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ON FILTERED MODULES AND THEIR ASSOCIATED GRADED MODULES

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Introduction.

This paper considers the relation between a filtered ring A and its filtered modules on one hand and their associated graded objects on the other. The first two sections are devoted to definitions and elaboration of our main tools. Let A be a filtered ring and N a filtered A-module. In section 2 the main result states that

$$\mathrm{l.w.dim}_A N \leqq \mathrm{gr.l.w.dim}_{E(A)} E(N) ,$$

where $E(\cdot)$ denotes the associated graded object, and that

$$\mathrm{gl.w.dim}\, A \leqq \mathrm{gr.gl.w.dim}\, E(A) .$$

In section 3 we discuss direct sums in the category of filtered modules. In section 4 we prove that, assuming certain finiteness conditions, then $E(N)$ injective over $E(A)$ implies that N is injective over A. Furthermore, we give an example which shows that some kind of finiteness condition is necessary. In section 5 we study the possibility of lifting projective modules over $E(A)$ to projective modules over A.

0. Definition of relevant concepts.

We shall study a filtered ring A and filtered (left or right) modules over A. All filtrations are supposed to be increasing and we always assume that $1 \in F_0 A$. If M is a filtered module then the filtration $F_* M$ is said to be:

1. discrete if $F_p M = 0$ for $p < p_0$, where p_0 depends on M,
2. separated if $\bigcap F_p M = 0$,
3. exhaustive if $M = \bigcup F_p M$,
4. complete if $M = \mathrm{proj\,lim}\, M/F_p M$.

Thus a discrete filtration is complete and a complete filtration is separated. Note that if we topologize M, using $F_* M$ as a fundamental

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system of neighborhoods of 0, then F_*M is discrete, separated or complete iff the associated topology is discrete, separated or complete respectively. The ring of integers ,Z, will always be assumed to be filtered by $F_p\mathsf{Z}=0$ for $p<0$, $F_p\mathsf{Z}=\mathsf{Z}$ for $p\geqq 0$. As a graded ring, Z will be given the grading $\mathsf{Z}^n=0$ for $n\neq 0$, $\mathsf{Z}^0=\mathsf{Z}$. Note that in particular "$E_0(\mathsf{Z})=\mathsf{Z}$".

Our interest will be in the three types of categories described below.

If C is a ring then ${}_C\mathfrak{M}$ ($\mathfrak{M}_C$) will denote the category of left (right) C-modules and the usual homorphisms.

If B is a graded ring then we let the objects in $\mathrm{gr}_B\mathfrak{M}$ ($\mathrm{gr}\,\mathfrak{M}_B$) be the graded left (right) modules over B. The grading M^* of a graded module will be said to be discrete if $M^p=0$ for $p<p_0$, where p_0 depends on M. A homomorphism $f\colon M\to N$ in ${}_B\mathfrak{M}$ ($\mathfrak{M}_C$), i.e. in the corresponding ungraded category, will be said to be of degree n if $f(M^p)\subset N^{p+n}$ for every integer p. This gives us a graded Z-module $\mathbf{Hom}(M,N)$, where $(\mathbf{Hom}(M,N))^n$ consists of the homomorphisms of degree n. As the morphisms in $\mathrm{gr}_B\mathfrak{M}$ ($\mathrm{gr}\,\mathfrak{M}_B$) we take $\hom_B(M,N)=(\mathbf{Hom}(M,N))^0$, Then $\mathrm{gr}_B\mathfrak{M}$ ($\mathrm{gr}\,\mathfrak{M}_B$) becomes an abelian category with "sufficiently many" projective and injective objects.

An object M in $\mathrm{gr}_B\mathfrak{M}$ ($\mathrm{gr}\,\mathfrak{M}_B$) is said to be gr-projective, -injective, -flat if it is projective etc. in $\mathrm{gr}_B\mathfrak{M}$ ($\mathrm{gr}\,\mathfrak{M}_B$). If we say that it is projective etc. without the prefix gr- then we mean that it is so in the corresponding ungraded category ${}_B\mathfrak{M}$ ($\mathfrak{M}_B$). We say that M is gr-free if it is free in ${}_B\mathfrak{M}$ ($\mathfrak{M}_B$) and has a basis consisting of homogeneous elements. It is easy to show that each M has a gr-free resolution and that M is gr-projective iff it is projective.

Note that ${\mathrm{Tor}^B}_n$ is a functor

$$\mathrm{gr}\,\mathfrak{M}_B\times \mathrm{gr}_B\mathfrak{M}\to \mathrm{gr}_{\mathsf{Z}}\mathfrak{M}\,.$$

Furthermore, ${\mathrm{Tor}^B}_n(M,N)$ considered as an ungraded object is our usual ${\mathrm{Tor}^B}_n(M,N)$ when B, M, N are regarded as ungraded. This follows immediately from the way we compute it. We have also defined a functor

$$\mathbf{Hom}_B\colon \mathrm{gr}_B\mathfrak{M}\times \mathrm{gr}_B\mathfrak{M}\to \mathrm{gr}_{\mathsf{Z}}\mathfrak{M}\,.$$

It is clear that $\mathbf{Hom}_B(\cdot,N)$ is exact iff N is gr-injective. Using the derived functors of $\mathbf{Hom}_B$ we get the functors

$${\mathrm{Ext}^B}_n\colon \mathrm{gr}_B\mathfrak{M}\times \mathrm{gr}_B\mathfrak{M}\to \mathrm{gr}_{\mathsf{Z}}\mathfrak{M}\,.$$

If A is a filtered ring then we let the objects in $\mathrm{filt}_A\mathfrak{M}$ ($\mathrm{filt}\,\mathfrak{M}_A$) be the filtered left (right) A-modules. Let M,N be filtered A-modules and let $f\colon M\to N$ be a homomorphism of A-modules. Then f is said to be of filt-degree n if $fF_pM\subset F_{p+n}N$ for every integer p. The homomorphisms $M\to N$

of finite filter-degree will be denoted by $\mathbf{Hom}_A(M,N)$. As the morphisms in $\mathrm{filt}_A\mathfrak{M}$ ($\mathrm{filt}\,\mathfrak{M}_A$) we take the homomorphisms of filt-degree 0 and the morphisms $M \to N$ will be denoted by $\hom_A(M,N)$. It is clear that we then get a category and that this category is not abelian in general.

It is well-known that there is a functor

$$E_0 \colon \mathrm{filt}_A\mathfrak{M} \to \mathrm{gr}_{E_0(A)}\mathfrak{M} .$$

We shall denote this functor simply by E. A morphism $f \in \hom_A(M,N)$ is said to be strict if $fF_pM = \mathrm{im} f \cap F_pN$. It is shown in [2, pp. 36–37] that if

$$K \xrightarrow{f} M \xrightarrow{g} N$$

is strict exact (i.e. it is exact and f and g are strict) then

$$E(K) \xrightarrow{E(f)} E(M) \xrightarrow{E(g)} E(N)$$

is exact.

Finally, all rings are supposed to have a unit element and all modules are assumed to be unitary.

1. Strict morphisms and exactness. Free objects in $\mathrm{filt}_A\mathfrak{M}$.

We want to somewhat extend the above mentioned and some related results in [2].

LEMMA 1. *Let A be a filtered ring and let*

$$(1) \qquad K \xrightarrow{f} M \xrightarrow{g} N$$

be a 0-sequence in $\mathrm{filt}_A\mathfrak{M}$. *Consider*

$$(2) \qquad E(K) \to E(M) \to E(N) .$$

We have

(a) *If* (1) *is strict exact then* (2) *is exact.*

(b) *If* (2) *is exact and F_*M is exhaustive then g is strict.*

(c) *If* (2) *is exact, F_*K is complete and F_*M is separated then f is strict.*

(d) *If* (2) *is exact and F_*M is discrete then f is strict.*

(e) *If F_*K is complete and F_*M is exhaustive and separated or if F_*M is exhaustive and discrete then* (1) *is strict exact iff* (2) *is exact.*

PROOF. (a) Let $E(g)m^p = 0$, where $m \in F_pM$ and m^p is its canonical image in

$$E^p(M) = F_pM/F_{p-1}M .$$

Then $g(m) \in F_{p-1}N$ so that $g(m)=g(m')$, where $m' \in F_{p-1}M$. It follows that $m-m'=f(k)$, where $k \in F_pK$, so that

$$m^p = (m-m')^p = f(k)^p = E(f)k^p .$$

Thus (2) is exact.

