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HARDY INEQUALITIES FOR LANDAU HAMILTONIAN AND FOR BAOUENDI-GRUSHIN OPERATOR WITH AHARONOV-BOHM TYPE MAGNETIC FIELD. PART I

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Abstract

In this paper we prove the Hardy inequalities for the quadratic form of the Laplacian with the Landau Hamiltonian type magnetic field. Moreover, we obtain a Poincaré type inequality and inequalities with more general families of weights. Furthermore, we establish weighted Hardy inequalities for the quadratic form of the magnetic Baouendi-Grushin operator for the magnetic field of Aharonov-Bohm type. For these, we show refinements of the known Hardy inequalities for the Baouendi-Grushin operator involving radial derivatives in some of the variables. The corresponding uncertainty type principles are also obtained.

1. Introduction

The purpose of this paper is to prove the weighted Hardy inequality for the quadratic form of the Landau Hamiltonian type and for the magnetic Baouendi-Grushin operator with Aharonov-Bohm type magnetic field. In Part II of this paper we investigate and present the corresponding Caffarelli-Kohn-Nirenberg inequalities for the Landau Hamiltonian and for the Baouendi-Grushin operator, with and without magnetic fields.

The classical Hardy inequality for functions $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ is

$$\int_{\mathbb{R}^n} |\nabla f(w)|^2 \, dw \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|f(w)|^2}{|w|^2} \, dw, \quad n \geq 3, \tag{1}$$

where the constant $((n-2)/2)^2$ is sharp but not attained. There exists a large literature concerning different versions of Hardy's inequalities and their applications. However, since we are interested in the inequalities associated with

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the Landau Hamiltonian and with the Baouendi-Grushin operator, let us only recall known results in these directions.

1.1. Baouendi-Grushin operator

The Hardy inequality (1) has been generalised for Baouendi-Grushin vector fields by Garofalo [10],

$$\int_{\mathbb{R}^n} |\nabla_x f|^2 + |x|^{2\gamma} |\nabla_y f|^2 \, dx \, dy \geq \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{R}^n} \left(\frac{|x|^{2\gamma}}{|x|^{2+2\gamma} + (1+\gamma)^2 |y|^2}\right) |f|^2 \, dx \, dy, \quad (2)$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$, with $n = m + k$, $m, k \geq 1$, $\gamma \geq 0$, $Q = m + (1+\gamma)k$ and $f \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^k \setminus \{(0,0)\})$. Here, $\nabla_x f$ and $\nabla_y f$ are the gradients of f in the variables x and y, respectively. The inequality (2) recovers (1) when $\gamma = 0$.

Let us put this result in perspective. Let $z = (x_1, \ldots, x_m, y_1, \ldots, y_k) = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$ with $k, m \geq 1$, $k + m = n$ and $\gamma \geq 0$. Let us consider the vector fields

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, m, \qquad Y_j = |x|^\gamma \frac{\partial}{\partial y_j}, \quad j = 1, \ldots, k.$$

The corresponding sub-elliptic gradient, which is the n-dimensional vector field, is then defined as

$$\nabla_\gamma := (X_1, \ldots, X_m, Y_1, \ldots, Y_k) = (\nabla_x, |x|^\gamma \nabla_y). \quad (3)$$

The Baouendi-Grushin operator on $\mathbb{R}^{m+k}$ is defined by

$$\Delta_\gamma = \sum_{i=1}^m X_i^2 + \sum_{j=1}^k Y_j^2 = \Delta_x + |x|^{2\gamma} \Delta_y = \nabla_\gamma \cdot \nabla_\gamma,$$

where Δ_x and Δ_y are the Laplace operators in the variables $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, respectively. The Baouendi-Grushin operator for an even positive integer γ is a sum of squares of C^∞ vector fields satisfying the Hörmander condition

$$\operatorname{rank} \operatorname{Lie}[X_1, \ldots, X_m, Y_1, \ldots, Y_k] = n.$$

We can define on $\mathbb{R}^{m+k}$ the anisotropic dilation attached to Δ_γ as

$$\delta_\lambda(x, y) = (\lambda x, \lambda^{1+\gamma} y)$$

for $\lambda > 0$, and the homogeneous dimension with respect to this dilation is

$$Q = m + (1+\gamma)k. \tag{4}$$

A change of variables formula for the Lebesgue measure implies that

$$d \circ \delta_\lambda(x, y) = \lambda^Q \, dx \, dy.$$

It is easy to check that

$$X_i(\delta_\lambda) = \lambda \delta_\lambda(X_i), \quad Y_i(\delta_\lambda) = \lambda \delta_\lambda(Y_i),$$

and hence

$$\nabla_\gamma \circ \delta_\lambda = \lambda \delta_\lambda \nabla_\gamma.$$

Let $\rho(z)$ be the corresponding distance function from the origin for $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$:

$$\rho = \rho(z) := (|x|^{2(1+\gamma)} + (1+\gamma)^2|y|^2)^{1/(2(1+\gamma))}. \tag{5}$$

By a direct calculation one obtains

$$|\nabla_\gamma \rho| = \frac{|x|^\gamma}{\rho^\gamma}. \tag{6}$$

The described setup may be thought of as a special case of the setting of homogeneous groups, see e.g. [8].

The weighted L^p-versions of (2) have been obtained by D'Ambrosio [5]: let $\Omega \subset \mathbb{R}^n$ be an open set. Let $p > 1$, $k, m \geq 1$, $\alpha, \beta \in \mathbb{R}$ be such that $m + (1+\gamma)k > \alpha - \beta$ and $m > \gamma p - \beta$. Then for every $f \in D_\gamma^{1,p}(\Omega, |x|^{\beta-\gamma p}\rho^{(1+\gamma)p-\alpha})$, we have

$$\int_\Omega |\nabla_\gamma f|^p |x|^{\beta-\gamma p} \rho^{(1+\gamma)p-\alpha} \, dx \, dy \geq \left(\frac{Q+\beta-\alpha}{p}\right)^p \int_\Omega |f|^p \frac{|x|^\beta}{\rho^\alpha} \, dx \, dy, \tag{7}$$

where $D_\gamma^{1,p}(\Omega, \omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the norm

$$\left(\int_\Omega |\nabla_\gamma f|^p \omega \, dz \, dy\right)^{1/p}$$

for $\omega \in L^1_{\text{loc}}(\Omega)$ with $\omega > 0$ a.e. on Ω.

If $0 \in \Omega$, then the constant $\left(\frac{Q+\beta-\alpha}{p}\right)^p$ in (7) is sharp. The inequality (7) has also been established in [15], and in [26] for $\Omega = \mathbb{R}^n$ with sharp constant.

Moreover, in [26], a Hardy-Rellich type inequality for the Baouendi-Grushin operator is obtained in L^2 with sharp constant:

$$\left(\frac{Q-\alpha-2}{2}\right)^2 \int_{\mathbb{R}^n} |\nabla_\gamma f|^2 \rho^\alpha \le \int_{\mathbb{R}^n} |\Delta_\gamma f|^2 \rho^{\alpha+2} |\nabla_\gamma \rho|^{-2},$$

where $p > 1$, $\frac{2-Q}{3} \le \alpha \le Q-2$ and $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$.

The inequalities of this type have been also studied for some sub-elliptic operators of different types (see e.g. [10], [11], [6], [5], [18], [14] and [7]). For Hardy and Caffarelli-Kohn-Nirenberg inequalities on more general homogeneous Carnot groups and the literature review including the Heisenberg group we refer to [23], [21], for the anisotropic versions of the usual L^2 and L^p Cafarelli-Kohn-Nirenberg inequalities we refer to [20] and [19], respectively, and for Hardy inequalities for more general sums of squares of vector fields we refer to [22].

Here we obtain the following refinement of Hardy inequalities for the Baouendi-Grushin operator:

Weighted refined Hardy inequalities for Grushin operators. *Let $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_k) \in \mathbb{R}^m \times \mathbb{R}^k$ with $k, m \ge 1$, $k + m = n$. Let $Q + \alpha_1 - 2 > 0$ and $m + \gamma\alpha_2 > 0$. Then we have the following Hardy type inequality for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$:*

$$\int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \left(\left| \frac{d}{d|x|} f \right|^2 + |x|^{2\gamma} |\nabla_y f|^2 \right) dx\, dy \ge \left(\frac{Q+\alpha_1-2}{2}\right)^2 \int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \frac{|\nabla_\gamma \rho|^2}{\rho^2} |f|^2\, dx\, dy, \quad (8)$$

where the constant $((Q + \alpha_1 - 2)/2)^2$ is sharp.

Already in the absence of weights, i.e. for $\alpha_1 = \alpha_2 = 0$, the estimate (8) is new. The obtained family of inequalities extends the known case of $k = 0$ when one has the classical Hardy inequality (1), and the case of $m = 0$ when one has the radial version established in [13], see also [17] (always for $\gamma = 0$). We note that since we can estimate $\left|\frac{d}{d|x|} f\right| \le |\nabla_x f|$, inequality (8) also gives a refinement to the inequality (7) for $p = 2$.

The estimate (8), in addition to its own interest, will play an important role in the derivation of estimates for magnetic operators.

1.2. Magnetic Baouendi-Grushin operator

In [16] and [4], Hardy inequalities for some magnetic forms were obtained. For example, in [4] for the quadratic form of the following magnetic Grushin operator

$$G_{\mathscr{A}} = -(\nabla_G + i\beta\mathscr{A}_0)^2,$$

the following Hardy inequality for $-\frac{1}{2} \leq \beta \leq \frac{1}{2}$ was proved:

$$\int_{\mathbb{R}^3} |(\nabla_G + i\beta\mathscr{A}_0) f|^2 \, dz \, dt \geq (1+\beta^2) \int_{\mathbb{R}^3} \frac{|z|^2}{d^4} |f|^2 \, dz \, dt, \tag{9}$$

where

$$\mathscr{A}_0 = (\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \mathscr{A}_4) = \left(-\frac{\partial_y d}{d}, \frac{\partial_x d}{d}, -2y\frac{\partial_t d}{d}, 2x\frac{\partial_t d}{d}\right),$$

$\nabla_G = (\partial_x, \partial_y, 2x\partial_t, 2y\partial_t)$ with $z = (x, y)$, $|z| = \sqrt{x^2 + y^2}$, $\beta \in \mathbb{R}$ is a “flux” and $d(z, t) = (|z|^4 + t^2)^{1/4}$ is the Kaplan distance.

The following results extend the estimate (9). While the most physical setting is $y \in \mathbb{R}^1$, we can obtain the results for any $y \in \mathbb{R}^k$.

