $=0$ otherwise. Then we have $\alpha kh+\beta h_X=h_{X-U}$, and $U \cap c(\mathscr{E}) \neq \emptyset$ implies $h_{X-U}f' \notin I$ by the definition of $c(\mathscr{E})$. But now $h_X f' \in I$ so $khf' \notin I$.

PROPOSITION 8. *If* $(I:gf')=(I:hf')$ *then* $gf'A \cap hf'A \not\subset I$.

PROOF. $(I:gf') \neq A$ implies that $gf' \notin I$ which in turn implies that $g(x) \notin p_m \mathsf{Z}$ for some x in $c(\mathscr{E})$ by lemma 6. But then lemma 7 gives that $h(x) \notin p_m \mathsf{Z}$. Now g and h are continuous and Z discrete so one can suppose that g and h are constant on some open and closed neighbourhood U of x. Thus on U we have $g(x)h(x) \notin p_m \mathsf{Z}$. In the same way as in lemma 7 for the function kh one can now deduce that $ghf' \notin I$ (the only properties of kh that were needed were (1) that $U \cap c(\mathscr{E}) \neq \emptyset$ and (2) that $(k(x)h(x),p_m)=1, x \in U$). But $ghf' \in gf'A \cap hf'A$ so the proposition is proved.

Now proposition 8 is the crucial result which together with the general considerations in section 2 immediately gives the following

THEOREM 1. $\operatorname{Spec}\operatorname{Mod} C(X,\mathsf{Z})$ *is locally distributive.*

PROOF. According to proposition 5 it suffices to show that if $f \in A-I$ and $f^2 \in I$, then $fA+I/I$ contains a non-zero cyclic sub-module which is distributive in $\operatorname{Spec}\operatorname{Mod} A$. But by propositions 4 and 8, $P(f'A+I/I)$ is distributive and $\neq 0$ and $f' \in fA$.

5.

5.1. The last aim of this paper is to study the ring of continuous real-valued functions on the unit interval which turns out to be unexpectedly complicated. Therefore I start with a similar but more easily handled ring. Let X be the set $\{1/n \mid n=1,2,\ldots\}\cup\{0\}$ and $A=C(X)$ the ring of continuous (real) functions on X. The goal of this section is to prove

THEOREM 2. *SpecMod$A$ is locally distributive.*

Let J be the ideal $\{g \mid g(0)=0\}$ in A. Then the following proposition plays the same essential part in proving theorem 2 as did proposition 8 in proving theorem 1.

PROPOSITION 9. *Let I be any ideal in A and suppose that $(I:f)=(I:g)\neq J$. Then $fA\cap gA \not\subset I$.*

PROOF. Choose $G\in J-(I:f)$. Then also $G'=|G|^{\frac{1}{2}}$ belongs to $J-(I:f)$ so one can suppose that $G\geqq 0$ and that $G^2\in J-(I:f)$. Let

$$V = \{1/n \mid f(1/n)g(1/n)\neq 0\}\cup\{0\}\,.$$

Prove to begin with that $Gf_V, Gg_V \notin I_V$ where I_V is the set of restrictions of functions in I to V. Let

$$V' = \{1/n \mid f(1/n)\neq 0\}\cup\{0\}$$

and define $H=G^2$ on V' and $=0$ otherwise. Then $Hf=G^2f\notin I$ so $Hg\notin I$. Let $H'=H$ on V and $H'=0$ otherwise. Then $Hg=H'g\notin I$ so $H'f\notin I$. If now $Gf=k$ on V where $k\in I$ then if $k'=G$ on V and $k'=0$ otherwise so $k'k=k'Gf=H'f\notin I$ a contradiction for $k\in I$. In the same way one proves $Gg_V\notin I_V$.

Next let

$$U = \{1/n\in V \mid |f(1/n)/g(1/n)|\leqq 1\}\cup\{0\} \quad\text{and}\quad W = (V-U)\cup\{0\}$$

and show that $G^{\frac{1}{2}}f_U\notin I_U$ or $G^{\frac{1}{2}}f_W\notin I_W$. Suppose not, that is, $G^{\frac{1}{2}}f=h$ on U and $G^{\frac{1}{2}}f=k$ on W where $h,k\in I$. Let $\lambda=G^{\frac{1}{2}}$ on U and $=0$ otherwise and let $\mu=G^{\frac{1}{2}}$ on W and $=0$ otherwise. Then λ and μ are in A, and $\lambda h+\mu k\in I$ and equals $G^{\frac{1}{2}}h=Gf$ on U and $=G^{\frac{1}{2}}k=Gf$ on W, so $\lambda h+\mu k=Gf$ on V, a contradiction since $Gf_V\notin I_V$. In the same way one shows that $G^{\frac{1}{2}}g_U\notin I_U$ or $G^{\frac{1}{2}}g_W\notin I_W$.