(b) Let $n \in \operatorname{im} g \cap F_pN$, $n=g(m)$. Since F_*M is exhaustive we can assume $m \in F_{p+s}M$. If $s=0$ we are done. Suppose $s>0$. Then $E(g)m^{p+s}=0$ and hence

$$m^{p+s} = E(f)k_{p+s}^{p+s}, \quad \text{where } k_{p+s} \in F_{p+s}K ,$$

$$m-f(k_{p+s}) \in F_{p+s-1}M \quad \text{and} \quad g(m-f(k_{p+s-1})) = n ,$$

i.e. we can reduce s to $s-1$. By induction we get $n \in gF_pM$.

(c), (d) Let $m \in \operatorname{im} f \cap F_pM$. Hence $E(g)m^p=g(m)^p=0$, which shows that $m^p=E(f)k^p{}_p$, that is,

$$m-f(k_p) \in \operatorname{im} f \cap F_{p-1}M, \quad \text{where } k_p \in F_pK .$$

By induction we get an element $k_{p-s} \in F_{p-s}K$ such that

$$m-f(k_p+\ldots+k_{p-s}) \in \operatorname{im} f \cap F_{p-s-1}M .$$

If F_*M is discrete we shall be done after a finite number of steps, which proves (d). If F_*K is complete we get

$$k = \textstyle\sum_{s=0}^{\infty} k_{p-s} \in F_pK$$

and

$$m-f(k) = \lim_{s\to\infty}(m-f(k_p+\ldots+k_{p-s})) = 0$$

if F_*M is discrete. This proves (c).

(e) It only remains to prove that if (2) is exact then (1) is exact. Let $m \in M$, $g(m)=0$. Since F_*M is exhaustive we can assume that $m \in F_pM$. We have $E(g)m^p=0$, so that $m-f(k_p) \in F_{p-1}M$ for some $k_p \in F_pK$, and $g(m-f(k_p))=0$. By induction we can choose $k_{p-s} \in F_{p-s}K$ such that

$$m-f(k_p+\ldots+k_{p-s}) \in F_{p-s-1}M .$$

If F_*M is discrete we shall be done after a finite number of steps. If F_*K is complete and F_*M is separated then we get

$$m = f\textstyle\sum_{0\le s<\infty} k_{p-s} .$$

Next we consider the free objects in $\operatorname{filt}_A\mathfrak{M}$ (cf. [7]).

Definition. Let $L \in \operatorname{filt}_A\mathfrak{M}$. L is then said to be filt-free if L is free in ${}_A\mathfrak{M}$ and has a basis $\{x_i\}_{i\in I}$ such that there are integers $p(i)$, $i \in I$ with the property

$$F_pL = \textstyle\sum_{s+p(i)=p} F_sA\cdot x_i .$$

The set $\{(x_i, p(i))\}_{i\in I}$ will then be called a filt-basis for L.

The following lemma is easy to prove.

LEMMA 2. *Let $L \in \mathrm{filt}_A\mathfrak{M}$. We have*

(a) *If L is* filt-*free with* filt-*basis $\{(x_i, p(i))\}_{i\in I}$ then $E(L)$ is* gr-*free and $\{x_i^{p(i)}\}_{i\in I}$ is a homogeneous basis.*

(b) *If $E(L)$ is* gr-*free with homogeneous basis $\{x_i^{p(i)}\}_{i\in I}$ and F_*L is discrete then L is* filt-*free with* filt-*basis $\{(x_i, p(i))\}_{i\in I}$.*

(c) *If $M \in \mathrm{gr}_{E(A)}\mathfrak{M}$ is* gr-*free then there is a* filt-*free $L \in \mathrm{filt}_A\mathfrak{M}$ such that $E(L) = M$.*

(d) *Let L be* filt-*free with* filt-*basis $\{(x_i, p(i))\}_{i\in I}$. Suppose that $M \in \mathrm{filt}_A\mathfrak{M}$ and that $f\colon \{x_i\}_{i\in I} \to M$ is a function such that $f(x_i) \in F_{s+p(i)}M$. Then there is unique homomorphism $L \to M$ of filter-degree s which extends f.*

(e) *Let L be* filt-*free, $M \in \mathrm{filt}_A\mathfrak{M}$ and suppose that $g\colon E(L) \to E(M)$ is a homomorphism of degree s. Then there is a homomorphism $f\colon L \to M$ of filter-degree s such that "$E(f)$" $= g$ (cf. Lemma* 15).

(f) *Let L be* filt-*free. Then F_*L is exhaustive or separated iff F_*A is exhaustive or separated respectively. If F_*A is discrete and $\{p(i)\}_{i\in I}$ (defined as above) is bounded below then F_*L is discrete. If I is finite and F_*A is complete then F_*L is complete.*

(g) *Let $M \in \mathrm{filt}_A\mathfrak{M}$. Then there is a resolution of M*

$$\to L_2 \xrightarrow{f_2} L_1 \xrightarrow{f_1} L_0 \xrightarrow{f_0} M \to 0$$

such that every L_n is filt-*free and every f_n is a strict homomorphism in* $\mathrm{filt}_A\mathfrak{M}$. *Such a resolution will be called a strict* filt-*free resolution of M. Note that if F_*A and F_*M are discrete then we can assume that every F_*L_n is discrete.*

Furthermore, if F_*A is exhaustive and complete, $E(A)$ is noetherian and $E(M)$ is f.g. *then we can assume that every L_n is* f.g.

2. Weak dimension.

Let A be a filtered ring and assume that $M \in \mathrm{filt}\,\mathfrak{M}_A$, $N \in \mathrm{filt}_A\mathfrak{M}$. Filter $M \otimes_A N$ by letting $F_p(M \otimes_A N)$ be the sub-Z-module of $M \otimes_A N$ generated by elements of type $m \otimes_A n$, where $m \in F_sM$, $n \in F_tN$ with $s + t \leq p$. This defines a functor

$$\otimes_A\colon \mathrm{filt}\,\mathfrak{M}_A \times \mathrm{filt}_A\mathfrak{M} \to \mathrm{filt}_{\mathbf{Z}}\mathfrak{M}\,.$$

Clearly $F_*(M \otimes_A N)$ is exhaustive or discrete if F_*M and F_*N are exhaustive or discrete respectively.

Let

$$\varkappa = \varkappa(M,N)\colon E(M)\otimes_{E(A)}E(N) \to E(M\otimes_A N)$$

be given by $\varkappa(m^s\otimes_{E(A)}n^t)=(m\otimes_A n)^{s+t}$. This obviously well-defines $\varkappa$ since the right-hand expression is independent of the choice of representatives m,n and is bilinear in m^s,n^t. Clearly $\varkappa$ is a natural transformation and $\varkappa(M,N)$ is always an epimorphism.