Hardy inequality for the magnetic Baouendi-Grushin operator. *Let $(x, y) = (x_1, x_2, y) \in \mathbb{R}^2 \times \mathbb{R}^k$. Let $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ be such that $\alpha_1 + k(\gamma + 1) > 0$, $\alpha_2\gamma + 2 > 0$ and $-1/2 \leq \beta \leq 1/2$. Let us define the Aharonov-Bohm type magnetic field*

$$\widetilde{\mathscr{A}} := \left(-\frac{\partial_{x_2}\rho}{\rho}, \frac{\partial_{x_1}\rho}{\rho}, -\frac{|x|^\gamma}{\sqrt{2}}\frac{\nabla_y\rho}{\rho}, \frac{|x|^\gamma}{\sqrt{2}}\frac{\nabla_y\rho}{\rho}\right),$$

and the corresponding gradient

$$\widetilde{\nabla}_\gamma = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{|x|^\gamma}{\sqrt{2}}\nabla_y, \frac{|x|^\gamma}{\sqrt{2}}\nabla_y\right).$$

Then for any complex-valued function $f \in C_0^\infty(\mathbb{R}^{2+k}\backslash\{0\})$, we have the following weighted Hardy inequality for the magnetic Baouendi-Grushin operator:

$$\int_{\mathbb{R}^{2+k}} \rho^{\alpha_1} \left|\widetilde{\nabla}_\gamma\rho\right|^{\alpha_2} \left|\left(\widetilde{\nabla}_\gamma + i\beta\widetilde{\mathscr{A}}\right)f\right|^2 dx\, dy$$
$$\geq \left(\left(\frac{\alpha_1 + k(\gamma+1)}{2}\right)^2 + \beta^2\right) \int_{\mathbb{R}^{2+k}} \rho^{\alpha_1} \left|\widetilde{\nabla}_\gamma\rho\right|^{\alpha_2} \frac{|x|^{2\gamma}}{\rho^{2\gamma+2}} |f|^2 \, dx\, dy,$$

where the constant $\left(\left(\frac{\alpha_1+k(\gamma+1)}{2}\right)^2 + \beta^2\right)$ is sharp (*so that the constant in* (9) *is actually also sharp*).

Uncertainty type principle. *Let $(x, y) = (x_1, x_2, y) \in \mathbb{R}^2 \times \mathbb{R}^k$. Let $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ be such that $\alpha_1 + k(\gamma + 1) > 0$, $\alpha_2\gamma + 2 > 0$ and $-1/2 \leq \beta \leq$*

$1/2$. *Then for any complex-valued function* $f \in C_0^\infty(\mathbb{R}^{2+k} \setminus \{0\})$, *we have*

$$\left\| \rho^{\alpha_1/2} \left|\widetilde{\nabla}_\gamma \rho\right|^{\alpha_2/2} \left(\widetilde{\nabla}_\gamma + i\beta\widetilde{\mathscr{A}}\right) f \right\|_{L^2(\mathbb{R}^{2+k})} \|f\|_{L^2(\mathbb{R}^{2+k})} \geq \left(\left(\frac{\alpha_1 + k(\gamma+1)}{2}\right)^2 + \beta^2\right)^{1/2} \int_{\mathbb{R}^{2+k}} \rho^{\alpha_1/2} \left|\widetilde{\nabla}_\gamma \rho\right|^{\alpha_2/2} \frac{|x|^\gamma}{\rho^{\gamma+1}} |f|^2 \, dx \, dy.$$

MAGNETIC BAOUENDI-GRUSHIN OPERATOR WITH CONSTANT MAGNETIC FIELD. In Remark 3.9, we also give inequalities for the magnetic Baouendi-Grushin operator on $\mathbb{C}^n$ with the constant magnetic field

$$\mathscr{L}_G = \sum_{j=1}^n \left(i\partial_{x_j} + \psi_{1,j}(y_j)\right)^2 + \left(i|x|^\gamma \partial_{y_j} + \psi_{2,j}(x_j)\right)^2.$$

1.3. Landau Hamiltonian

Let us recall that the Landau Hamiltonian (or the twisted Laplacian) on $\mathbb{C}^n$ is defined as

$$\mathscr{L} = \sum_{j=1}^n \left[\left(i\partial_{x_j} + \frac{1}{2}y_j\right)^2 + \left(i\partial_{y_j} - \frac{1}{2}x_j\right)^2\right].$$

Setting

$$\widetilde{X}_j = \partial_{x_j} - \frac{1}{2}iy_j \quad \text{and} \quad \widetilde{Y}_j = \partial_{y_j} + \frac{1}{2}ix_j,$$

we have

$$\mathscr{L} = -\sum_{j=1}^n (\widetilde{X}_j^2 + \widetilde{Y}_j^2).$$

The twisted Laplacian can be also written as $\mathscr{L} = -\Delta + \frac{1}{4}(|x|^2 + |y|^2) + iN$, where

$$N = \sum_{j=1}^n (y_j \partial_{x_j} - x_j \partial y_j)$$

is the rotation field. Let $\nabla_{\mathscr{L}}$ be the gradient operator associated with $\mathscr{L}$:

$$\nabla_{\mathscr{L}} f = \left(\widetilde{X}_1 f, \dots, \widetilde{X}_n f, \widetilde{Y}_1 f, \dots, \widetilde{Y}_n f\right).$$

Let $W_{\mathscr{L}}^{1,2}(\mathbb{C}^n)$ be the Sobolev space defined by

$$W_{\mathscr{L}}^{1,2}(\mathbb{C}^n) = \left\{f \in L^2(\mathbb{C}^n) : \widetilde{X}_j f, \widetilde{Y}_j f \in L^2(\mathbb{C}^n),\ 1 \leq j \leq n\right\}. \tag{10}$$

In the recent paper [2], a version of the Hardy inequality for the twisted Laplacian with Landau Hamiltonian magnetic field was established for *real-valued* functions $f \in W^{1,2}_{\mathscr{L}}(\mathbb{C}^n)$, namely, the inequality

$$\frac{1}{4}\int_{\mathbb{C}^n} |f|^2 \omega(z)\, dz \le \int_{\mathbb{C}^n} |\nabla_{\mathscr{L}} f|^2\, dz,$$

with the weight

$$\omega(z) = \frac{|\nabla_{\mathscr{L}} \mathbb{E}|^2}{\mathbb{E}^2} + \frac{|z|^2}{4},$$

where $\mathbb{E}$ is a fundamental solution to the twisted Laplacian on $\mathbb{C}^n$.

In this paper we obtain the versions of Hardy inequalities for the Landau Hamiltonian for both *complex-valued* and *real-valued functions*.

In fact, we obtain results for operators $\widetilde{\mathscr{L}}_\psi$ of the form

$$\widetilde{\mathscr{L}}_\psi = \sum_{j=1}^{n}\Big[\big(i\partial_{x_j} + \psi(|z|)y_j\big)^2 + \big(i\partial_{y_j} - \psi(|z|)x_j\big)^2\Big],$$

where $\psi(|z|)$ is a radial real-valued differentiable function satisfying some conditions, and $z = (x, y)$. Setting

$$\check{X}_j = \partial_{x_j} - i\psi(|z|)y_j \quad \text{and} \quad \check{Y}_j = \partial_{y_j} + i\psi(|z|)x_j,$$

we write

$$\widetilde{\nabla_{\mathscr{L}_\psi}} f = \big(\check{X}_1 f, \dots, \check{X}_n f, \check{Y}_1 f, \dots, \check{Y}_n f\big).$$

As usual, we will identify $\mathbb{C} \cong \mathbb{R}^2$.

HARDY INEQUALITIES FOR THE LANDAU-HAMILTONIAN $\widetilde{\mathscr{L}}_\psi$. *Let $\psi = \psi(|z|)$ be a radial real-valued function such that $\psi \in L^2_{\mathrm{loc}}(\mathbb{C} \setminus \{0\})$ with*

$$|\psi(r)|r^2 \le \tfrac{1}{2}, \quad \forall r \in (0, \infty). \tag{11}$$

Then we have the following inequalities:

(i) HARDY-SOBOLEV INEQUALITY. *For any $\theta_1 \in \mathbb{R} \setminus \{0\}$ we have*

$$\int_{\mathbb{C}} \frac{\big|\widetilde{\nabla_{\mathscr{L}}} f\big|^2}{|z|^{2\theta_1}}\, dz - \theta_1^2 \int_{\mathbb{C}} \frac{|f|^2}{|z|^{2\theta_1+2}}\, dz \ge \int_{\mathbb{C}} \frac{\psi(|z|)^2}{|z|^{2\theta_1-2}} |f|^2\, dz,$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$.

(ii) LOGARITHMIC HARDY INEQUALITY. *We have*

$$\int_{\mathbb{C}} \big|\widetilde{\nabla_{\mathscr{L}}} f\big|^2 \big|\log|z|\big|^2\, dz - \frac{1}{4}\int_{\mathbb{C}} |f|^2\, dz \ge \int_{\mathbb{C}} \psi(|z|)^2 |z|^2 \big|\log|z|\big|^2 |f|^2 dz,$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$.

(iii) POINCARÉ INEQUALITY. *Let* Ω *be a bounded domain in* $\mathbb{C}$ *and* $R = \sup_{z\in\Omega}\{|z|\}$. *Then we have*

$$\int_{\Omega} \left|\widetilde{\nabla_{\mathscr{L}}} f\right|^2 dz - \frac{1}{R^2}\int_{\Omega} |f|^2\, dz \geq \int_{\Omega} \psi(|z|)^2 |z|^2 |f|^2\, dz,$$

for all complex-valued functions $f \in \widehat{\mathfrak{L}}_0^{1,2}(\Omega)$ *satisfying* $df/d|z| \in L^2(\Omega)$, *where the space* $\widehat{\mathfrak{L}}_0^{1,2}(\Omega)$ *is defined in* (40).

(iv) HARDY-SOBOLEV INEQUALITY WITH SUPERWEIGHTS. *Let* $\theta_2, \theta_3, \theta_4, a, b \in \mathbb{R}$ *with* $a, b > 0$, $\theta_2\theta_3 < 0$ *and* $2\theta_4 \leq \theta_2\theta_3$. *Then we have*

$$\int_{\mathbb{C}} \frac{(a+b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4}} \left|\widetilde{\nabla_{\mathscr{L}}} f\right|^2 dz \geq \frac{\theta_2\theta_3 - 2\theta_4}{2} \int_{\mathbb{C}} \frac{(a+b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4+2}} |f|^2\, dz,$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. *Weights of this type have appeared in* [12], *as well as in* [24], *and are called the superweights due to the freedom in the choice of indices.*

If one is interested in all the inequalities above only for real-valued functions f *then assumption* (11) *is not needed, see Remark 3.6.*

The Hardy inequalities for a magnetic Baouendi-Grushin operator with Aharonov-Bohm type magnetic field are proved in Section 2. In Section 3 we prove the Hardy inequalities for the twisted Laplacian with the Landau-Hamiltonian type magnetic field.

2. Weighted Hardy inequalities for magnetic Baouendi-Grushin operator with Aharonov-Bohm type magnetic field

In this section we establish weighted Hardy inequalities for the quadratic form of the magnetic Baouendi-Grushin operator with Aharonov-Bohm type magnetic field. We adapt all the notation introduced in Sections 1.1 and 1.2, namely, ∇_γ, ρ and Q defined in (3), (5) and (4), respectively. Recalling these for convenience of the reader, we have

$$\nabla_\gamma = (\nabla_x, |x|^\gamma \nabla_y), \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^k, \qquad Q = m + (1+\gamma)k,\ \gamma \geq 0,$$

and the magnetic field $\mathscr{A}$ is defined here as

$$\mathscr{A} = \frac{\nabla_\gamma \rho}{\rho} = \left(\frac{\nabla_x \rho}{\rho}, |x|^\gamma \frac{\nabla_y \rho}{\rho}\right) \in \mathbb{R}^m \times \mathbb{R}^k. \tag{12}$$

We start with a simple inequality showing the best constants one can expect.