Now one wants to show that $G^{\frac{1}{2}}f_U\notin I_U$ or $G^{\frac{1}{2}}g_W\notin I_W$. Suppose the contrary. Then by the above result we have $G^{\frac{1}{2}}g_U\notin I_U$ and we shall use the hypotheses $(I:f)=(I:g)$ to get a contradiction. Let $h=G^{\frac{1}{2}}$ on U and $=0$ otherwise and let $k=G^{\frac{1}{4}}$ on U and $=0$ otherwise. h and k belong to A and $hf=kG^{\frac{1}{4}}f=kf'$ where $f'\in I$ is some element such that $G^{\frac{1}{4}}f=f'$ on U. Thus we have $h\in(I:f)=(I:g)$ so $hg\in I$. But $hg=G^{\frac{1}{2}}g$ on U while $G^{\frac{1}{2}}g_U\notin I_U$ so we get what we wanted.

Hence one has $G^{\frac{1}{2}}f_U \notin I_U$ or $G^{\frac{1}{2}}g_W \notin I_W$. Suppose the first relation holds (the argument is quite analogous if $G^{\frac{1}{2}}g_W \notin I_W$). Let $h=G^{\frac{1}{2}}$ on U and $h=0$ otherwise. h is continuous and $hf \notin I$. Let $k=hf/g$ on U and $k=0$ otherwise. k is continuous since $|f/g| \leqq 1$ on U and $h(0)=0$. Finally $hf=kg$ so $hf \in fA \cap gA - I$.

PROOF OF THEOREM 2. Let I be any ideal in A and f an arbitrary element in $A-I$. It is enough to show that $fA+I/I$ contains a non-zero cyclic module $fgA+I/I$ which is distributive in SpecMod$A$ (by proposition 5). If $fA+I/I$ has a simple sub-module this is clear. If $fA+I/I$ does not contain any simple sub-module, then we cannot have $(I:fg)=J$ for any g; for J is a maximal ideal and we would then have $fgA+I/I \cong A/J$ which is a simple module. Suppose now that $(I:fg)=(I:fh) \neq A$. Then also $(I:fg) \neq J$, and proposition 9 implies that

$$fgA+I/I \cap fhA+I/I \neq 0 ,$$

so $P(fA+I/I)$ is distributive by proposition 4.

5.2. Let X be any closed subset of the closed unit-interval $[0,1]$ and let f be a continuous function $X \to \mathsf{R}$. Then f can be extended to a continuous function F on $[0,1]$ (see e.g. [2]). This can be restated in the following way: let $A=C([0,1])$ and $B=C(X)$. Then the natural restriction homomorphism $A \to B$ is surjective. Let now I be an ideal in B. Then one can find an ideal J in A such that $B/I \cong A/J$. If now $P(A/J)$ is a direct sum of distributive objects in SpecMod$A$ the same is true for $P(B/I)$ in SpecMod$B$ and since the objects $P(B/I)$ form a system of generators of SpecMod$B$ we conclude that SpecMod$B$ is locally distributive if SpecMod$A$ is so. The following proposition gives some sort of a converse.

PROPOSITION 10. *If* SpecMod$C(X)$ *is locally distributive for every totally disconnected compact subset* X *of* $[0,1]$ *then so is* SpecMod$C([0,1])$.

PROOF. Let $A=C([0,1])$ and let f and g be two arbitrary elements in A. Define

$$U = U_{f,g} = \{x \mid |f(x)| < |g(x)| \text{ or } |f(y)| = |g(y)| \neq 0 \text{ for all } y \text{ in some neighbourhood of } x\}$$

$$V = V_{f,g} = \{x \mid |f(x)| > |g(x)|\} .$$

U and V are open in $[0,1]$ and can be represented as the intersection of $[0,1]$ with the at most countable disjoint unions of open intervals $\bigcup (a_i,b_i)$ and $\bigcup (c_i,d_i)$ respectively.