For $M=A$ we have a morphism $\tau\colon A\otimes_A N \to N$ in $\mathrm{filt}_A\mathfrak{M}$ given by $\tau(a\otimes_A n)=an$. Now τ is an isomorphism in ${}_A\mathfrak{M}$ and

$$F_pN = 1\cdot F_pN \subset F_0A\cdot F_pN \subset \tau F_p(A\otimes_A N) \subset F_pN\ .$$

Consequently $\tau F_p(A\otimes_A N)=F_pN$ and thus $\tau^{-1}F_pN=F_p(A\otimes_A N)$ so that τ is an isomorphism in $\mathrm{filt}_A\mathfrak{M}$. We have a commutative diagram in $\mathrm{gr}_{E(A)}\mathfrak{M}$

$$\begin{array}{ccc} E(A)\otimes_A E(N) & \xrightarrow{\ \varphi\ } & E(N) \\ {\scriptstyle \varkappa(A,N)}\big\downarrow & & \big\uparrow{\scriptstyle E(\tau)} \\ E(A\otimes_A N) & \longrightarrow & \end{array}$$

where $\varphi(b\otimes_{E(A)}y)=by$. It follows that $\varkappa(A,N)$ is an isomorphism. Similarly $\varkappa(M,A)$ is always an isomorphism. We summarize in

Lemma 3. *There is a natural epimorphism*

$$\varkappa(M,N)\colon E(M)\otimes_{E(A)}E(N) \to E(M\otimes_A N)$$

given by $\varkappa(m^s\otimes_{E(A)}n^t)=(m\otimes_A n)^{s+t}$. *Furthermore,* $\varkappa(A,N)$ *and* $\varkappa(M,A)$ *are isomorphisms in* $\mathrm{gr}_{E(A)}\mathfrak{M}$ $(\mathrm{gr}\,\mathfrak{M}_{E(A)})$.

Using $\varkappa$ we get the following result

Lemma 4. *Let A be a filtered ring and let $N\in\mathrm{filt}_A\mathfrak{M}$. Suppose that F_*A, F_*N are discrete and exhaustive and that $E(N)$ is* gr*-flat. Then N is flat.*

Proof. Let J be a right ideal in A. Filter J by $F_pJ=J\cap F_pA$ that is, give J the filtration making the inclusion $i\colon J\to A$ a strict morphism in $\mathrm{filt}\,\mathfrak{M}_A$. Then $E(i)\colon E(J)\to E(A)$ is a monomorphism by lemma 1. Consider the commutative diagram

$$\begin{array}{ccc} E(J)\otimes_{E(A)}E(N) & \xrightarrow{\ E(i)\otimes_{E(A)}1\ } & E(A)\otimes_{E(A)}E(N) \\ {\scriptstyle \varkappa(J,N)}\big\downarrow{\scriptstyle \text{epi}} & & {\scriptstyle \text{iso}}\big\downarrow{\scriptstyle \varkappa(A,N)} \\ E(J\otimes_A N) & \xrightarrow{\ E(i\otimes_A 1)\ } & E(A\otimes_A N) \end{array}$$

Since $E(N)$ is gr-flat we infer that $E(i)\otimes_{E(A)}1$ is a monomorphism. Thus $\varkappa(J,N)$ is a monomorphism and hence an isomorphism. It follows that $E(i\otimes_A 1)$ is a monomorphism. Now $F_*(J\otimes_A N)$ is discrete and exhaustive and so, by lemma 1, $i\otimes_A 1$ is a monomorphism. This proves that N is flat.

LEMMA 5. *Let A be filtered ring and let $N \in \text{filt}_A\mathfrak{M}$. Suppose that F_*A and F_*N are discrete and exhaustive. Then*

$$\text{l.w.dim}_A N \leqq \text{gr.l.w.dim}_{E(A)} E(N)$$

PROOF. Suppose that the right-hand side is $\leqq n$. Let

$$0 \to L \xrightarrow{f} L_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} L_1 \xrightarrow{f_1} L_0 \xrightarrow{f_0} N \to 0$$

be a strict exact sequence, where $L_0,\ldots,L_{n-1}$ are filt-free and $F_*L_0,\ldots,$ F_*L_{n-1}, F_*L are discrete and exhaustive. This gives us an exact sequence

$$0 \to E(L) \xrightarrow{E(f)} E(L_{n-1}) \to \cdots \to E(L_1) \xrightarrow{E(f_1)} E(L_0) \xrightarrow{E(f_0)} E(N) \to 0$$

in $\text{gr}_{E(A)}\mathfrak{M}$, where $E(L_0),\ldots,E(L_{n-1})$ are gr-free and consequently gr-flat. Thus $E(L)$ is gr-flat, which by lemma 4 implies that L is flat. Consequently $\text{l.w.dim}_A N \leqq n$ and the proof is complete.

THEOREM 1. *Let A be a discretely and exhaustively filtered ring. Then*

$$\text{gl.w.dim} A \leqq \text{gr.gl.w.dim} E(A)$$

PROOF. Suppose that the right-hand side is $\leqq n$. Let $N \in {}_A\mathfrak{M}$. Filter N by $F_pN = F_pA\cdot N$. Then $N \in \text{filt}_A\mathfrak{M}$ and F_*N is discrete and exhaustive. Thus $\text{l.w.dim}_A N \leqq \text{gr.l.w.dim}_{E(A)} E(N) \leqq n$ and it follows that $\text{gl.w.dim} A \leqq n$ and this concludes the proof.

Now let us consider a graded ring B. Let $M \in \text{gr}\,\mathfrak{M}_B$, $N \in \text{gr}_B\mathfrak{M}$. Then $M\otimes_B N$ is a graded $\mathbf{Z}$-module (this is due to the fact that there is an exact sequence

$$M \otimes_{\mathbf{Z}} B \otimes_{\mathbf{Z}} N \xrightarrow{\varphi} M \otimes_{\mathbf{Z}} N \to M \otimes_B N \to 0$$

where $\varphi(m \otimes_{\mathbf{Z}} b \otimes_{\mathbf{Z}} n) = mb \otimes_{\mathbf{Z}} n - m \otimes_{\mathbf{Z}} bn$). Give B, M, N the associated filtrations i.e. $F_pB = \oplus_{n\leqq p} B^n$ etc. Note that $(M \otimes_B N)^q$ is the $\mathbf{Z}$-submodule of $M \otimes_B N$ generated by elements of type $m \otimes_B n$, where $m \in M^s$, $n \in N^t$ and $s+t=q$. Thus the associated filtration of $M \otimes_B N$ is given by letting $F_p(M \otimes_B N)$ be the $\mathbf{Z}$-submodule of $M \otimes_B N$ generated by elements of type $m \otimes_B n$, where $m \in M^s$, $n \in N^t$ and $s+t \leqq p$. But this submodule $F_p(M \otimes_B N)$ is then also the Z-submodule of $M \otimes_B N$ generated by elements of type $m \otimes_B n$, where $m \in F_s M$, $n \in F_t N$ and $s+t \leqq p$ that is, the earlier introduced filtration, induced by F_*M, F_*N, coincides with the associated filtration. Note that the associated filtration is exhaustive and separated for any graded module. In particular $F_*(M \otimes_B N)$ is exhaustive and separated. If we re-examine the proof of lemma 4 we find that in order to conclude that $i \otimes_A 1$ is a monomorphism we really only need $F_*(J \otimes_A N)$ to be separated (instead of discrete). Furthermore it is clear that $N \in \mathrm{gr}_B\mathfrak{M}$ is gr-flat iff $J \otimes_B N \to B \otimes_B N$ is a monomorphism for each graded (even f.g. graded) right ideal J. Furthermore, "$E(B)=B$, $E(M)=M$, $E(N)=N$" as graded objects.

All this adds up to,

LEMMA 6. *Let B be a graded ring and let $N \in \mathrm{gr}_B\mathfrak{M}$. Then N is* gr-*flat iff it is flat.*

In the same way as before we obtain

THEOREM 2. *Let B be a graded ring and let $N \in \mathrm{gr}_B\mathfrak{M}$. Then*

$$\mathrm{gr.l.w.dim}_B N = \mathrm{l.w.dim}_B N ,$$

$$\mathrm{gr.gl.w.dim} B \leqq \mathrm{gl.w.dim} B$$

with equality also on the last line if B^* is discrete.