LEMMA 2.1. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set. Let* $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_k) \in \mathbb{R}^m \times \mathbb{R}^k$ *with* $k, m \geq 1$, $k + m = n$. *Let* $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ *be such that*

$$Q + \alpha_1 - 2 > 0 \quad \textit{and} \quad m + \alpha_2\gamma > 0.$$

Then for any real-valued function $f \in C_0^\infty(\Omega)$, *we have the following weighted Hardy inequality for the magnetic Baouendi-Grushin operator*

$$\int_\Omega \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} |(\nabla_\gamma + i\beta\mathscr{A}) f|^2 \, dx \, dy \geq \left(\left(\frac{Q + \alpha_1 - 2}{2} \right)^2 + \beta^2 \right) \int_\Omega \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \frac{|x|^{2\gamma}}{\rho^{2\gamma+2}} |f|^2 \, dx \, dy, \quad (13)$$

Moreover, if $0 \in \Omega$, *then the constant in* (13) *is sharp.*

PROOF. By opening brackets we have

$$\int_\Omega \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} |(\nabla_\gamma + i\beta\mathscr{A}) f|^2 \, dx \, dy = \int_\Omega \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} |\nabla_\gamma f|^2 \, dx \, dy + \beta^2 \int_\Omega \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} |\mathscr{A} f|^2 \, dx \, dy. \quad (14)$$

Taking into account (6) and putting $p = 2$, $\alpha = \gamma(\alpha_2 + 2) + 2 - \alpha_1$ and $\beta = \gamma(\alpha_2 + 2)$ in (7), we have for any $Q + \alpha_1 - 2 > 0$ and $m + \alpha_2\gamma > 0$ that

$$\int_\Omega \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} |\nabla_\gamma f|^2 \, dx \, dy \geq \left(\frac{Q + \alpha_1 - 2}{2} \right)^2 \int_\Omega \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \frac{|\nabla_\gamma \rho|^2}{\rho^2} |f|^2 \, dx \, dy. \quad (15)$$

Taking into account the form of the magnetic field $\mathscr{A} = (\mathscr{A}_1, \mathscr{A}_2) = \left(\frac{\nabla_x \rho}{\rho}, |x|^\gamma \frac{\nabla_y \rho}{\rho} \right)$ in (12), and by a direct calculation one finds

$$\begin{aligned} \beta^2 \int_\Omega \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} |\mathscr{A} f|^2 \, dx \, dy &= \beta^2 \int_\Omega \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \frac{|\nabla_x \rho|^2 + |x|^{2\gamma} |\nabla_y \rho|^2}{\rho^2} |f|^2 \, dx \, dy \\ &= \beta^2 \int_\Omega \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \frac{|\nabla_\gamma \rho|^2}{\rho^2} |f|^2 \, dx \, dy. \end{aligned} \quad (16)$$

Then by (14), (15) and (16) we obtain

$$\int_\Omega \rho^{\alpha_1}|\nabla_\gamma\rho|^{\alpha_2}|(\nabla_\gamma + i\beta\mathscr{A})f|^2\,dz \geq \left(\left(\frac{Q+\alpha_1-2}{2}\right)^2+\beta^2\right)\int_\Omega \rho^{\alpha_1}|\nabla_\gamma\rho|^{\alpha_2}\frac{|\nabla_\gamma\rho|^2}{\rho^2}|f|^2\,dx\,dy. \quad (17)$$

Then using (6), we observe that (17) yields (13). Since the constant in (15) is sharp when $0 \in \Omega$ by (7), then the constant in the obtained inequality is sharp when $0 \in \Omega$. The proof of Lemma 2.1 is complete.

We obtain the following corollary in $\mathbb{R}^{2+k}$ for the Aharonov-Bohm potential of the type considered in [4].

COROLLARY 2.2. *Let $\Omega \subset \mathbb{R}^{2+k}$ be an open set. Let $(x, y) = (x_1, x_2, y) \in \mathbb{R}^2 \times \mathbb{R}^k$. Let $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ be such that $\alpha_1 + k(\gamma+1) > 0$ and $\alpha_2\gamma + 2 > 0$. Then for any real-valued function $f \in C_0^\infty(\Omega)$ and for the following Aharonov-Bohm type magnetic field*

$$\widetilde{\mathscr{A}} = \left(-\frac{\partial_{x_2}\rho}{\rho}, \frac{\partial_{x_1}\rho}{\rho}, -\frac{|x|^\gamma}{\sqrt{2}}\frac{\nabla_y\rho}{\rho}, \frac{|x|^\gamma}{\sqrt{2}}\frac{\nabla_y\rho}{\rho}\right), \quad (18)$$

we have the following weighted Hardy inequality for the magnetic Baouendi-Grushin operator

$$\int_\Omega \rho^{\alpha_1}\left|\widetilde{\nabla_\gamma}\rho\right|^{\alpha_2}\left|\left(\widetilde{\nabla_\gamma} + i\beta\widetilde{\mathscr{A}}\right)f\right|^2 dx\,dy \geq \left(\left(\frac{\alpha_1+k(\gamma+1)}{2}\right)^2+\beta^2\right)\int_\Omega \rho^{\alpha_1}\left|\widetilde{\nabla_\gamma}\rho\right|^{\alpha_2}\frac{|x|^{2\gamma}}{\rho^{2\gamma+2}}|f|^2\,dx\,dy, \quad (19)$$

where

$$\widetilde{\nabla_\gamma} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{|x|^\gamma}{\sqrt{2}}\nabla_y, \frac{|x|^\gamma}{\sqrt{2}}\nabla_y\right). \quad (20)$$

Moreover, if $0 \in \Omega$, then the constant $\left(\left(\frac{\alpha_1+k(\gamma+1)}{2}\right)^2+\beta^2\right)$ in (19) is sharp.

PROOF. In this case $m = 2$, then $Q = 2 + k(1+\gamma)$. Since we have $\left|\widetilde{\mathscr{A}}f\right|^2 = |\mathscr{A}f|^2$, $\left|\widetilde{\nabla_\gamma}\rho\right| = |\nabla_\gamma\rho|$ and $\left|\widetilde{\nabla_\gamma}f\right|^2 = |\nabla_\gamma f|^2$, then in the exact same way as in the proof of the Lemma 2.1 one obtains (19). The proof of Corollary 2.2 is complete.

As another corollary of Lemma 2.1, we obtain the following uncertainty principle.

COROLLARY 2.3 (Uncertainty type principle). *Let $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_k) \in \mathbb{R}^m \times \mathbb{R}^k$ with $k, m \geq 1$, $k + m = n$. Let $\Omega \subset \mathbb{R}^n$ be an open* *set. Let* $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ *be such that* $Q + \alpha_1 - 2 > 0$ *and* $m + \alpha_2\gamma > 0$*. Then for any real-valued function* $f \in C_0^\infty(\Omega)$*, we have*

$$\big\|\rho^{\alpha_1/2}|\nabla_\gamma \rho|^{\alpha_2/2}(\nabla_\gamma + i\beta\mathscr{A})f\big\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} \geq \left(\left(\frac{Q+\alpha_1-2}{2}\right)^2 + \beta^2\right)^{1/2} \int_\Omega \rho^{\alpha_1/2}|\nabla_\gamma\rho|^{\alpha_2/2}\frac{|x|^\gamma}{\rho^{\gamma+1}}|f|^2\,dx\,dy.$$

PROOF. By Lemma 2.1 we get

$$\begin{aligned}\big\|\rho^{\alpha_1/2}|\nabla_\gamma \rho|^{\alpha_2/2}(\nabla_\gamma + i\beta\mathscr{A})f\big\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} &\geq \left(\left(\frac{Q+\alpha_1-2}{2}\right)^2 + \beta^2\right)^{1/2}\left\|\rho^{\alpha_1/2}|\nabla_\gamma\rho|^{\alpha_2/2}\frac{|x|^\gamma}{\rho^{\gamma+1}}f\right\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)}\\ &\geq \left(\left(\frac{Q+\alpha_1-2}{2}\right)^2 + \beta^2\right)^{1/2} \int_\Omega \rho^{\alpha_1/2}|\nabla_\gamma\rho|^{\alpha_2/2}\frac{|x|^\gamma}{\rho^{\gamma+1}}|f|^2\,dx\,dy.\end{aligned}$$

The proof of Corollary 2.3 is complete.

We now give the main theorem of this section for complex-valued functions f.

THEOREM 2.4. *Let* $(x, y) = (x_1, x_2, y) \in \mathbb{R}^2 \times \mathbb{R}^k$*. Let* $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ *be such that* $\alpha_1 + k(\gamma+1) > 0$*,* $\alpha_2 + 2\gamma > 0$ *and* $-1/2 \leq \beta \leq 1/2$*. Recall* $\widetilde{\nabla}_\gamma$*,* $\widetilde{\mathscr{A}}$ *and* ρ *defined in* (20)*,* (18) *and* (5)*, respectively.*

Then for any complex-valued function $f \in C_0^\infty(\mathbb{R}^{2+k} \setminus \{0\})$*, we have the following weighted Hardy inequality for the magnetic Baouendi-Grushin operator:*

$$\int_{\mathbb{R}^{2+k}} \rho^{\alpha_1}\big|\widetilde{\nabla}_\gamma\rho\big|^{\alpha_2}\big|\big(\widetilde{\nabla}_\gamma + i\beta\widetilde{\mathscr{A}}\big)f\big|^2\,dx\,dy \geq \left(\left(\frac{\alpha_1+k(\gamma+1)}{2}\right)^2+\beta^2\right)\int_{\mathbb{R}^{2+k}}\rho^{\alpha_1}\big|\widetilde{\nabla}_\gamma\rho\big|^{\alpha_2}\frac{|x|^{2\gamma}}{\rho^{2\gamma+2}}|f|^2\,dx\,dy, \quad (21)$$

and the constant $\big(\frac{\alpha_1+k(\gamma+1)}{2}\big)^2 + \beta^2$ *is sharp.*

The proof of Theorem 2.4 will be based on the following theorem:

THEOREM 2.5. *Let* $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_k) \in \mathbb{R}^m \times \mathbb{R}^k$ *with* $k, m \geq 1$*,* $k + m = n$*. Let* $\alpha_1, \alpha_2 \in \mathbb{R}$ *be such that*

$$Q + \alpha_1 - 2 > 0 \quad \textit{and} \quad m + \gamma\alpha_2 > 0.$$

Then we have the following Hardy type inequality for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$:

$$\int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \left(\left| \frac{d}{d|x|} f \right|^2 + |x|^{2\gamma} |\nabla_y f|^2 \right) dx\, dy \geq \left(\frac{Q + \alpha_1 - 2}{2} \right)^2 \int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \frac{|\nabla_\gamma \rho|^2}{\rho^2} |f|^2 \, dx\, dy, \quad (22)$$

with sharp constant $\left(\frac{Q+\alpha_1-2}{2}\right)^2$.

REMARK 2.6. Theorem 2.5 implies the following inequality

$$\int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \left(|\nabla_x f|^2 + |x|^{2\gamma} |\nabla_y f|^2 \right) dx\, dy \geq \left(\frac{Q + \alpha_1 - 2}{2} \right)^2 \int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \frac{|\nabla_\gamma \rho|^2}{\rho^2} |f|^2 \, dx\, dy, \quad (23)$$

with sharp constant, which gives the result of D'Ambrosio (7) when $p = 2$ and $\Omega = \mathbb{R}^n$. We also mention that inequality (23) has been established in [15] and [26].