Let h be an element in A with $h(a_i)=h(b_i)=0$ for all i. Then define a function h_U on $[0,1]$ by $h_U=h$ on U and $h_U=0$ otherwise. I claim that h_U is continuous on $[0,1]$, hence is an element of A. This is clearly true for every inner and outer point of U. Let now x belong to the border of U. Then there are $x_j \in U$ converging to x. Now $x_j \in (a_{i(j)}, b_{i(j)})$ for some $i(j)$, so we have either $x \leqq a_{i(j)} < x_j$ or $x_j > b_{i(j)} \geqq x$. By choosing in the first case $a_{i(j)}$ and in the other $b_{i(j)}$ one can find a sequence of a_i and b_i converging to x. But now $h(a_i)=h(b_i)=0$ so by continuity $h(x)=0$. Moreover we have

$$|h_U(y)-h_U(x)| \leqq |h(y)-h(x)| ,$$

so h continuous in x implies that h_U is continuous in x and hence everywhere in $[0,1]$. Similarly if $h(c_i)=h(d_i)=0$ for all i, define $h_V=h$ on V and $=0$ otherwise. One shows in the same way that h_V is continuous. Finally if both h_U and h_V are defined let $h'=h_U+h_V$.

Let now I be an arbitrary ideal in A and $f \in A-I$. According to proposition 5 it suffices to prove that $fA+I/I$ contains a non-zero cyclic sub-module which is distributive as an object in $\mathrm{Spec\,Mod}\,A$. First suppose there is a $gf \in A-I$ such that

$$(I:fg) \supset I(X) = \{f \mid f(X)=0\} ,$$

where X is some compact and totally disconnected subset of $[0,1]$. Then

$$fgA+I/I \cong A/(I:fg) \cong B/J$$

where $B=C(X)\cong A/I(X)$. But according to the hypotheses $\mathrm{Spec\,Mod}\,B$ is locally distributive, so B/J is a direct sum of distributive objects in $\mathrm{Spec\,Mod}\,B$, and therefore the same conclusion holds for $fgA+I/I$ in $\mathrm{Spec\,Mod}\,A$. In particular $fgA+I/I$ contains a non-zero cyclic sub-module C such that $P(C)$ is distributive.

Secondly suppose that there is no $fg \in A-I$ such that $(I:fg)\supset I(X)$ for some compact totally disconnected subset X of $[0,1]$. Suppose that $(I:fg)=(I:fh)\neq A$. We want to show that $fgA \cap fhA$ is not contained in I. Define $U=U_{fg,fh}$, $V=V_{fg,fh}$ and a_i,b_i,c_i,d_i as above for the functions fg and fh in place of f and g. Take

$$\alpha \in I(\overline{\{a_i,b_i,c_i,d_i\}})-(I:fg) .$$

(This is possible since the closure of $\{a_i,b_i,c_i,d_i\}$ is compact and totally disconnected.) Then we have

$$\alpha fg = \alpha' fg = \alpha_U fg + \alpha_V fg \notin I ,$$

so either $\alpha_U fg$ or $\alpha_V fg$ is not in I. Suppose for example that $\alpha_U fg \notin I$ (if not the reasoning is quite analogous, we only have to change the roles of fg and fh). Define the function β by $\beta=\alpha_U fg$ on U and $\beta=0$ otherwise. β is continuous on U and if $x\notin U$ then $\beta(x)=\alpha_U(x)=0$. Moreover $|\beta(y)|\leqq|\alpha_U(y)|$ for all y in $[0,1]$ since $|g/h|\leqq 1$ on U. Hence β is continuous everywhere since α_U is so. But now $\beta h=\alpha_U g$ on U and outside U both β and α_U are $=0$ so equality holds everywhere. That is, we have found a function $\alpha_U fg\in fgA\cap fhA-I$ as required. Now the proposition follows from proposition 4 which implies that $P(fA+I/I)$ is distributive.

Proposition 11. *Let $A=C([0,1])$ and let I be an ideal in A such that I contains a function G with $Z(G)=\{x\mid G(x)=0\}$ having at most one cluster point. Then $P(A/I)$ is a direct sum of distributive objects.*

Proof. Let f and g be two arbitrary elements in A. For every $\lambda\in\mathsf{R}$ define

$$U_\lambda = U_{\lambda,f,g} = \{x \mid |f(x)+\lambda G(x)|<|g(x)| \text{ or } |f(y)+\lambda G(y)|=|g(y)|\neq 0 \text{ for all } y \text{ in some neighbourhood of } x\},$$

$$V_\lambda = V_{\lambda,f,g} = \{x \mid |f(x)+\lambda G(x)|>|g(x)|\},$$

and let E_λ be the border of $V_\lambda\cap U_\lambda$. Note that E_λ is closed and that we have

$$|f(x)+\lambda G(x)| = |g(x)| \quad \text{on } E_\lambda .$$

If $\lambda\neq 0$ we have

$$\begin{aligned} E_0\cap E_\lambda &\subset \{x \mid |f(x)+\lambda G(x)|=|f(x)|\} \\ &= \{x \mid G(x)=0 \text{ or } G(x)\neq 0 \text{ and } \lambda=-2f(x)/G(x)\}. \end{aligned}$$

Hence if $\lambda\neq\mu\neq 0$ we have $E_0\cap E_\lambda\cap E_\mu\subset Z(G)$.