PROOF. The inequalities $\leqq$ are both trivial. The reverse inequality on the first line follows from lemma 6 using a gr-flat resolution of N. On the second line the reverse inequality follows from theorem 1 when B^* is discrete since then the associated filtration of B is discrete.

In analogy with [7] we also obtain,

THEOREM 3. *Let B be a positively graded ring. Suppose that* $\mathrm{gl.w.dim} B^0 = 0$ *that is B^0 is von-Neumann regular. Then*

$$\mathrm{gl.w.dim} B = \mathrm{l.w.dim}_B B^0 = \mathrm{r.w.dim}_B B^0 .$$

PROOF. Let $M \in {}_B\mathfrak{M}$ and give B the associated filtration. Filter M by $F_p M = 0$ for $p < 0$ and $F_p M = M$ for $p \geqq 0$. Then $E^p(M) = 0$ for $p \neq 0$ and hence $E(M)$ is annihilated by the ideal $J = \oplus_{n>0} B^n$, which shows that we can consider $E(M)$ as a module over $B^0 = B/J$. By lemma 5 and [3, exercise 5 page 360] we have

$$\text{l.w.dim}_B M \leqq \text{l.w.dim}_B E(M) \leqq \text{l.w.dim}_B B^0 + \text{l.w.dim}_{B^0} E(M) = \\ = \text{l.w.dim}_B B^0 .$$

Since M is arbitrary we get $\text{gl.w.dim}\, B = \text{l.w.dim}_B B^0$ and by symmetry also $\text{gl.w.dim}\, B = \text{r.w.dim}_B B^0$.

3. Direct sums, suspensions and the natural transformation $\varkappa$.

In the following A is supposed to be a filtered ring.

LEMMA 7. *Let*

$$f_\alpha \colon M_\alpha \to M, \quad \alpha \in I \tag{1}$$

be morphisms in $\text{filt}_A\mathfrak{M}$. *Then* (1) *is a direct sum system in* $\text{filt}_A\mathfrak{M}$ *iff it is a direct sum system in* ${}_A\mathfrak{M}$ *and*

$$F_p M = \textstyle\sum_\alpha f_\alpha(F_p M_\alpha) .$$

PROOF. Suppose that (1) is a direct sum system in ${}_A\mathfrak{M}$ and

$$F_p M = \textstyle\sum_\alpha f_\alpha(F_p M_\alpha) .$$

Let $g_\alpha \colon M_\alpha \to N$, $\alpha \in I$ be morphisms in $\text{filt}_A\mathfrak{M}$. Then there is a unique morphism $g \colon M \to N$ such that $gf_\alpha = g_\alpha$ for each $\alpha \in I$. But

$$gF_p M = g \textstyle\sum_\alpha f_\alpha F_p M_\alpha = \sum_\alpha g_\alpha F_p M_\alpha \subset F_p N .$$

Thus (1) is a direct sum system in $\text{filt}_A\mathfrak{M}$. Now assume that (1) is a direct sum system in $\text{filt}_A\mathfrak{M}$. Let $g_\alpha \colon M_\alpha \to N$, $\alpha \in I$ be a direct sum system in ${}_A\mathfrak{M}$. Filter N by

$$F_p N = \textstyle\sum_\alpha g_\alpha F_p M_\alpha .$$

Clearly this makes N an object in $\text{filt}_A\mathfrak{M}$ and it follows from what we have already proved that it is a direct sum system in $\text{filt}_A\mathfrak{M}$. Thus there is a unique isomorphism $f \colon N \to M$ in $\text{filt}_A\mathfrak{M}$ such that $fg_\alpha = f_\alpha$ for each $\alpha \in I$ and this implies that

$$F_p M = \textstyle\sum_\alpha f_\alpha F_p M_\alpha .$$

Note that the proof above at the same time shows that $\text{filt}_A\mathfrak{M}$ has arbitrary direct sums.

LEMMA 8. *Let*

$$f_\alpha \colon M_\alpha \to M, \qquad \alpha \in I$$

be a direct sum system in $\text{filt}_A\mathfrak{M}$. *Then*

$$E(f_\alpha): E(M_\alpha) \to E(M), \quad \alpha \in I$$

is a direct sum system in $\mathrm{gr}_{E(A)}\mathfrak{M}$ *(and thus also in* ${}_{E(A)}\mathfrak{M}$*).*

PROOF. Let $m^p \in E^p(M)$. Then $m \in F_pM$ and $m = \sum_\alpha f_\alpha m_\alpha$, where $m_\alpha \in F_pM_\alpha$ and all but a finite number is 0. Consequently

$$m^p = (\textstyle\sum_\alpha f_\alpha m_\alpha)^p = \sum_\alpha (f_\alpha m_\alpha)^p = \sum_\alpha E(f_\alpha)m_\alpha{}^p .$$

On the other hand suppose that $E(f_\alpha)m_\alpha{}^p = 0$. Then

$$f_\alpha m_\alpha \in F_{p-1}M = \textstyle\sum_\beta f_\beta F_{p-1}M_\beta$$

and it follows that $f_\alpha m_\alpha = \sum_\beta f_\beta n_\beta$, where $n_\beta \in F_{p-1}M_\beta$. Then $m_\beta = 0$ for $\beta \neq \alpha$ and $m_\alpha = n_\alpha \in F_{p-1}M_\alpha$. Thus we get $m_\alpha{}^p = 0$, which concludes the proof.

DEFINITION. Let $M \in \mathrm{filt}_A\mathfrak{M}$. Then the nth suspension $s^nM \in \mathrm{filt}_A\mathfrak{M}$ is defined by $s^nM = M$ as A-module but with filtration

$$F_p s^n M = F_{p+n}M .$$

Similarly if B is a graded ring and $N \in \mathrm{gr}_B\mathfrak{M}$ then the nth suspension s^nN is N as a B-module but with grading $(s^nM)^p = M^{p+n}$.

It is clear that $L \in \mathrm{filt}_A\mathfrak{M}$ is filt-free iff L is the direct sum in $\mathrm{filt}_A\mathfrak{M}$ of suspensions of A considered as a filtered left A-module. Furthermore, if $\{(x_\alpha, p(\alpha))\}_{\alpha\in I}$ is a filt-basis for L then $L = \bigoplus_{\alpha\in I} s^{-p(\alpha)}A$ in $\mathrm{filt}_A\mathfrak{M}$. We have $E(s^nM) = s^nE(M)$ and

$$s^mM \otimes_A s^nN = s^{m+n}(M \otimes_A N)$$

in $\mathrm{filt}_Z\mathfrak{M}$ (or $\mathrm{gr}_Z\mathfrak{M}$) if A is a filtered (or graded) ring. If $f: M \to N$ is a morphism of filtered (graded) modules then we get a morphism $s^nf: s^nM \to s^nN$, where $s^nf = f$ as a morphism of modules. It is clear that under these identifications we also have

$$\varkappa(s^mM, s^nN) = s^{m+n}\varkappa(M,N) .$$

In particular $\varkappa(s^mM, s^nN)$ is an isomorphism iff $\varkappa(M,N)$ is an isomorphism.