PROOF OF THEOREM 2.5. We write $r = |x|$ and $B(r, y) = \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2}$. Then, using (5) and (6), one has

$$B(r, y) = \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} = r^{\alpha_2 \gamma} \rho^{\alpha_1 - \alpha_2 \gamma} = r^{\alpha_2 \gamma} \left(r^{2(1+\gamma)} + (1+\gamma)^2 |y|^2 \right)^{\frac{\alpha_1 - \alpha_2 \gamma}{2(1+\gamma)}}. \quad (24)$$

Let us first calculate the following

$$\begin{aligned}
&\int_{\mathbb{R}^k} \int_0^\infty \left(\left| \left(\partial_r + \alpha \frac{\partial_r \rho}{\rho} \right) f \right|^2 + r^{2\gamma} \left| \left(\nabla_y + \alpha \frac{\nabla_y \rho}{\rho} \right) f \right|^2 \right) r^{m-1} B(r, y) \, dr\, dy \\
&\quad = \int_{\mathbb{R}^k} \int_0^\infty \left(|\partial_r f|^2 + r^{2\gamma} |\nabla_y f|^2 \right) r^{m-1} B(r, y) \, dr\, dy \\
&\qquad + \alpha^2 \int_{\mathbb{R}^k} \int_0^\infty \left(\left| \frac{\partial_r \rho}{\rho} \right|^2 + r^{2\gamma} \left| \frac{\nabla_y \rho}{\rho} \right|^2 \right) |f|^2 r^{m-1} B(r, y) \, dr\, dy \\
&\qquad + 2\alpha \operatorname{Re} \int_{\mathbb{R}^k} \int_0^\infty \frac{\partial_r \rho}{\rho} r^{m-1} B(r, y) \, \overline{\partial_r f} \cdot f \, dr\, dy \\
&\qquad + 2\alpha \operatorname{Re} \int_{\mathbb{R}^k} \int_0^\infty \frac{\nabla_y \rho}{\rho} \cdot \overline{\nabla_y f} \, r^{2\gamma + m - 1} B(r, y) f \, dr\, dy \\
&\quad =: I_1 + I_2 + I_3 + I_4.
\end{aligned} \quad (25)$$

Using
$$\frac{\partial_r \rho}{\rho} = \frac{r^{2\gamma+1}}{\rho^{2\gamma+2}} \quad \text{and} \quad \frac{\nabla_y \rho}{\rho} = \frac{(\gamma+1)y}{\rho^{2\gamma+2}},$$
we calculate
$$\left|\frac{\partial_r \rho}{\rho}\right|^2 + r^{2\gamma}\left|\frac{\nabla_y \rho}{\rho}\right|^2 = \frac{r^{4\gamma+2} + r^{2\gamma}(\gamma+1)^2|y|^2}{\rho^{4\gamma+4}} = \frac{r^{2\gamma}}{\rho^{2\gamma+2}} = \frac{|\nabla_\gamma \rho|^2}{\rho^2}. \tag{26}$$
Thus, we obtain
$$I_2 = \alpha^2 \int_{-\infty}^{\infty}\int_0^\infty \frac{|\nabla_\gamma \rho|^2}{\rho^2}|f|^2 r^{m-1} B(r,y)\,dr\,dy. \tag{27}$$
Now we proceed using integration by parts for I_3,
$$\begin{aligned} I_3 = &-\alpha \int_{\mathbb{R}^k}\int_0^\infty (2\gamma+m+\gamma\alpha_2)\rho^{\alpha_1-\alpha_2\gamma-2\gamma-2} r^{2\gamma+m-1+\gamma\alpha_2}|f|^2\,dr\,dy \\ &-\alpha \int_{\mathbb{R}^k}\int_0^\infty (\alpha_1-\alpha_2\gamma-2\gamma-2)\rho^{\alpha_1-\alpha_2\gamma-4\gamma-4} r^{4\gamma+m+\gamma\alpha_2+1}|f|^2\,dr\,dy. \end{aligned}$$
Since $B(r,y) = r^{\alpha_2\gamma}\rho^{\alpha_1-\alpha_2\gamma}$ by (24), then we have
$$\begin{aligned} I_3 &= -\alpha \int_{\mathbb{R}^k}\int_0^\infty \left((2\gamma+m+\gamma\alpha_2)\frac{r^{2\gamma}}{\rho^{2\gamma+2}} + (\alpha_1-\alpha_2\gamma-2\gamma-2)\frac{r^{4\gamma+2}}{\rho^{4\gamma+4}}\right) \\ &\qquad \times r^{m-1}B(r,y)|f|^2\,dr\,dy \\ &= -\alpha \int_{\mathbb{R}^k}\int_0^\infty \left(2\gamma+m+\gamma\alpha_2 + (\alpha_1-\alpha_2\gamma-2\gamma-2)\frac{r^{2\gamma+2}}{\rho^{2\gamma+2}}\right)\frac{|\nabla_\gamma \rho|^2}{\rho^2} \\ &\qquad \times |f|^2 r^{m-1}B(r,y)\,dr\,dy. \end{aligned}$$
Similarly, we have for I_4
$$\begin{aligned} I_4 &= -\alpha \int_{\mathbb{R}^k}\int_0^\infty \operatorname{div}_y\left(B(r,y)\frac{\nabla_y \rho}{\rho}\right) r^{2\gamma+m-1}|f|^2\,dr\,dy \\ &= -\alpha(\gamma+1)\int_{\mathbb{R}^k}\int_0^\infty \operatorname{div}_y\left(\rho^{\alpha_1-\alpha_2\gamma-2\gamma-2}y\right) r^{\alpha_2\gamma+2\gamma+m-1}|f|^2\,dr\,dy \\ &= -\alpha \int_{\mathbb{R}^k}\int_0^\infty \left((\alpha_1-\alpha_2\gamma-2\gamma-2)\rho^{\alpha_1-\alpha_2\gamma-2\gamma-3}\frac{(\gamma+1)^2|y|^2}{\rho^{2\gamma+1}}\right) \\ &\qquad \times r^{\alpha_2\gamma+2\gamma+m-1}|f|^2\,dr\,dy \\ &\quad -\alpha \int_{\mathbb{R}^k}\int_0^\infty k(\gamma+1)\rho^{\alpha_1-\alpha_2\gamma-2\gamma-2} r^{\alpha_2\gamma+2\gamma+m-1}|f|^2\,dr\,dy. \end{aligned}$$

Since $\frac{r^{2\gamma}}{\rho^{2\gamma+2}} = \frac{|\nabla_\gamma \rho|^2}{\rho^2}$ and $B(r, y) = r^{\alpha_2\gamma}\rho^{\alpha_1-\alpha_2\gamma}$ by (26) and (24), respectively, then we obtain

$$I_4 = -\alpha \int_{\mathbb{R}^k}\int_0^\infty \left((\alpha_1 - \alpha_2\gamma - 2\gamma - 2)\frac{(\gamma+1)^2|y|^2}{\rho^{2\gamma+2}} + k(\gamma+1)\right) \times \frac{|\nabla_\gamma \rho|^2}{\rho^2}|f|^2 r^{m-1} B(r, y)\, dr\, dy.$$

Then, taking into account the definition (5), we get

$$\begin{aligned} I_3 + I_4 &= -\alpha \int_{\mathbb{R}^k}\int_0^\infty \big(\alpha_1 - \alpha_2\gamma - 2\gamma - 2 + 2\gamma + m + \gamma\alpha_2 + k(\gamma+1)\big)\frac{|\nabla_\gamma \rho|^2}{\rho^2} \\ &\quad \times |f|^2 r^{m-1} B(r, y)\, dr\, dy, \end{aligned}$$

and using that $Q = m + (1+\gamma)k$ in (4), one has

$$I_3 + I_4 = -\alpha \int_{\mathbb{R}^k}\int_0^\infty (Q + \alpha_1 - 2)\frac{|\nabla_\gamma \rho|^2}{\rho^2}|f|^2 r^{m-1} B(r, y)\, dr\, dy. \quad (28)$$

Putting (27) and (28) in (25), we have

$$\begin{aligned} &\int_{\mathbb{R}^k}\int_0^\infty \left(\left|\left(\partial_r + \alpha\frac{\partial_r \rho}{\rho}\right) f\right|^2 + r^{2\gamma}\left|\left(\nabla_y + \alpha\frac{\nabla_y \rho}{\rho}\right) f\right|^2\right) r^{m-1} B(r, y)\, dr\, dy \\ &= \int_{\mathbb{R}^k}\int_0^\infty \big(|\partial_r f|^2 + r^{2\gamma}|\nabla_y f|^2\big) r^{m-1} B(r, y)\, dr\, dy \\ &\quad - \big((Q + \alpha_1 - 2)\alpha - \alpha^2\big)\int_{\mathbb{R}^k}\int_0^\infty \frac{|\nabla_\gamma \rho|^2}{\rho^2}|f|^2 r^{m-1} B(r, y)\, dr\, dy. \end{aligned}$$

By substituting $\alpha = \frac{Q+\alpha_1-2}{2}$ and taking into account (24), we obtain (22).

Let us now show the sharpness of the constant in (22). Taking into account (15) and (22), we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \rho^{\alpha_1}|\nabla_\gamma \rho|^{\alpha_2}\big(|\nabla_x f|^2 + |x|^{2\gamma}|\nabla_y f|^2\big)\, dx\, dy \\ &\geq \int_{\mathbb{R}^n} \rho^{\alpha_1}|\nabla_\gamma \rho|^{\alpha_2}\left(\left|\frac{d}{d|x|} f\right|^2 + |x|^{2\gamma}|\nabla_y f|^2\right) dx\, dy \\ &\geq \left(\frac{Q + \alpha_1 - 2}{2}\right)^2 \int_{\mathbb{R}^n} \rho^{\alpha_1}|\nabla_\gamma \rho|^{\alpha_2}\frac{|\nabla_\gamma \rho|^2}{\rho^2}|f|^2\, dx\, dy, \end{aligned}$$

which proves the constant $\left(\frac{Q+\alpha_1-2}{2}\right)^2$ in (22) is sharp. Thus, we have completed the proof of Theorem 2.5.

We are now ready to prove Theorem 2.4.

PROOF OF THEOREM 2.4. Using polar coordinates for the x-plane, $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, we have $r = |x|$ and

$$\frac{\partial_{x_1}\rho}{\rho} = \frac{r^{2\gamma+1}\cos\phi}{r^{2(1+\gamma)} + (1+\gamma)^2|y|^2},$$

$$\frac{\partial_{x_2}\rho}{\rho} = \frac{r^{2\gamma+1}\sin\phi}{r^{2(1+\gamma)} + (1+\gamma)^2|y|^2},$$

$$\frac{\nabla_y\rho}{\rho} = \frac{(1+\gamma)y}{r^{2(1+\gamma)} + (1+\gamma)^2|y|^2}.$$

Thus, we can write

$$\int_{\mathbb{R}^{2+k}} \rho^{\alpha_1}\left|\widetilde{\nabla}_\gamma\rho\right|^{\alpha_2}\left|\left(\widetilde{\nabla}_\gamma + i\beta\widetilde{\mathscr{A}}\right)f\right|^2 dx_1\,dx_2\,dy =: I_1 + I_2, \tag{29}$$

where

$$\begin{aligned} I_1 = &\int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \left|\left(\cos\phi\,\partial_r - \frac{\sin\phi}{r}\partial_\phi - i\beta\frac{r^{2\gamma+1}\sin\phi}{r^{2(1+\gamma)} + (1+\gamma)^2|y|^2}\right)f\right|^2 \\ &\quad\times rB(r,y)\,dr\,d\phi\,dy \\ &+ \int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \left|\left(\sin\phi\,\partial_r + \frac{\cos\phi}{r}\partial_\phi + i\beta\frac{r^{2\gamma+1}\cos\phi}{r^{2(1+\gamma)} + (1+\gamma)^2|y|^2}\right)f\right|^2 \\ &\quad\times rB(r,y)\,dr\,d\phi\,dy \end{aligned}$$

and

$$\begin{aligned} I_2 = &\frac{1}{2}\int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \left|\left(\nabla_y - i\beta\frac{(1+\gamma)y}{r^{2(1+\gamma)} + (1+\gamma)^2|y|^2}\right)f\right|^2 \\ &\quad\times r^{2\gamma+1}B(r,y)\,dr\,d\phi\,dy \\ &+ \frac{1}{2}\int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \left|\left(\nabla_y + i\beta\frac{(1+\gamma)y}{r^{2(1+\gamma)} + (1+\gamma)^2|y|^2}\right)f\right|^2 \\ &\quad\times r^{2\gamma+1}B(r,y)\,dr\,d\phi\,dy. \end{aligned}$$