For E an arbitrary closed subset of $[0,1]$ let $I(E)=\{f\mid f(E)=0\}$. If $E=\bigcap_1^n E_i$ then we have trivially $I(E)\supset\sum_1^n I(E_i)$. But the opposite inequality also holds. For $E_i{}^c=[0,1]-E_i$ is an open covering of E^c so 1_{E^c} can be written $\sum_1^n H_j$, where $H_j\geqq 0$ are continuous on E^c and support $(H_j)\subset E_j{}^c$ (see for example [4 p. 171]). If now $h\in I(E)$ then $h=h1_{E^c}=\sum hH_i$ on E^c and $hH_i=0$ on E_i-E. If one lets $k_i=hH_i$ on E^c and $k_i=0$ on E then $h=\sum k_i$ everywhere and the k_i are continuous. In fact they are obviously so on E^c, and if $x\in E$ we have

$$|k_i(y)-k_i(x)| = |k_i(y)| \leqq |h(y)| = |h(y)-h(x)|$$

since $H_i\leqq 1$, and as h is continuous in x it follows that k_i is so. Finally $k_i=0$ on E_i so $k_i\in I(E_i)$, and thus $h\in\sum_1^n I(E_i)$.

In particular we have

$$I(Z(G)) \subset I(E_0\cap E_\lambda\cap E_\mu) = I(E_0)+I(E_\lambda)+I(E_\mu)$$

for $\lambda \neq \mu \neq 0$, so we have two possibilities for $(I:f)$: either (1) $I(Z(G)) \subset (I:f)$ or (2) there is a $\lambda \in \mathsf{R}$ such that $I(E_\lambda) \not\subset (I:f)$.

In case (1) we have

$$fA+I/I \cong A/(I:f) \cong A/I(Z(G))/(I:f)/I(Z(G)),$$

and since the ring $A/I(Z(G))$ has a locally distributive spectral category (see section 5.1.), it follows that $P(fA+I/I)$ is a direct sum of distributive objects.

In case (2) choose $h \in I(E_\lambda)-(I:f)$. Let $h_U = h$ on U and $h_U = 0$ otherwise; let $h_V = h$ on V and $h_V = 0$ otherwise. h_U and h_V are continuous (see the proof of proposition 10). Suppose also that $(I:f)=(I:g)$. Then

$$h(f+\lambda G) = h_U(f+\lambda G)+h_V(f+\lambda G) \notin I,$$

so one can suppose that for example $h_U(f+\lambda G) \notin I$ (if $h_V(f+\lambda G) \notin I$ one continues quite analogously). Let $h' = h(f+\lambda G)/g$ on U and $h'=0$ otherwise. Then h' is continuous (see the proof of proposition 10), and

$$h'g = h(f+\lambda G) \in (fA+I) \cap (gA+I)-I$$

so $fA+I/I \cap gA+I/I \neq 0$.

Let now $f \in A-I$ and suppose there is no $fg \in A-I$ such that $(I:fg) \supset I(Z(G))$ and let fg and fh be such that $(I:fg)=(I:fh)\neq A$. Then there is a $\lambda \in \mathsf{R}$ such that $I(E_\lambda) \not\subset (I:fg)$ (E_λ is here constructed with respect to fg and fh instead of f and g above) and according to case (2) above we have

$$fgA+I/I \cap fhA+I/I \neq 0,$$

that is $P(fA+I/I)$ is distributive by proposition 4.

If on the other hand fg is such that $A \neq (I:fg) \supset I(Z(G))$ then $fgA+I/I$ is a direct sum of distributive objects in Spec Mod A according to case (1) above. In both cases $P(fA+I/I)$ contains a non-zero distributive subobject, so $P(A/I)$ is a direct sum of distributive objects by Zorn's lemma (cf. the proof of proposition 5).

NOTE. Proposition 11 holds with the same proof if $Z(G)$ has at most finitely many cluster points. One only has to modify the proof of theorem 2 to hold also for $C(X)$ where X is closed in $[0,1]$ and has finitely many cluster points.

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