Now suppose that $f_\alpha: M_\alpha \to M$, $\alpha \in \mathfrak{A}$ and $g_\beta: N_\beta \to N$, $\beta \in \mathfrak{B}$ are direct sum systems in $\mathrm{filt}\,\mathfrak{M}_A$ and $\mathrm{filt}_A\mathfrak{M}$ respectively. This gives us a direct sum system

$$f_\alpha \otimes_A g_\beta: M_\alpha \otimes_A N_\beta \to M \otimes_A N$$

in ${}_Z\mathfrak{M}$ and $f_\alpha \otimes g_\beta$ are morphisms in $\mathrm{filt}_Z\mathfrak{M}$. Let $x \in F_p(M \otimes_A N)$. Then

$$x = \textstyle\sum_{i\in I} m^{(i)} \otimes_A n^{(i)} ,$$

where $m^{(i)} \in F_{s(i)}M$, $n^{(i)} \in F_{t(i)}N$ and $s(i)+t(i)=p$. Consequently

$$m^{(i)} = \sum_\alpha f_\alpha m_\alpha^{(i)} ,$$

where $m_\alpha^{(i)} \in F_{s(i)}M_\alpha$ and $n^{(i)}=\sum_\beta g_\beta n_\beta^{(i)}$, where $n_\beta^{(i)} \in F_{t(i)}N_\beta$. Thus

$$m_\alpha^{(i)} \otimes_A n_\beta^{(i)} \in F_p(M_\alpha \otimes_A N_\beta)$$

and

$$x = \sum_i m^{(i)} \otimes_A n^{(i)} = \sum_{\alpha,\beta} f_\alpha \otimes g_\beta \sum_i m_\alpha^{(i)} \otimes_A n_\beta^{(i)} ,$$

which shows that

$$F_p(M \otimes_A N) = \sum_{\alpha,\beta} f_\alpha \otimes_A g_\beta F_p(M_\alpha \otimes_A N_\beta) .$$

We have proved,

LEMMA 9. *Let* $f_\alpha \colon M_\alpha \to M$, $\alpha \in \mathfrak{A}$ and $g_\beta \colon N_\beta \to N$, $\beta \in \mathfrak{B}$ *be direct sum systems in* filt $\mathfrak{M}_A$, filt$_A\mathfrak{M}$ *respectively. Then*

$$f_\alpha \otimes_A g_\beta \colon M_\alpha \otimes_A N_\beta \to M \otimes_A N, \quad (\alpha,\beta) \in \mathfrak{A}\times\mathfrak{B}$$

is a direct sum system in filt$_Z\mathfrak{M}$.

Let the situation be as in lemma 7 and consider the commutative diagram

$$\begin{array}{ccc} E(M_\alpha) \otimes_{E(A)} E(N_\beta) & \xrightarrow{\varkappa(M_\alpha, N_\beta)} & E(M_\alpha \otimes_A N_\beta) \\ \big\downarrow {\scriptstyle E(f_\alpha)\otimes_{E(A)}E(g_\beta)} & & \big\downarrow {\scriptstyle E(f_\alpha \otimes_A g_\beta)} \\ E(M) \otimes_{E(A)} E(N) & \xrightarrow{\varkappa(M,N)} & E(M \otimes_A N) \end{array}$$

for $\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$. By lemmas 8 and 9 it follows that both systems of vertical arrows are direct sum systems. Hence $\varkappa(M,N) = \bigoplus \varkappa(M_\alpha, N_\beta)$ is an isomorphism if every $\varkappa(M_\alpha, N_\beta)$ is an isomorphism. If we combine this result with lemma 3 we get,

LEMMA 10. *If either* $M \in$ filt $\mathfrak{M}_A$ *or* $N \in$ filt$_A\mathfrak{M}$ *is* filt-*free then* $\varkappa(M,N)$ *is an isomorphism.*

Lemma 10 gives us an alternative proof of lemma 5 as follows: Let

$$\to L_2 \to L_1 \to L_0 \to N \to 0$$

be a strict filt-free resolution of N, where every F_*L_n is discrete and exhaustive. Let $M \in$ filt $\mathfrak{M}_A$. We get a commutative diagram

$$\begin{array}{ccccccc} \to E(M)\otimes_{E(A)}E(L_2) & \to & E(M)\otimes_{E(A)}E(L_1) & \to & E(M)\otimes_{E(A)}E(L_0) & \to & 0 \\ \text{iso}\downarrow\varkappa & & \text{iso}\downarrow\varkappa & & \text{iso}\downarrow\varkappa & & \\ \to E(M\otimes_A L_2) & \to & E(M\otimes_A L_1) & \to & E(M\otimes_A L_0) & \to & 0 \end{array}$$

which implies that

$$\begin{aligned} E_1{}^{*,n}(M\otimes_A L_*) &= H_n E(M\otimes_A L_*) = H_n(E(M)\otimes_{E(A)}E(L_*)) \\ &= \mathrm{Tor}^{E(A)}{}_n(E(M),E(N)) . \end{aligned}$$

Thus $\mathrm{Tor}^{E(A)}{}_n(E(M),E(N))=0$ yields $E_1{}^{*,n}(M\otimes_A L)=0$, whence

$$\mathrm{Tor}^A{}_n(M,N) = H_n(M\otimes_A L_*) = 0$$

if $F_*(M\otimes_A L_*)$ is exhaustive and complete (cf. [4, page 18]). In our particular case we even have $F_*(M\otimes_A L_*)$ exhaustive and discrete.

We have proved,

Lemma 11. *Suppose that F_*A is discrete and exhaustive. Let $M\in \mathrm{filt}\,\mathfrak{M}_A$ and $N\in \mathrm{filt}_A\mathfrak{M}$, where F_*M and F_*N are discrete and exhaustive. Then* $\mathrm{Tor}^{E(A)}{}_n(E(M),E(N))=0$ *implies that* $\mathrm{Tor}^A{}_n(M,N)=0$.

Now lemma 5 follows easily from lemma 11 since we can filter any $M\in\mathfrak{M}_A$ by putting $F_pM=M\cdot F_pA$.

The following result implies proposition 7 in [8] by considering K as a filtered ring with $F_pK=0$ for $p<0$ and $F_pK=K$ for $p\geqq 0$.

Lemma 12. *Let A be a filtered ring and let $M\in\mathrm{filt}\,\mathfrak{M}_A$, $N\in\mathrm{filt}_A\mathfrak{M}$. Suppose that F_*A, F_*M, F_*N are all discrete and exhaustive. Then $\varkappa(M,N)$ is an isomorphism if either $E(M)$ or $E(N)$ is flat (which is equivalent to* gr-*flat by lemma* 5).

Proof. Assume that $E(M)$ is flat. Let

$$0\to S\xrightarrow{f} L\xrightarrow{g} N\to 0 \tag{1}$$

be strict exact, where L is filt-free and all filtrations are discrete and exhaustive. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0\to & E(M)\otimes_{E(A)}E(S) & \xrightarrow{1\otimes_{E(A)}E(f)} & E(M)\otimes_{E(A)}E(L) & \xrightarrow{1\otimes_{E(A)}E(g)} & E(M)\otimes_{E(A)}E(N) & \to 0 \\ & \text{epi}\downarrow\varkappa(M,S) & & \text{iso}\downarrow\varkappa(M,L) & & \text{epi}\downarrow\varkappa(M,N) & \\ 0\to & E(M\otimes_A S) & \xrightarrow{E(1\otimes_A f)} & E(M\otimes_A L) & \xrightarrow{E(1\otimes_A g)} & E(M\otimes_A N) & \to 0 \end{array}$$

Since (1) is strict exact and $E(M)$ is flat it follows that the upper row is exact. Hence $E(1 \otimes_A f)$ is a monomorphism and $E(1 \otimes_A g)$ is an epimorphism. Now $F_*(M \otimes_A S)$, $F_*(M \otimes_A L)$, $F_*(M \otimes_A N)$ are all discrete and exhaustive. Thus by lemma 1 (b) (with $L=0$) $1 \otimes_A f$ is strict and by lemma 1 (c) or (d) (with $N=0$) $1 \otimes_A g$ is strict.

By lemma 4 M is flat so that

$$0 \to M \otimes_A S \xrightarrow{1 \otimes_A f} M \otimes_A L \xrightarrow{1 \otimes_A g} M \otimes_A N \to 0$$

is strict exact. Thus the lower row above is exact by lemma 1 (a). The 5-lemma now shows that $\varkappa(M,N)$ is an isomorphism.

The proof that $\varkappa(M,N)$ is an isomorphism when $E(N)$ is flat is of course analogous.