By opening brackets, we obtain

$$
\begin{aligned}
I_1 = & \int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \left(|\partial_r f|^2 + \frac{|\partial_\phi f|^2}{r^2} + \frac{\beta^2 r^{4\gamma+2}|f|^2}{(r^{2(1+\gamma)}+(1+\gamma)^2|y|^2)^2}\right) \\
& \times r B(r,y)\,dr\,d\phi\,dy \\
& + 2\,\mathrm{Re}\int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \left(\frac{\partial_\phi f}{r}\cdot \overline{\frac{i\beta r^{2\gamma+1} f}{r^{2(1+\gamma)}+(1+\gamma)^2|y|^2}}\right) r B(r,y)\,dr\,d\phi\,dy \\
= & \int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \left(|\partial_r f|^2 + \frac{1}{r^2}\left|\partial_\phi f + i\beta \frac{r^{2\gamma+2}}{r^{2(1+\gamma)}+(1+\gamma)^2|y|^2} f\right|^2\right) \\
& \times r B(r,y)\,dr\,d\phi\,dy
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 = & \int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty |\nabla_y f|^2 r^{2\gamma+1} B(r,y)\,dr\,d\phi\,dy \\
& + \beta^2(1+\gamma)^2 \int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \frac{|y|^2|f|^2}{(r^{2(1+\gamma)}+(1+\gamma)^2|y|^2)^2} \\
& \times r^{2\gamma+1} B(r,y)\,dr\,d\phi\,dy.
\end{aligned}
$$

Using the Fourier series for f, we can expand

$$
f(r,\phi,y) = \sum_{k=-\infty}^{\infty} f_k(r,y) e^{ik\phi},
$$

and by a direct calculation, we get

$$
\begin{aligned}
& \frac{1}{r^2}\int_0^{2\pi}\left|\partial_\phi f + i\beta\frac{r^{2\gamma+2}}{r^{2(1+\gamma)}+(1+\gamma)^2|y|^2} f\right|^2 d\phi \\
& \quad = \frac{2\pi}{r^2}\sum_{k=-\infty}^{\infty}\left(k+\beta\frac{r^{2\gamma+2}}{r^{2(1+\gamma)}+(1+\gamma)^2|y|^2}\right)^2 |f_k(r,y)|^2 \\
& \quad \geq \frac{2\pi}{r^2}\min_k\left(k+\beta\frac{r^{2\gamma+2}}{r^{2(1+\gamma)}+(1+\gamma)^2|y|^2}\right)^2 \sum_{k=-\infty}^{\infty}|f_k(r,y)|^2 \\
& \quad = \frac{1}{r^2}\min_k\left(k+\beta\frac{r^{2\gamma+2}}{r^{2(1+\gamma)}+(1+\gamma)^2|y|^2}\right)^2 \int_0^{2\pi}|f|^2\,d\phi \\
& \quad = \beta^2\frac{r^{4\gamma+2}}{(r^{2(1+\gamma)}+(1+\gamma)^2|y|^2)^2}\int_0^{2\pi}|f|^2\,d\phi.
\end{aligned}
$$

Now putting the obtained estimates for I_1 and I_2 in (29), we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2+k}} \rho^{\alpha_1} \big|\widetilde{\nabla_\gamma}\rho\big|^{\alpha_2} \big|\big(\widetilde{\nabla_\gamma} + i\beta\widetilde{\mathscr{A}}\big) f\big|^2 \, dx_1\, dx_2\, dy \\
&\quad \geq \int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \big(|\partial_r f|^2 + r^{2\gamma}|\nabla_y f|^2\big) r B(r,y)\, dr\, d\phi\, dy \\
&\qquad + \beta^2 \int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \left(\frac{r^{4\gamma+2} + (1+\gamma)^2|y|^2 r^{2\gamma}}{(r^{2\gamma+2} + (1+\gamma)^2|y|^2)^2}\right)|f|^2 r B(r,y)\, dr\, d\phi\, dy \\
&\quad = \int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \big(|\partial_r f|^2 + r^{2\gamma}|\nabla_y f|^2\big) r B(r,y)\, dr\, d\phi\, dy \\
&\qquad + \beta^2 \int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \frac{r^{2\gamma}}{r^{2\gamma+2} + (1+\gamma)^2|y|^2}|f|^2 r B(r,y)\, dr\, d\phi\, dy.
\end{aligned} \tag{30}$$

Since $|\widetilde{\nabla_\gamma}\rho| = |\nabla_\gamma \rho|$, then putting $m = 2$ in (15) and taking into account (24), we obtain the following estimate for the first integral in (30):

$$\begin{aligned}
&\int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \big(|\partial_r f|^2 + r^{2\gamma}|\nabla_y f|^2\big) r B(r,y)\, dr\, d\phi\, dy \\
&\qquad\qquad \geq \left(\frac{\alpha_1 + k(\gamma+1)}{2}\right)^2 \int_{\mathbb{R}^{2+k}} \rho^{\alpha_1}|\widetilde{\nabla_\gamma}\rho|^{\alpha_2} \frac{|\widetilde{\nabla_\gamma}\rho|^2}{\rho^2}|f|^2\, dx\, dy.
\end{aligned} \tag{31}$$

Let us calculate the second integral in (30)

$$\begin{aligned}
&\beta^2 \int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \frac{r^{2\gamma}}{r^{2\gamma+2} + (1+\gamma)^2|y|^2}|f|^2 r B(r,y)\, dr\, d\phi\, dy \\
&\quad = \beta^2 \int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \frac{r^{4\gamma+2}\cos^2\phi}{(r^{2\gamma+2} + (1+\gamma)^2|y|^2)^2}|f|^2 r B(r,y)\, dr\, d\phi\, dy \\
&\qquad + \beta^2 \int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \frac{r^{4\gamma+2}\sin^2\phi}{(r^{2\gamma+2} + (1+\gamma)^2|y|^2)^2}|f|^2 r B(r,y)\, dr\, d\phi\, dy \\
&\qquad + \beta^2 \int_{\mathbb{R}^k}\int_0^{2\pi}\int_0^\infty \frac{(1+\gamma)^2 r^{2\gamma}|y|^2}{(r^{2\gamma+2} + (1+\gamma)^2|y|^2)^2}|f|^2 r B(r,y)\, dr\, d\phi\, dy \\
&\quad = \beta^2 \int_{\mathbb{R}^{2+k}} \rho^{\alpha_1}|\widetilde{\nabla_\gamma}\rho|^{\alpha_2}\left|\left(\frac{\partial_{x_1}\rho}{\rho}, \frac{\partial_{x_2}\rho}{\rho}, \frac{|x|x^\gamma}{\sqrt{2}}\frac{\nabla_y\rho}{\rho}, \frac{|x|^\gamma}{\sqrt{2}}\frac{\nabla_y\rho}{\rho}\right)\right|^2 |f|^2\, dx\, dy \\
&\quad = \beta^2 \int_{\mathbb{R}^{2+k}} \rho^{\alpha_1}|\widetilde{\nabla_\gamma}\rho|^{\alpha_2}\frac{|\widetilde{\nabla_\gamma}\rho|^2}{\rho^2}|f|^2\, dx\, dy.
\end{aligned} \tag{32}$$

Putting the estimates (31) and (32) in (30), we obtain (21). Since we have (21) with a sharp constant for all real-valued functions by Corollary 2.2, then this constant is sharp also in the class of complex-valued functions in (21). Thus, we have completed the proof of Theorem 2.4.

We record the corresponding uncertainty principle.

COROLLARY 2.7 (Uncertainty type principle). *Let* $(x, y) = (x_1, x_2, y) \in \mathbb{R}^2 \times \mathbb{R}^k$. *Let* $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ *be such that* $\alpha_1 + k(\gamma + 1) > 0$, $\alpha_2\gamma + 2 > 0$ *and* $-1/2 \le \beta \le 1/2$. *Then for any complex-valued function* $f \in C_0^\infty(\mathbb{R}^{2+k} \setminus \{0\})$ *we have*

$$\left\| \rho^{\alpha_1/2} \left| \widetilde{\nabla}_\gamma \rho \right|^{\alpha_2/2} \left(\widetilde{\nabla}_\gamma + i\beta \widetilde{\mathscr{A}} \right) f \right\|_{L^2(\mathbb{R}^{2+k})} \| f \|_{L^2(\mathbb{R}^{2+k})}$$
$$\ge \left(\left(\frac{\alpha_1 + k(\gamma+1)}{2} \right)^2 + \beta^2 \right)^{1/2} \int_{\mathbb{R}^{2+k}} \rho^{\alpha_1/2} \left| \widetilde{\nabla}_\gamma \rho \right|^{\alpha_2/2} \frac{|x|^\gamma}{\rho^{\gamma+1}} |f|^2 \, dx \, dy. \quad (33)$$

PROOF. Using (21) and in a similar way as in the proof of the Corollary 2.3, one obtains (33).

3. Hardy inequalities for Landau-Hamiltonian

In this section we show the Hardy inequalities for the twisted Laplacian with Landau-Hamiltonian type magnetic field.

Let us introduce now the generalised form of the twisted Laplacian

$$\widetilde{\mathscr{L}} = \sum_{j=1}^{n} \left[\left(i\partial_{x_j} + \psi(|z|) y_j \right)^2 + \left(i\partial_{y_j} - \psi(|z|) x_j \right)^2 \right],$$

where $\psi(|z|)$ is a radial real-valued differentiable function. Setting

$$\check{X}_j = \partial_{x_j} - i\psi(|z|) y_j \quad \text{and} \quad \check{Y}_j = \partial_{y_j} + i\psi(|z|) x_j,$$

we write

$$\widetilde{\nabla}_{\mathscr{L}} f = \left(\check{X}_1 f, \dots, \check{X}_n f, \check{Y}_1 f, \dots, \check{Y}_n f \right).$$

We then have the following result for complex-valued functions f.