4. Injectivity.

Let $M,N \in \mathrm{filt}_A\mathfrak{M}$ and filter $\mathbf{Hom}_A(M,N)$ by letting $F_p\mathbf{Hom}_A(M,N)$ be the homomorphisms $M \to N$ of filt-degree p (cf. [4, page 19] and [6]). This gives us a functor

$$\mathrm{filt}_A\mathfrak{M} \times \mathrm{filt}_A\mathfrak{M} \to \mathrm{filt}_Z\mathfrak{M} .$$

It is clear that $F_*\,\mathbf{Hom}_A(M,N)$ is exhaustive. However, we need not have $\mathbf{Hom}_A(M,N) = \mathrm{Hom}_A(M,N)$ even if M is filt-free and $N=A$ is discrete and exhaustive (let M has a filt-basis $\{(x_i,0)\}_{1 \le i < \infty}$ and let $f\colon M \to A$ be given by $f(x_i)=a_i$, where $a_i \notin F_iA$, which is possible if $A \neq F_pA$ for every p). We have,

Lemma 13. *Let $M,N \in \mathrm{filt}_A\mathfrak{M}$. Then*

(a) *If F_*M is exhaustive and F_*N is separated then $F_*\,\mathbf{Hom}_A(M,N)$ is separated.*

(b) *If F_*M is exhaustive and F_*N is discrete then $F_*\,\mathbf{Hom}_A(M,N)$ is discrete.*

(c) *If F_*M is exhaustive and F_*N is complete then $F_*\,\mathbf{Hom}_A(M,N)$ is complete.*

Proof. (a) Suppose that $f \in \bigcap_p F_p\mathbf{Hom}_A(M,N)$. Then $fF_pM \subset \bigcap_s F_{p+s}N = 0$. Thus $fM = f\bigcup_p F_pM = 0$. The same argument proves (b).

(c) We have to show that the projective system

$$p>q: \qquad \begin{array}{ccc} & \mathbf{Hom}_A(M,N) & \\ {\scriptstyle\varphi_p}\swarrow & & \searrow{\scriptstyle\varphi_q} \\ \mathbf{Hom}_A(M,N)/F_p\mathbf{Hom}_A(M,N) & \xrightarrow{\varphi_q{}^p} & \mathbf{Hom}_A(M,N)/F_q\mathbf{Hom}_A(M,N) \end{array}$$

represents $\mathbf{Hom}_A(M,N)$ as $\operatorname{proj\,lim} \mathbf{Hom}_A(M,N)/F_p\mathbf{Hom}_A(M,N)$.

Suppose that we are given $f_p \in \mathbf{Hom}_A(M,N)/F_p\mathbf{Hom}_A(M,N)$, for each $p \in \mathsf{Z}$, such that $\varphi_q{}^p f_p = f_q$ when $p > q$. Then we have to show that there is a unique $f \in \mathbf{Hom}_A(M,N)$ such that $\varphi_q f = f_q$, for each $q \in \mathsf{Z}$. Since each φ_p is an epimorphism we have $f_p = \varphi(g_p)$ where $g_p \in \mathbf{Hom}_A(M,N)$. The uniqueness of f, if it exists, follows from (a). As for existence, let g_p be given as above. Then $g_p - g_q \in F_q\mathbf{Hom}_A(M,N)$ for $q > p$ and consequently $(g_p - g_q)F_sM \subset F_{s+q}N$ for each integer s. Let $x \in M$. We may assume that $x \in F_tM$, for some t, since F_*M is exhaustive. Then

$$g_p(x) - g_q(x) = (g_p - g_q)(x) \in F_{t+q}N ,$$

which shows that $(g_p(x))$ is a Cauchy-sequence in N. Since N is complete we may define a function $f: M \to N$ by

$$f(x) = \lim_{p\to\infty} g_p(x) .$$

It is clear that $f \in \mathrm{Hom}_A(M,N)$. Note that for $x \in F_tM$

$$g_p(x) \in g_q(x) + F_{t+q}N, \quad p \geqq q ,$$

whence $f(x) \in g_q(x) + F_{t+q}N$. Thus $(f - g_q)F_tM \subset F_{t+q}N$ so that $f - g_q \in F_q\mathbf{Hom}_A(M,N)$. We conclude that $f \in \mathbf{Hom}_A(M,N)$ and that $\varphi_q f = f_q$ for every integer q.

Definition. Let $M \in \mathrm{filt}_A\mathfrak{M}$. We say that M is filt f.g. if there exists a finite set $\{(x_i, p(i))\}_{1 \leqq i \leqq n}$, where $x_i \in M$ and $p(i)$ are integers, such that

$$F_pM = \sum_{s+p(i) \leqq p} F_sA \cdot x_i .$$

In particular, every f.g. filt-free object in $\mathrm{filt}_A\mathfrak{M}$ is filt f.g.

Following [2] page 41 we get,

Lemma 14. *Let $M \in \mathrm{filt}_A\mathfrak{M}$. Assume that F_*A is complete and F_*M is separated and exhaustive. Then M is* tilt f.g. *if $E(M)$ is* f.g.

Proof. Let $\{x_i{}^{p(i)}\}_{1 \leqq i \leqq n}$ be a homogeneous set of generators of $E(M)$. Let L be filt-free with filt-basis $\{(y_i, p(i))\}_{1 \leqq i \leqq n}$. Define $f: L \to M$ by $f(y_i) = x_i$. Then $E(f)$ is an epimorphism. By lemma 2(f) F_*L is complete and thus, by lemma 1(e) (with $N = 0$), f is an epimorphism, which shows that M is filt f.g.

Lemma 15. *Let $M,N \in \mathrm{filt}_A\mathfrak{M}$. Assume that M is filt* f.g. *and F_*M is exhaustive. Then*

$$\mathrm{Hom}_A(M,N) = \mathbf{Hom}_A(M,N) .$$

PROOF. Let $\{(x_i, p(i))\}_{1 \leqq i \leqq n}$ be as in the definition of filt f.g. and let $f \in \mathrm{Hom}_A(M, N)$. Then there is an integer s, such that $f(x_i) \in F_{p(i)+s}N$ for $1 \leqq i \leqq n$. It is then obvious that

$$f \in F_s \mathbf{Hom}_A(M, N) \subset \mathbf{Hom}_A(M, N) .$$

We can construct a natural transformation (cf. [6])

$$\varphi = \varphi(M, N) \colon E\mathbf{Hom}_A(M, N) \to \mathbf{Hom}_{E(A)}(E(M), E(N))$$

by $\varphi(f^p)(x^q) = f(x)^{p+q}$, between functors from $\mathrm{filt}_A\mathfrak{M} \times \mathrm{filt}_A\mathfrak{M}$ to $\mathrm{gr}_Z\mathfrak{M}$.

LEMMA 16. *Let $\varphi(M, N)$ be as above. Then $\varphi(M, N)$ is always a monomorphism and it is an isomorphism if M is* filt-*free.*

PROOF. Suppose that $\varphi(f^p) = 0$. Then $f(x)^{p+q} = 0$ for every $x \in F_qM$ and every integer q and hence $fF_qM \subset F_{p+q-1}N$, for every integer q. Thus $f \in F_{p-1}\mathbf{Hom}_A(M, N)$, which shows that $f^p = 0$.

Now assume that $\{(x_i, p(i))\}_{i \in I}$ is a filt-basis in M. Then $\{x_i^{p(i)}\}_{i \in I}$ is a homogeneous basis in $E(M)$. Thus, if $g \in \mathbf{Hom}_{E(A)}(E(M), E(N))^p$, then we have

$$g x_i^{p(i)} = y_i^{p(i)+p}$$

so that if we define $f \colon M \to N$ by $f(x_i) = y_i$ then $\varphi(f^p) = g$.