THEOREM 3.1. *Let* $\theta_1, \theta_2, \theta_3, \theta_4, a, b \in \mathbb{R}$ *with* $a, b > 0$, $\theta_1 \neq 0$, $\theta_2\theta_3 < 0$ *and* $2\theta_4 \le \theta_2\theta_3$. *Let* $\psi = \psi(|z|)$ *be a radial real-valued function such that* $\psi \in L^2_{\mathrm{loc}}(\mathbb{C} \setminus \{0\})$ *with*

$$|\psi(r)| r^2 \le \frac{1}{2}, \quad \forall r \in (0, \infty).$$

Let Ω *be a bounded domain in* $\mathbb{C}$ *and* $R = \sup_{z\in\Omega}\{|z|\}$. *Then we have the following inequalities:*

(i) WEIGHTED HARDY-SOBOLEV INEQUALITY*:*

$$\int_{\mathbb{C}} \frac{|\widetilde{\nabla_{\mathscr{L}}} f|^2}{|z|^{2\theta_1}}\,dz - \theta_1^2 \int_{\mathbb{C}} \frac{|f|^2}{|z|^{2\theta_1+2}}\,dz \geq \int_{\mathbb{C}} \frac{(\psi(|z|))^2}{|z|^{2\theta_1-2}} |f|^2\,dz, \tag{34}$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$*;*

(ii) LOGARITHMIC HARDY INEQUALITY*:*

$$\int_{\mathbb{C}} |\widetilde{\nabla_{\mathscr{L}}} f|^2 \big|\log|z|\big|^2\,dz - \frac{1}{4}\int_{\mathbb{C}} |f|^2\,dz \geq \int_{\mathbb{C}} \psi(|z|)^2 |z|^2 \big|\log|z|\big|^2 |f|^2\,dz, \tag{35}$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$*;*

(iii) POINCARÉ INEQUALITY*:*

$$\int_{\Omega} \big|\widetilde{\nabla_{\mathscr{L}}} f\big|^2\,dz - \frac{1}{R^2}\int_{\Omega} |f|^2\,dz \geq \int_{\Omega} \psi(|z|)^2 |z|^2 |f|^2\,dz, \tag{36}$$

for all complex-valued functions $f \in \widehat{\mathfrak{L}}_0^{1,2}(\Omega)$ *satisfying* $df/d|z| \in L^2(\Omega)$, *where the space* $\widehat{\mathfrak{L}}_0^{1,2}(\Omega)$ *is defined in* (40)*;*

(iv) HARDY-SOBOLEV INEQUALITY WITH MORE GENERAL WEIGHTS*:*

$$\int_{\mathbb{C}} \frac{(a+b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4}} \big|\widetilde{\nabla_{\mathscr{L}}} f\big|^2\,dz \geq \frac{\theta_2\theta_3 - 2\theta_4}{2} \int_{\mathbb{C}} \frac{(a+b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4+2}} |f|^2\,dz + \int_{\mathbb{C}} \frac{\psi(|z|)^2 (a+b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4-2}} |f|^2\,dz, \tag{37}$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$.

The proof of Theorem 3.1 will be based on the following family of weighted Hardy inequalities and the Poincaré type inequality that were obtained in [25, Theorem 3.4 and Theorem 5.1] and [24, Theorem 8.1], where $\mathbb{E} = |x|\mathscr{R}$ is the Euler operator and $\mathscr{R} := d/d|x|$ is the radial derivative.

THEOREM 3.2 ([25, Theorem 3.4]). *Let* $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* Q *and let* $\theta \in \mathbb{R}$. *Then for any complex-valued function* $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, $1 < p < \infty$, *and any homogeneous quasi-norm* $|\cdot|$ *on* $\mathbb{G}$ *for* $\theta p \neq Q$, *we have*

$$\left\| \frac{f}{|x|^\theta} \right\|_{L^p(\mathbb{G})} \leq \left| \frac{p}{Q - \theta p} \right| \left\| \frac{1}{|x|^\theta} \mathbb{E} f \right\|_{L^p(\mathbb{G})}. \tag{38}$$

If $\theta p \neq Q$ then the constant $\left|\frac{p}{Q-\theta p}\right|$ is sharp. For $\theta p = Q$, we have

$$\left\| \frac{f}{|x|^{Q/p}} \right\|_{L^p(\mathbb{G})} \leq p \left\| \frac{\log|x|}{|x|^{Q/p}} \mathbb{E} f \right\|_{L^p(\mathbb{G})} \tag{39}$$

with sharp constant.

Let $\Omega \subset \mathbb{G}$ be an open set and let $\widehat{\mathfrak{L}}_0^{1,p}(\Omega)$ be the completion of $C_0^\infty(\Omega \setminus \{0\})$ with respect to

$$\|f\|_{\widehat{\mathfrak{L}}^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\mathbb{E} f\|_{L^p(\Omega)}, \quad 1 < p < \infty.$$

In the abelian case, when $\mathbb{G} = (\mathbb{R}^2, +)$ and $p = 2$, let us give this definition: let $\Omega \subset \mathbb{R}^2$ be an open set and let $\widehat{\mathfrak{L}}_0^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega \setminus \{0\})$ with respect to

$$\|f\|_{\widehat{\mathfrak{L}}^{1,2}(\Omega)} = \|f\|_{L^2(\Omega)} + \|\mathbb{E} f\|_{L^2(\Omega)}. \tag{40}$$

THEOREM 3.3 ([25, Theorem 5.1]). *Let Ω be a bounded open subset of $\mathbb{G}$. If $1 < p < \infty$, $f \in \widehat{\mathfrak{L}}_0^{1,p}(\Omega)$ and $\mathcal{R} f \in L^p(\Omega)$, then we have*

$$\|f\|_{L^p(\Omega)} \leq \frac{Rp}{Q} \|\mathcal{R} f\|_{L^p(\Omega)} = \frac{Rp}{Q} \left\| \frac{1}{|x|} \mathbb{E} f \right\|_{L^p(\Omega)}, \tag{41}$$

where $R = \sup_{x \in \Omega} |x|$.

THEOREM 3.4 ([24, Theorem 8.1]). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $a, b > 0$, $\theta_2\theta_3 < 0$ and $p\theta_4 - \theta_2\theta_3 \leq Q - p$. Then for any complex-valued function $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, $1 < p < \infty$, and any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$, we have*

$$\frac{Q - p\theta_4 + \theta_2\theta_3 - p}{p} \left\| \frac{\left(a + b|x|^{\theta_2}\right)^{\theta_3/p}}{|x|^{\theta_4+1}} f \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{\left(a + b|x|^{\theta_2}\right)^{\theta_3/p}}{|x|^{\theta_4}} \mathcal{R} f \right\|_{L^p(\mathbb{G})}. \tag{42}$$

If $Q \neq p\theta_4 + p - \theta_2\theta_3$, then the constant $\frac{Q - p\theta_4 + \theta_2\theta_3 - p}{p}$ is sharp.

We briefly recall their proof for the convenience of the reader, but also since these will be useful in our argument.

PROOF OF THEOREM 3.2. Integrating by parts gives for $\theta p \neq Q$

$$\begin{aligned}
&\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\theta p}}\,dx \\
&\quad= \int_0^\infty \int_{\mathfrak{S}} |f(ry)|^p r^{Q-1-\theta p}\,d\sigma(y)\,dr \\
&\quad= -\frac{p}{Q-\theta p}\int_0^\infty r^{Q-\theta p}\,\mathrm{Re}\int_{\mathfrak{S}} |f(ry)|^{p-2} f(ry)\overline{\frac{df(ry)}{dr}}\,d\sigma(y)\,dr \\
&\quad\leq \left|\frac{p}{Q-\theta p}\right| \int_{\mathbb{G}} \frac{|\mathbb{E}f(x)||f(x)|^{p-1}}{|x|^{\theta p}}\,dx \\
&\quad= \left|\frac{p}{Q-\theta p}\right| \int_{\mathbb{G}} \frac{|\mathbb{E}f(x)||f(x)|^{p-1}}{|x|^{\theta+\theta(p-1)}}\,dx.
\end{aligned}$$

Using Hölder's inequality, we obtain

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\theta p}}\,dx \leq \left|\frac{p}{Q-\theta p}\right| \left(\int_{\mathbb{G}} \frac{|\mathbb{E}f(x)|^p}{|x|^{\theta p}} dx\right)^{1/p} \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\theta p}}\,dx\right)^{(p-1)/p},$$

which implies (38).

In order to show the sharpness of the constant, let us check the equality condition in the above Hölder inequality. We consider the function

$$g_1(x) = \frac{1}{|x|^C},$$

where $C \in \mathbb{R}$, $C \neq 0$, and $\theta p \neq Q$. Then, a direct calculation implies

$$\left|\frac{1}{C}\right|^p \left(\frac{|\mathbb{E}g_1(x)|}{|x|^\theta}\right)^p = \left(\frac{|g_1(x)|^{p-1}}{|x|^{\theta(p-1)}}\right)^{p/(p-1)},$$

which satisfies the equality condition in Hölder's inequality. Thus, the constant $\left|\frac{p}{Q-\theta p}\right|$ is sharp in (38).

Now we show (39). Taking into account $\mathfrak{S} := \{x \in \mathbb{G} : |x| = 1\}$ and applying the polar decomposition on homogeneous groups $\mathbb{G}$, then using integration by parts, one calculates

$$\begin{aligned}\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q}\,dx &= \int_0^\infty \int_{\mathfrak{S}} |f(ry)|^p r^{Q-1-Q}\,d\sigma(y)\,dr \\ &= -p \int_0^\infty \log r \,\mathrm{Re} \int_{\mathfrak{S}} |f(ry)|^{p-2} f(ry) \overline{\frac{df(ry)}{dr}}\,d\sigma(y)\,dr \\ &\leq p \int_{\mathbb{G}} \frac{|\mathbb{E}f(x)||f(x)|^{p-1}}{|x|^Q} \big|\log|x|\big|\,dx \\ &= p \int_{\mathbb{G}} \frac{|\mathbb{E}f(x)|\big|\log|x|\big|}{|x|^{Q/p}} \frac{|f(x)|^{p-1}}{|x|^{Q(p-1)/p}}\,dx.\end{aligned}$$

Using again Hölder's inequality, we get

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q}\,dx \leq p\bigg(\int_{\mathbb{G}} \frac{|\mathbb{E}f(x)|^p\big|\log|x|\big|^p}{|x|^Q}\,dx\bigg)^{1/p} \bigg(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q}\,dx\bigg)^{(p-1)/p},$$

which gives (39).

As in the case of (38), we consider the following function to show the sharpness of the constant in (39):

$$g_2(x) = (\log|x|)^C,$$

where $C \in \mathbb{R}$ and $C \neq 0$. Then, one has

$$\left|\frac{1}{C}\right|^p \bigg(\frac{|\mathbb{E}g_2(x)|\big|\log|x|\big|}{|x|^{Q/p}}\bigg)^p = \bigg(\frac{|g_2(x)|^{p-1}}{|x|^{Q(p-1)/p}}\bigg)^{p/(p-1)},$$

which satisfies the equality condition in Hölder's inequality. Thus, we have completed the proof of Theorem 3.2.

Before proving Theorem 3.3, we first show the following proposition.

PROPOSITION 3.5 ([25, Proposition 5.2]). *Let $\Omega \subset \mathbb{G}$ be an open set. If $1 < p < \infty$, $f \in \widehat{\mathfrak{L}}_0^{1,p}(\Omega)$ and $\mathbb{E}f \in L^p(\Omega)$, then we have*

$$\|f\|_{L^p(\Omega)} \leq \frac{p}{Q}\|\mathbb{E}f\|_{L^p(\Omega)}. \tag{43}$$

PROOF. Let us consider the function $\zeta\colon \mathbb{R} \to \mathbb{R}$, which is an even and smooth function, satisfying

- $0 \leq \zeta \leq 1$,

- $\zeta(r) = 1$ if $|r| \leq 1$,
- $\zeta(r) = 0$ if $|r| \geq 2$.

We set $\zeta_\lambda(x) := \zeta(\lambda|x|)$ for $\lambda > 0$. We have the inequality (43) for $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ by (38) when $\theta = 0$. There is $\{f_\ell\}_{\ell=1}^\infty \in C_0^\infty(\Omega \setminus \{0\})$ such that $f_\ell \to f$ in $\widehat{\mathfrak{L}}_0^{1,p}(\Omega)$ as $\ell \to \infty$. Let $\lambda > 0$. From (38) when $\theta = 0$, one gets

$$\|\zeta_\lambda f_\ell\|_{L^p(\Omega)} \leq \frac{p}{Q}\big(\|(\mathbb{E}\zeta_\lambda) f_\ell\|_{L^p(\Omega)} + \|\zeta_\lambda(\mathbb{E} f_\ell)\|_{L^p(\Omega)}\big)$$

for any $\ell \geq 1$. Then, we can immediately see that

$$\lim_{\ell\to\infty} \zeta_\lambda f_\ell = \zeta_\lambda f, \quad \lim_{\ell\to\infty} (\mathbb{E}\zeta_\lambda) f_\ell = (\mathbb{E}\zeta_\lambda) f, \quad \lim_{\ell\to\infty} \zeta_\lambda(\mathbb{E} f_\ell) = \zeta_\lambda(\mathbb{E} f)$$

in $L^p(\Omega)$. By these properties, we obtain

$$\|\zeta_\lambda f\|_{L^p(\Omega)} \leq \frac{p}{Q}\big\{\|(\mathbb{E}\zeta_\lambda) f\|_{L^p(\Omega)} + \|\zeta_\lambda(\mathbb{E} f)\|_{L^p(\Omega)}\big\}.$$

Since

$$|(\mathbb{E}\zeta_\lambda)(x)| \leq \begin{cases} \sup|\mathbb{E}\zeta|, & \text{if } \lambda^{-1} < |x| < 2\lambda^{-1}, \\ 0, & \text{otherwise}, \end{cases}$$

one obtains (43) in the limit as $\lambda \to 0$. The proof of Proposition 3.5 is completed.

PROOF OF THEOREM 3.3. Since $R = \sup_{x\in\Omega} |x|$, by Proposition 3.5 one has

$$\|f\|_{L^p(\Omega)} \leq \frac{p}{Q}\|\mathbb{E} f\|_{L^p(\Omega)} \leq \frac{Rp}{Q}\|\mathcal{R} f\|_{L^p(\Omega)} = \frac{Rp}{Q}\left\|\frac{1}{|x|}\mathbb{E} f\right\|_{L^p(\Omega)},$$

which implies (41).

PROOF OF THEOREM 3.4. Since for $Q = p\theta_4 + p - \theta_2\theta_3$ there is nothing to prove, let us only consider the case $Q \neq p\theta_4 + p - \theta_2\theta_3$. Using polar coordinates $(r, y) = (|x|, x/|x|) \in (0, \infty) \times \mathfrak{S}$ on $\mathbb{G}$, where $\mathfrak{S}$ is the unit quasi-sphere

$$\mathfrak{S} := \{x \in \mathbb{G} : |x| = 1\},$$

and by the polar decomposition on $\mathbb{G}$ (see, for example, [9] or [8]) and using integration by parts, one calculates

$$\int_{\mathbb{G}} \frac{(a + b|x|^{\theta_2})^{\theta_3}}{|x|^{p\theta_4+p}}|f(x)|^p\,dx = \int_0^\infty \int_{\mathfrak{S}} \frac{(a + br^{\theta_2})^{\theta_3}}{r^{p\theta_4+p}}|f(ry)|^p r^{Q-1}\,d\sigma(y)\,dr.$$

Since $\theta_2\theta_3 < 0$ and $p\theta_4 - \theta_2\theta_3 < Q - p$, we get

$$\begin{aligned}
&\int_{\mathbb{G}} \frac{(a+b|x|^{\theta_2})^{\theta_3}}{|x|^{p\theta_4+p}}|f(x)|^p\,dx \\
&\quad\leq \int_0^\infty \int_{\mathfrak{S}} (a+br^{\theta_2})^{\theta_3} r^{Q-1-p\theta_4-p} \\
&\quad\quad\times \left(\frac{br^{\theta_2}}{a+br^{\theta_2}} + \frac{a}{a+br^{\theta_2}}\cdot\frac{Q-p\theta_4-p}{Q-p\theta_4-p+\theta_2\theta_3}\right)|f(ry)|^p\,d\sigma(y)\,dr \\
&\quad= \int_0^\infty \int_{\mathfrak{S}} \frac{(a+br^{\theta_2})^{\theta_3} r^{Q-1-p\theta_4-p}}{Q-p\theta_4-p+\theta_2\theta_3} \\
&\quad\quad\times \left(\frac{\theta_2\theta_3 br^{\theta_2}}{a+br^{\theta_2}} + Q - p\theta_4 - p\right)|f(ry)|^p\,d\sigma(y)\,dr \\
&\quad= \int_0^\infty \int_{\mathfrak{S}} \frac{d}{dr}\left(\frac{(a+br^{\theta_2})^{\theta_3} r^{Q-p\theta_4-p}}{Q-p\theta_4-p+\theta_2\theta_3}\right)|f(ry)|^p\,d\sigma(y)\,dr \\
&\quad= -\frac{p}{Q-p\theta_4-p+\theta_2\theta_3}\int_0^\infty (a+br^{\theta_2})^{\theta_3} r^{Q-p\theta_4-p} \\
&\quad\quad\times \operatorname{Re}\int_{\mathfrak{S}} |f(ry)|^{p-2} f(ry)\overline{\frac{df(ry)}{dr}}\,d\sigma(y)\,dr \\
&\quad\leq \left|\frac{p}{Q-p\theta_4-p+\theta_2\theta_3}\right| \int_{\mathbb{G}} \frac{(a+b|x|^{\theta_2})^{\theta_3}|\mathcal{R}f(x)||f(x)|^{p-1}}{|x|^{p\theta_4+p-1}}\,dx \\
&\quad= \frac{p}{Q-p\theta_4-p+\theta_2\theta_3}\int_{\mathbb{G}} \frac{(a+b|x|^{\theta_2})^{\theta_3(p-1)/p}|f(x)|^{p-1}}{|x|^{(\theta_4+1)(p-1)}} \\
&\quad\quad\times \frac{(a+b|x|^{\theta_2})^{\theta_3/p}}{|x|^{\theta_4}}|\mathcal{R}f(x)|\,dx.
\end{aligned}$$

Here Hölder's inequality gives

$$\begin{aligned}
&\int_{\mathbb{G}} \frac{(a+b|x|^{\theta_2})^{\theta_3}}{|x|^{p\theta_4+p}}|f(x)|^p\,dx \\
&\quad\leq \frac{p}{Q-p\theta_4-p+\theta_2\theta_3}\left(\int_{\mathbb{G}} \frac{(a+b|x|^{\theta_2})^{\theta_3}}{|x|^{p\theta_4+p}}|f(x)|^p\,dx\right)^{(p-1)/p} \\
&\quad\quad\times \left(\int_{\mathbb{G}} \frac{(a+b|x|^{\theta_2})^{\theta_3}}{|x|^{p\theta_4}}|\mathcal{R}f(x)|^p\,dx\right)^{1/p},
\end{aligned}$$

which implies (42).

In order to show the sharpness of the constant, we check the equality condition in above Hölder's inequality. We consider the function

$$g_3(x) = |x|^C,$$

where $C \in \mathbb{R}$, $C \neq 0$ and $Q \neq p\theta_4 + p - \theta_2\theta_3$. Then, a direct calculation gives

$$\left|\frac{1}{C}\right|^p \left(\frac{(a+b|x|^{\theta_2})^{\theta_3/p}|\mathcal{R}g_3(x)|}{|x|^{\theta_4}}\right)^p = \left(\frac{(a+b|x|^{\theta_2})^{\theta_3(p-1)/p}|g_3(x)|^{p-1}}{|x|^{(\theta_4+1)(p-1)}}\right)^{p/(p-1)},$$

which satisfies the equality condition in Hölder's inequality. This shows the sharpness of the constant $(Q - p\theta_4 - p + \theta_2\theta_3)/p$ in (42). Thus, we have completed the proof of Theorem 3.4.

We are now ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Let $\kappa(|z|) \neq 0$ be a radial function. Using polar coordinates for the z-plane $x = r\cos\phi$, $y = r\sin\phi$, we have

$$\begin{aligned}
&\int_{\mathbb{C}} \frac{|\widetilde{\nabla_{\mathscr{L}}} f|^2}{\kappa(|z|)}\,dz \\
&= \int_{\mathbb{C}} \Big(\big|\big(i\partial_x + y\psi(|z|)\big)f\big|^2 + \big|\big(i\partial_y - x\psi(|z|)\big)f\big|^2 \Big)\frac{dz}{\kappa(|z|)} \\
&= \int_0^\infty \int_0^{2\pi} \left(\left| \left(i\cos\phi\,\partial_r - \frac{i\sin\phi}{r}\partial_\phi + \psi(r) r\sin\phi \right) f \right|^2 \right) r\,d\phi \frac{dr}{\kappa(r)} \\
&\quad + \int_0^\infty \int_0^{2\pi} \left(\left| \left(i\sin\phi\,\partial_r + \frac{i\cos\phi}{r}\partial_\phi - \psi(r) r\cos\phi \right) f \right|^2 \right) r\,d\phi \frac{dr}{\kappa(r)} \\
&= \int_0^\infty \int_0^{2\pi} \left(|\partial_r f|^2 + \frac{|\partial_\phi f|^2}{r^2} + \psi(r)^2 r^2 |f|^2 \right) r\,d\phi \frac{dr}{\kappa(r)} \\
&\quad - 2\,\mathrm{Re} \int_0^\infty \int_0^{2\pi} \left(\frac{i\partial_\phi f}{r} \cdot \overline{\psi(r) r f} \right) r\,d\phi \frac{dr}{\kappa(r)} \\
&= \int_0^\infty \int_0^{2\pi} \left(|\partial_r f|^2 + \frac{1}{r^2} \big| i\partial_\phi f - \psi(r) r^2 f \big|^2 \right) r\,d\phi \frac{dr}{\kappa(r)}.
\end{aligned}$$

Let us represent f via its Fourier series

$$f(r,\phi)=\sum_{k=-\infty}^{\infty} f_k(r)e^{ik\phi}.$$

Then from the assumptions on ψ, we obtain

$$\begin{aligned}\frac{1}{r^2}\int_0^{2\pi}|i\partial_\phi f-\psi(r)r^2 f|^2\,d\phi &= \frac{2\pi}{r^2}\sum_k\big(k+\psi(r)r^2\big)^2|f_k(r)|^2\\ &\geq \frac{2\pi}{r^2}\min_k\big(k+\psi(r)r^2\big)^2\sum_k|f_k(r)|^2\\ &= \frac{1}{r^2}\min_k\big(k+\psi(r)r^2\big)^2\int_0^{2\pi}|f|^2\,d\phi\\ &= r^2\psi(r)^2\int_0^{2\pi}|f|^2\,d\phi.\end{aligned}$$

Thus, we arrive at

$$\int_{\mathbb{C}}\frac{\big|\widetilde{\nabla_{\mathscr{L}}}f\big|^2}{\kappa(|z|)}\,dz\geq\int_{\mathbb{C}}\frac{1}{\kappa(|z|)}\left|\frac{d}{d|z|}f\right|^2dz+\int_{\mathbb{C}}\frac{|z|^2\psi(|z|)^2}{\kappa(|z|)}|f|^2\,dz. \tag{44}$$

Putting $\kappa(|z|)=|z|^{2\theta_1}$, it follows that

$$\int_{\mathbb{C}}\frac{\big|\widetilde{\nabla_{\mathscr{L}}}f\big|^2}{|z|^{2\theta_1}}\,dz\geq\int_{\mathbb{C}}\frac{1}{|z|^{2\theta_1}}\left|\frac{d}{d|z|}f\right|^2dz+\int_{\mathbb{C}}\frac{\psi(|z|)^2}{|z|^{2\theta_1-2}}|f|^2\,dz. \tag{45}$$

In the abelian case $\mathbb{G}=(\mathbb{R}^2,+)$, $p=2$ and $\theta=\theta_1+1$ with $\theta_1\neq 0$, (38) implies

$$\int_{\mathbb{C}}\frac{1}{|z|^{2\theta_1}}\left|\frac{d}{d|z|}f\right|^2dz\geq\theta_1^2\int_{\mathbb{C}}\frac{|f|^2}{|z|^{2\theta_1+2}}\,dz.$$

Putting this in (45), we obtain (34).