LEMMA 17. *Suppose that B is a graded ring. Let $N \in \mathrm{gr}_B\mathfrak{M}$. Then the following statements are equivalent,*

(a) *N is* gr-*injective (i.e. injective in* $\mathrm{gr}_B\mathfrak{M}$).

(b) $\mathrm{hom}_B(\cdot, N)$ *is exact.*

(c) $\mathbf{Hom}_B(\cdot, N)$ *is exact.*

(d) $\mathbf{Hom}_B(i, 1) \colon \mathbf{Hom}_B(B, N) \to \mathbf{Hom}_B(J, N)$ *is an epimorphism for every homogeneous left ideal J in B.*

PROOF. Obviously (a) and (b) are equivalent and (c) implies (b) and (d). That (b) implies (c) follows by using suspensions, noting that

$$\mathbf{Hom}_B(M, N)^p = \mathrm{hom}_B(s^{-p}M, N) .$$

Finally (a) follows from (d) as in [3, page 9].

LEMMA 18. *Let B be a graded ring and let $N \in \mathrm{gr}_B\mathfrak{M}$. Then N is* gr-*injective if N is injective.*

PROOF. Consider a diagram

$$\begin{array}{ccccc} & & N & & \\ & & \uparrow{\scriptstyle f} & & \\ 0 & \longrightarrow & M & \xrightarrow{g} & K \end{array}$$

in $\operatorname{gr}_B\mathfrak{M}$, with exact row. Then there is an $h \in \operatorname{Hom}_B(L,N)$ such that $hg=f$. Let $H\colon K \to N$ be given by $H(k_n)=$ the component in N_n of $h(k_n)$, where $k_n \in K_n$. Then $H \in \operatorname{gr}_B\mathfrak{M}$ and $Hg=f$. Thus N is gr-injective.

THEOREM 4. *Let $N \in \operatorname{filt}_A\mathfrak{M}$, where F_*A is exhaustive and F_*N complete. Suppose that $E(N)$ is* gr-*injective. Then*

$$\mathbf{Hom}_A(i,1)\colon \operatorname{Hom}_A(A,N) = \mathbf{Hom}_A(A,N) \to \mathbf{Hom}_A(J,N)$$

is an epimorphism for every left ideal J of A i.e. every homomorphism $f\colon J \to N$ of finite filt-*degree can be extended to a homomorphism $A \to N$.*

PROOF. Filter J by $F_pJ=J\cap F_pA$ so that $i\colon J \to A$ is a strict monomorphism. Consider the commutative diagram

$$\begin{array}{ccc} E\,\mathbf{Hom}_A(A,N) & \xrightarrow{E\mathbf{Hom}_A(i,1)} & E\,\mathbf{Hom}_A(J,N) \\ {\scriptstyle \varphi(A,N)}\Big\downarrow{\scriptstyle \text{iso}} & & {\scriptstyle \text{mono}}\Big\downarrow{\scriptstyle \varphi(J,N)} \\ \mathbf{Hom}_{E(A)}(E(A),E(N)) & \xrightarrow{\mathbf{Hom}_{E(A)}(E(i),1)} & \mathbf{Hom}_{E(A)}(E(J),E(N)) \end{array}$$

$E(i)$ is mono and $E(N)$ is gr-injective. Thus $\mathbf{Hom}_{E(A)}(E(i),1)$ is epi. Hence $\varphi(J,N)$ is an epimorphism and therefore also an isomorphism. It follows that $E\,\mathbf{Hom}_A(i,1)$ is an epimorphism. Now $F_*\mathbf{Hom}_A(A,N)$ and $F_*\mathbf{Hom}_{(A)}(J,N)$ are exhaustive and complete by lemma 13(c). Thus $\mathbf{Hom}_A(i,1)$ is an epimorphism by lemma 1(e) (with $N=0$).

DEFINITION. Let C be a ring. Then $N \in {}_C\mathfrak{M}$ is said to be principal-injective if every homomorphism

$$f\colon Cx \to N, \quad x \in C$$

can be extended to C.

THEOREM 5. *Let $N \in \operatorname{filt}_A\mathfrak{M}$, where F_*A is exhaustive, F_*N complete and $F_pN=N$ for $p \geqq t$. Suppose that N is principal-injective and $E(N)$ is* gr-*injective. Then N is injective.*

PROOF. Let J be a left ideal in A and give J the induced filtration. Suppose that $f: J \to N$ is a homomorphism. By theorem 4 it is sufficient to prove that f is of finite filt-degree. Assume the contrary. Then there is an $a \in F_p J$ such that $f(a) \notin F_{p+t}N$. Consider $f/Ca: Ca \to N$. This homomorphism has an extension $A \to N$ and consequently $f(a) = ac$ for some $c \in N = F_t N$. Thus $f(a) \in F_{p+t}N$ which is a contradiction.

REMARKS. (a) A filtration which stops as above is obtained e.g. if one filters with ideals.

(b) By lemmas 14, 15 and theorem 4 it follows that if F_*A is exhaustive and complete and $E(A)$ is noetherian and self-injective that is a QF-ring then A is a QF-ring (thus F_*A will automatically be discrete if $F_0 A = A$ since A is artinian). In this way J.-E. Roos in [6] gives a very short proof of the main result in [1]. At the same time he strenghtens this, using a spectral sequence, to:

$E(A)$ noetherian, F_*A exhaustive and discrete yields

$$\text{l.inj.dim}_A A \leqq \text{gr.l.inj.dim}_{E(A)} E(A) .$$

This works even if F_*A is only assumed to be complete instead of discrete.

(c) Let B be any non-left-noetherian ring. Grade B by $B^0 = B$ and $B^n = 0$ for $n \neq 0$. Let $J_1 \subset J_2 \subset \ldots$ be a strictly increasing sequence of left ideals in B and put $J = \bigcup_{1 \leqq i < \infty} J_i$. Define $M \in \text{gr}_B \mathfrak{M}$ by

$$M = \bigoplus_{1 \leqq i < \infty} I(J/J_i) ,$$

where $I(J/J_i)$ is the injective envelope of J/J_i in ${}_B\mathfrak{M}$ (as a matter of fact any injective module in ${}_B\mathfrak{M}$ containing J/J_i will do). Let $f: J \to M$ be given by

$$f(j) = \bigoplus_{1 \leqq i < \infty} (j + J_i) .$$

Then f is well-defined, since $j + J_i = 0$ for $i \geqq n(j)$, and f is a homomorphism in ${}_B\mathfrak{M}$. This homomorphism has no extension $F: B \to M$ since then we would have

$$F(1) \in \bigoplus_{1 \leqq i < n} E(J/J_i)$$

and thus $f(J) \subset \bigoplus_{1 \leqq i < n} E(J/J_i)$, whence

$$f(J) \subset \bigoplus_{1 \leqq i < n} J/J_i .$$

This implies $J \subset J_n$, which is a contradiction. Consequently M is not injective. Now let L be a left ideal in B and let $g \in \mathbf{Hom}_B(L, M)^p$ that is $g: L \to E(J/J_p)$. Since $E(J/J_p)$ is injective there is an extension $G: B \to E(J/J_p)$ of g, which shows that M is gr-injective. Hence M is a positively graded ring which is gr-injective but not injective. If we give B and M

the associated filtrations we get a discretely and exhaustively filtered ring B and a discretely and exhaustively filtered $M \in \mathrm{filt}_B\mathfrak{M}$ such that $E(M)$ is gr-injective but M is not injective.

5. More about direct sums.

We have the following slight strengthening of [5, page 72].

LEMMA 19. *Let A be a complete topological ring, which has a fundamental neighborhood system of 0 consisting of additive subgroups. Let $\varphi: A \to B$ be a ring-homomorphism such that every $x \in \ker\varphi$ is topologically nilpotent that is $x^n \to 0$ when $n \to \infty$. Then every idempotent in $\operatorname{im}\varphi$ can be lifted to A.*

PROOF. Let $b \in \operatorname{im}\varphi$ be an idempotent. Suppose that $b = \varphi(a)$. Thus $\varphi(a^2 - a) = 0$ so that $n = a^2 - a \in \ker\varphi$. Put

$$c = a + d(1-2a)$$

where d is to be determined (note that $1-2a = f'(a)$, where $f(x) = x - x^2$). Suppose that d commutes with a. Then $c^2 = c$ is equivalent to

$$(d^2 - d)(1+4n) + n = 0 \tag{1}$$

which has the formal solution

$$d = \tfrac{1}{2}(1-(1+4n)^{-\frac{1}{2}}) = \tfrac{1}{2}\textstyle\sum_{1\leq k<\infty}(-1)^{k-1}\binom{2k}{k}n^k .$$

Now $\frac{1}{2}\binom{2k}{k}$ is an integer since $\binom{2k}{k}$ is the coefficient of $1^k(-1)^k$ in the binomial expansion of $(1-1)^{2k} = 0$ and the other terms occur pairwise and thus have an even sum. It follows that the expression obtained for d is a power-series in n with integer coefficients and since any sequence $a_k n^k$, where a_k are integers, tends to 0 we conclude that the power-series converges. We find that d given by this series really commutes with a and hence satisfies (1). Consequently $c = a + d(1-2a)$ is an idempotent and since

$$d = n\textstyle\sum_{1\leq k<\infty}\tfrac{1}{2}(-1)^{k-1}\binom{2k}{k}n^{k-1} \in \ker\varphi$$

we conclude that $c - a \in \ker\varphi$. This shows that c lifts a.

LEMMA 20. *Let A be a filtered ring and let $M \in \mathrm{filt}_A\mathfrak{M}$, where F_*M is exhaustive and complete. Assume that $f: M \to M$ is a homomorphism in $\mathrm{filt}_A\mathfrak{M}$ such that $E(f)^2 = E(f)$. Then there is a homomorphism $g: M \to M$ in $\mathrm{filt}_A\mathfrak{M}$ such that $g^2 = g$ and $E(g) = E(f)$. The homomorphism g is automatically strict.*

PROOF. Consider the mapping

$$\varphi\colon \hom_A(M,M) \to \hom_{E(A)}(E(M),E(M))$$

given by $\varphi(h)=E(h)$. Note that $\hom_A(M,M)$ and $\hom_{E(A)}(E(M),E(M))$ are rings with multiplication defined by composition and that φ is a ring-homomorphism. Filter $\hom_A(M,M)\subset \mathbf{Hom}_A(M,M)$ by the induced filtration. Since

$$\hom_A(M,M) = F_0\mathbf{Hom}_A(M,M)$$

we see that $\hom_A(M, M)$ is closed in $\mathbf{Hom}_A(M,M)$. Thus $\hom_A(M,M)$ is complete by lemma 13(c). Note that $\hom_A(M,M)$ is a topological ring since

$$f \in F_{-p}\hom_A(M,M), \quad g \in F_{-q}\hom_A(M,M)$$

implies that $fg \in F_{-p-q}\hom_A(M,M)$. Now

$$\ker\varphi = F_{-1}\hom_A(M,M)\,.$$

Let $h \in F_{-1}\hom_A(M,M)$. Then $h^nF_pM\subset F_{p-n}M$ that is

$$h^n \in F_{-n}\hom_A(M,M)$$

and we find that $h^n \to 0$ when $n\to\infty$. Hence, by lemma 19, every idempotent in $\operatorname{im}\varphi$ can be lifted. It only remains to prove that any idempotent g in $\hom_A(M,M)$ is strict. Suppose that $g(x)\in F_pM$. Then $g(g(x))=g(x)$ so that $g(x)\in gF_pM$ that is g is strict.

Let M be as in lemma 20. Then every $f\in F_{-1}\hom_A(M,M)$ is topologically nilpotent. Thus $1-f$ has an inverse $\sum_{0\leqq n<\infty}f^n$, which shows that

$$F_{-1}\hom_A(M,M) \subset \operatorname{Rad}\hom_A(M,M)\,.$$

Following [5, page 73], we obtain,

COROLLARY. *Let M be as in the previous lemma. Then any finite or countable orthogonal set of idempotents in*

$$\operatorname{im}\varphi \subset \hom_{E(A)}(E(M),E(M))$$

can be lifted to $\hom_A(M,M)$.

Note that φ is an epimorphism if M is filt-free. Furthermore we know that each gr-free object $N\in \operatorname{gr}_{E(A)}\mathfrak{M}$ equals $E(L)$ for some filt-free object $L\in \operatorname{filt}_A\mathfrak{M}$.

THEOREM 6. *Let A be a filtered ring, where F_*A is exhaustive. Let $\bar{P} \in \mathrm{gr}_{E(A)}\mathfrak{M}$ be projective (which is equivalent to* gr*-projective). Assume that either F_*A and $\bar{P}^*$ are discrete or $\bar{P}$ finitely generated and F_*A complete. Then there is a $P \in \mathrm{filt}_A\mathfrak{M}$ such that $E(P)=\bar{P}$. Furthermore, if $M \in \mathrm{filt}_A\mathfrak{M}$ then any homomorphism $\bar{g}\colon \bar{P} \to E(M)$ of degree p can be written $\bar{g}=$ "$E(g)$", where $g\colon P \to M$ is a homomorphism in ${}_A\mathfrak{M}$ of filt-degree p.*

PROOF. Let $\bar{L}$ be gr-free and let $\bar{f}\colon \bar{L} \to \bar{P}$ be an epimorphism in $\mathrm{gr}_{E(A)}\mathfrak{M}$. If $\bar{P}$ is f.g. we assume that $\bar{L}$ is f.g. There is a right inverse $\bar{\imath}\colon \bar{P} \to \bar{L}$ of f in $\mathrm{gr}_{E(A)}\mathfrak{M}$. Put $\bar{h}=\bar{\imath}\circ\bar{f}\colon \bar{L} \to \bar{L}$. Then $\bar{h}^2=\bar{h}$. We can assume that $\bar{L}=E(L)$, where L is filt-free and f.g. if $\bar{P}$ is. Since F_*A is exhaustive it follows that F_*L is exhaustive. If F_*A is complete and $\bar{P}$, and therefore $\bar{L}$, is f.g. then F_*L is complete. If F_*A and $\bar{P}^*$ are discrete then we may assume that F_*L is discrete. Thus in either case we may apply lemma 20 to find an idempotent $h \in \hom_A(L,L)$ such that $E(h)=\bar{h}$. Let $P=\mathrm{im}\, h$. We get

$$h\colon L \xrightarrow[\text{epi}]{f} P \xrightarrow[\text{mono}]{i} L \,.$$

Filter P by the filtration induced by f. Since h is strict we have

$$F_pP \;=\; fF_pL \;=\; hF_pL \;=\; \mathrm{im}\, h \cap F_pL \;=\; P \cap F_pL$$

that is the filtration on P is also induced by i. Thus f and i are strict and this gives us

$$E(h)\colon E(L) \xrightarrow[\text{epi}]{E(f)} E(P) \xrightarrow[\text{mono}]{E(i)} E(L)$$

which shows that we can identify $E(P)$ with $\bar{P}$ and put $\bar{f}=E(f)$, $\bar{\imath}=E(i)$.

Now suppose that $M \in \mathrm{filt}_A\mathfrak{M}$ and that $\bar{g}\colon E(P) \to E(M)$ is a homomorphism of degree p. By using a suitable suspension of M we may assume that $p=0$. We have

$$\bar{g} \;=\; \bar{g}\circ E(f)\circ E(i)\,.$$

But $\bar{g}\circ E(f)\colon E(L) \to E(M)$. Thus we can put $\bar{g}\circ E(f)=E(k)$, where $k \in \hom_A(L,M)$. It follows that $\bar{g}=E(ki)$, where $ki \in \hom_A(P,M)$.

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