Putting $\kappa(|z|)=|\log|z||^{-2}$ in (44), one has

$$\begin{aligned}&\int_{\mathbb{C}}\big|\log|z|\big|^2\big|\widetilde{\nabla_{\mathscr{L}}}f\big|^2\,dz\\ &\quad\geq\int_{\mathbb{C}}\big|\log|z|\big|^2\left|\frac{d}{d|z|}f\right|^2dz+\int_{\mathbb{C}}\psi(|z|)^2|z|^2\big|\log|z|\big|^2|f|^2\,dz.\end{aligned}\tag{46}$$

In the abelian case $\mathbb{G}=(\mathbb{R}^2,+)$ and $p=2$, (39) implies

$$\int_{\mathbb{C}}\big|\log|z|\big|^2\left|\frac{d}{d|z|}f\right|^2dz\geq\frac{1}{4}\int_{\mathbb{C}}|f|^2\,dz.$$

Putting this in (46), we obtain (35).

Putting $\kappa(|z|) = 1$ in (44), we get

$$\int_{\mathbb{C}} \left|\widetilde{\nabla_{\mathscr{L}}} f\right|^2 dz \geq \int_{\mathbb{C}} \left|\frac{d}{d|z|} f\right|^2 dz + \int_{\mathbb{C}} \psi(|z|)^2 |z|^2 |f|^2 \, dz. \tag{47}$$

In the abelian case, when Ω is a bounded domain in $(\mathbb{R}^2, +)$, and $p = 2$, (41) implies

$$\int_{\Omega} \left|\frac{d}{d|z|} f\right|^2 dz \geq \frac{1}{R^2} \int_{\Omega} |f|^2 \, dz.$$

Putting this in (47), we obtain (36).

Putting $\kappa(|z|) = (a + b|z|^{\theta_2})^{-\theta_3}/|z|^{-2\theta_4}$ in (44), we have

$$\begin{aligned} &\int_{\mathbb{C}} \frac{(a + b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4}} \left|\widetilde{\nabla_{\mathscr{L}}} f\right|^2 dz \\ &\geq \int_{\mathbb{C}} \frac{(a + b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4}} \left|\frac{d}{d|z|} f\right|^2 dz + \int_{\mathbb{C}} \frac{\psi^2(|z|)(a + b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4 - 2}} |f|^2 \, dz. \end{aligned} \tag{48}$$

Again, in the abelian case $\mathbb{G} = (\mathbb{R}^2, +)$ and $p = 2$, (42) gives

$$\frac{\theta_2\theta_3 - 2\theta_4}{2} \int_{\mathbb{C}} \frac{(a + b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4 + 2}} |f|^2 \, dz \leq \int_{\mathbb{C}} \frac{(a + b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4}} \left|\frac{d}{d|z|} f\right|^2 dz,$$

for $a, b > 0$, $\theta_2\theta_3 < 0$ and $2\theta_4 \leq \theta_2\theta_3$. Putting this in (48), we obtain (37). Thus, we have completed the proof of Theorem 3.1.

Now we give some inequalities for real-valued functions to show the best estimates one can expect. While estimates for real-valued functions have less physical meaning than those for complex-valued functions in questions of the spectral theory, they also find their use in applications to the existence of real (or positive) solutions to some nonlinear equations.

REMARK 3.6. Let $\psi = \psi(|z|)$ be a radial real-valued function such that $\psi \in L^2_{\mathrm{loc}}(\mathbb{C} \setminus \{0\})$. By a direct calculation, we have for all real-valued functions $f \in W^{1,2}_{\mathscr{L}}(\mathbb{C}^n)$

$$\begin{aligned} \int_{\mathbb{C}^n} \left|\widetilde{\nabla_{\mathscr{L}}} f\right|^2 dz &= \sum_{j=1}^{n} \int_{\mathbb{C}^n} \left(\left|\left(i\partial_{x_j} + y_j \psi(|z|)\right) f\right|^2 + \left|\left(i\partial_{y_j} - x_j \psi(|z|)\right) f\right|^2 \right) dz \\ &= \int_{\mathbb{C}^n} |\nabla f|^2 \, dz + \int_{\mathbb{C}^n} |z|^2 \psi(|z|)^2 |f|^2 \, dz. \end{aligned} \tag{49}$$

Here $W^{1,2}_{\mathscr{L}}$ is defined in (10). Using the well-known Hardy inequality for $n \geq 1$, we obtain

$$\int_{\mathbb{C}^n} \left|\widetilde{\nabla_{\mathscr{L}}} f\right|^2 dz \geq (n-1)^2 \int_{\mathbb{C}^n} \frac{|f|^2}{|z|^2} dz + \int_{\mathbb{C}^n} |z|^2 \psi(|z|)^2 |f|^2 dz, \tag{50}$$

for all real-valued functions $f \in W^{1,2}_{\mathscr{L}}(\mathbb{C}^n)$ and $n \geq 1$. We note that for $\psi(|z|) = 1/2$, we recover the classical Landau Hamiltonian, i.e.,

$$\widetilde{\mathscr{L}}_{1/2} = -\mathscr{L} \quad \text{and} \quad \widetilde{\nabla_{\mathscr{L}_{1/2}}} = \nabla_{\mathscr{L}}.$$

Let Ω be a bounded domain in $\mathbb{C}$ with $0 \in \Omega$ and $R \geq e \cdot \sup_{z\in\Omega}\{|z|\}$. Then, in the case $n = 1$ we can use the critical Hardy inequality with sharp constant for $f \in W^{1,2}_{\mathscr{L}}(\Omega)$ (see for example [1], [3]):

$$\int_{\Omega} \left|\widetilde{\nabla_{\mathscr{L}}} f\right|^2 dz \geq \frac{1}{4} \int_{\Omega} \frac{|f|^2}{|z|^2(\log(R/|z|))^2} dz + \int_{\Omega} |z|^2 \psi(|z|)^2 |f|^2 dz, \tag{51}$$

for all real-valued functions $f \in W^{1,2}_{\mathscr{L}}(\Omega)$ and $n = 1$. Since the constants in the Hardy and critical Hardy inequalities are sharp, then the constants in the obtained inequalities (50) and (51) are sharp.

COROLLARY 3.7 (Uncertainty type principle). *Let Ω be a bounded domain in $\mathbb{C}$ with $0 \in \Omega$ and $R \geq e \cdot \sup_{z\in\Omega}\{|z|\}$. Let $\psi = \psi(|z|)$ be a radial real-valued function such that $\psi \in L^2_{\mathrm{loc}}(\mathbb{C} \setminus \{0\})$. Then we have*

$$\left\|\widetilde{\nabla_{\mathscr{L}}} f\right\|_{L^2(\mathbb{C}^n)} \|f\|_{L^2(\mathbb{C}^n)} \geq \int_{\mathbb{C}^n} \left(\sqrt{\frac{(n-1)^2}{|z|^2} + |z|^2\psi(|z|)^2}\right) |f|^2 dz \tag{52}$$

for $n \geq 1$ and all real-valued functions $f \in W^{1,2}_{\mathscr{L}}(\mathbb{C}^n)$, and

$$\left\|\widetilde{\nabla_{\mathscr{L}}} f\right\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} \geq \int_{\Omega} \left(\sqrt{\frac{1}{4|z|^2(\log(R/|z|))^2} + |z|^2\psi(|z|)^2}\right) |f|^2 dz \tag{53}$$

for $n = 1$ and all real-valued functions $f \in W^{1,2}_{\mathscr{L}}(\Omega)$.

PROOF. By (50) we get for $n \geq 1$

$$\begin{aligned}\left\|\widetilde{\nabla_{\mathscr{L}}} f\right\|_{L^2(\mathbb{C}^n)} \|f\|_{L^2(\mathbb{C}^n)} &\geq \left\| \left(\sqrt{\frac{(n-1)^2}{|z|^2} + |z|^2\psi(|z|)^2}\right) f \right\|_{L^2(\mathbb{C}^n)} \|f\|_{L^2(\mathbb{C}^n)} \\ &\geq \int_{\mathbb{C}^n} \left(\sqrt{\frac{(n-1)^2}{|z|^2} + |z|^2\psi(|z|)^2}\right) |f|^2 dz,\end{aligned}$$

which gives (52). Similarly, using (51) we obtain (53). The proof is complete.

REMARK 3.8. In the case $n = 1$ of the Remark 3.6, we also can use the another type of critical Hardy inequality (see for example Solomyak [27]) in (49):

$$\int_{\mathbb{C}} \left|\widetilde{\nabla_{\mathscr{L}}} f\right|^2 dz \geq C \int_{\mathbb{C}} \frac{|f|^2}{|z|^2(1+\log^2|z|)} dz + \int_{\mathbb{C}} |z|^2 \psi(|z|)^2 |f|^2 dz,$$

where C is a positive constant.

REMARK 3.9. Let $\mathscr{L}_G$ be the magnetic Baouendi-Grushin operator on $\mathbb{C}^n$ with the constant magnetic field

$$\mathscr{L}_G = \sum_{j=1}^{n} \big(i\partial_{x_j} + \psi_{1,j}(y_j)\big)^2 + \big(i|x|^\gamma \partial_{y_j} + \psi_{2,j}(x_j)\big)^2,$$

where $|x| = \sqrt{|x_1|^2 + \ldots + |x_n|^2}$ and $\psi_{1,j}, \psi_{2,j} \in L^2_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$. Setting $\widehat{X}_j = i\partial_{x_j} + \psi_{1,j}(y_j)$ and $\widehat{Y}_j = i|x|^\gamma \partial_{y_j} + \psi_{2,j}(x_j)$, we write

$$\nabla_{G\mathscr{L}} f = (\widehat{X}_1 f, \ldots, \widehat{X}_n f, \widehat{Y}_1 f, \ldots, \widehat{Y}_n f).$$

Then, a direct calculation gives for all real-valued functions $f \in C_0^\infty(\mathbb{R}^{2n} \setminus \{0\})$

$$\begin{aligned}
&\int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} |\nabla_{G\mathscr{L}} f|^2 dz \\
&= \sum_{j=1}^{n} \int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \Big(\big|\big(i\partial_{x_j} + \psi_{1,j}(y_j)\big) f\big|^2 + \big|\big(i|x|^\gamma \partial_{y_j} + \psi_{2,j}(x_j)\big) f\big|^2 \Big) dz \\
&= \int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} |\nabla_\gamma f|^2 dz \\
&\qquad + \sum_{j=1}^{n} \int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \big(|\psi_{2,j}(x_j)|^2 + |\psi_{1,j}(y_j)|^2\big) |f|^2 dz.
\end{aligned}$$

Then putting $m = k = n$ in (15), and hence $Q = n(2+\gamma)$, we obtain the following Hardy inequality for magnetic Baouendi-Grushin operator on $\mathbb{C}^n$ with the constant magnetic field and for any real-valued function $f \in$

$C_0^\infty(\mathbb{R}^{2n} \setminus \{0\})$ with sharp constant

$$\int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} |\nabla_{G\mathscr{L}} f|^2 \, dz \geq \left(\frac{n(2+\gamma)+\alpha_1-2}{2}\right) \int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \frac{|x|^{2\gamma}}{\rho^{2\gamma+2}} |f|^2 \, dz + \sum_{j=1}^n \int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \big(|\psi_{2,j}(x_j)|^2 + |\psi_{1,j}(y_j)|^2\big) |f|^2 \, dz,$$

where $n(2+\gamma)+\alpha_1-2 > 0$ and $n+\alpha_2\gamma > 0$.

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