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HOMOTOPY 4-SPHERES HAVE LITTLE SYMMETRY

PETER ORLIK*

Introduction.

The degree of symmetry, $N(M)$ of a topological (smooth) manifold M is defined as the maximal dimension of a compact Lie group G that can act continuously (smoothly) and effectively on M. It is well-known that if M^n is smooth and $N(M^n)=n(n+1)/2$ then M is S^n or RP^n. It was shown in [3] that if Σ^n, $n \geqq 40$ is an exotic sphere then $N(\Sigma)<n^2/8+1$. A theorem of Seifert [8] implies that if Σ^3 is a counterexample to the Poincaré conjecture then $N(\Sigma^3)=0$. The purpose of this note is to find all simply connected 4-manifolds M^4 with $N(M^4)>1$ and obtain as a corollary that if Σ^4 is a counterexample to the Poincaré conjecture then $N(\Sigma^4) \leqq 1$.

Groups.

Given a compact Lie group G a theorem of Mann [4] computes the smallest dimension $m(G)$ of a manifold admitting an effective G action. Note that for computing $N(G)$ it is sufficient to consider almost effective actions of groups $G=T^r \times G'$ where G' is semi-simple.

PROPOSITION. *If G acts almost effectively on M^4, then the maximal torus of G is at most 2-dimensional.*

PROOF. If the maximal torus T is 3-dimensional then M^4 has a 3-dimensional orbit and a theorem of Mostert [5] applies. The orbit space cannot be a circle, so it is a closed interval, and the action is equivalent to a smooth action. The non-principal isotropy groups of the induced T action must be 1-dimensional toruses and together they do not annihilate $\pi_1(T)$.

The following is a list of all compact Lie groups $G=T^r \times G'$ where G' is semi-simple with maximal torus T^q so that $\dim G \leqq 10$, $m(G) \leqq 4$ and $r+q \leqq 2$.

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G	Spin 5	SU(3)	Spin 3 × Spin 3	S^1 × Spin 3	Spin 3	T^2	S^1
dim G	10	8	6	4	3	2	1
$m(G)$	4	4	3	3	2	2	1

Note that Spin 4 = Spin 3 × Spin 3.

Transitive actions.

These are clearly equivalent to smooth actions and the only possibilities are:

$$\text{Spin}\,5/\text{Spin}\,4 = S^4, \quad \text{SU}(3)/\text{U}(2) = \text{CP}^2, \quad \text{Spin}\,4/T^2 = S^2 \times S^2 .$$

Actions with 3-dimensional orbits.

According to Mostert [5] the action is equivalent to a smooth action. The orbit space is an interval with isotropy types $\{(H); (U_0), (U_1)\}$ and we may assume $H \subset U_i$. Now U_i/H is an r_i-sphere so M is homeomorphic to

$$G \times_{U_0} D^{r_0+1} \cup G \times_{U_1} D^{r_1+1}$$

by an equivariant homeomorphism of the common boundary G/H. The manifolds thus obtained are classified by the components of the double coset space $N_0 \setminus N(H)/N_1$ where $N_i = N(H) \cap N(U_i)$, see [6].

G = Spin 4 admits the following isotropy structures:

$$\{(\text{Spin}\,3); \text{Spin}\,4, \text{Spin}\,4\} = S^4$$

$$\{(\text{Spin}\,3); (\text{Spin}\,3 \times S^1), \text{Spin}\,4\} = \text{CP}^2$$

$$\{(\text{Spin}\,3); (\text{Spin}\,3 \times S^1), (\text{Spin}\,3 \times S^1)\} = \text{CP}^2 \# \overline{\text{CP}}^2$$

where $\overline{\text{CP}}^2$ is the reverse orientation of CP^2. The manifolds are determined by the isotropy structures.

$$G = S^1 \times \text{Spin}\,3 \text{ can act as a subgroup of Spin}\,4 .$$

In addition we have the following possibilities:

$$\{(S^1); (\text{Spin}\,3), (T^2)\} = S^4, \quad \{(S^1); (T^2), (T^2)\} = S^2 \times S^2 ,$$

and the manifolds are again determined by the isotropy structure.

G = Spin 3 has finite principal orbit type (H). If $H = 1$ then we obtain restrictions of the above actions. If $H = \mathbf{Z}_p$ then we have

$$\{(\mathbf{Z}_p); (S^1), (S^1)\} = Q_p$$

where Q_p is the double of the D^2-bundle over S^2 with euler class p and boundary the lens space $L(p,1)$.

Finally, if

$$H = D_8^* = \{x,y \mid x^2=(xy)^2=y^2\},$$

the binary dihedral group, then we have the following possible isotropy structure $\{(D_8^*); (\mathrm{Pin}\,2), (\mathrm{Pin}\,2)\}$. The normalizer of D_8^* in Spin 3 is the binary octahedral group O^* and the double coset $N_0 \backslash N(H)/N_1$ has two components. The component of the identity gives a non-simply connected manifold. The other component corresponds to the irreducible 5-dimensional representation of SO(3) given in [2, p. 43] so the total space is S^4. I am indebted to G. Bredon for explaining this example.

Actions with 2-dimensional orbits.

$G = \mathrm{Spin}\,3$ must have principal isotropy type (S^1) and principal orbit type S^2. The slice is a 2-dimensional cohomology manifold, hence a 2-manifold and may be taken as a disk. The only other orbits are fixed points so the orbit space, M^* is a 2-manifold. Note that M^* is simply connected because M is. If $M^*=S^2$ then all orbits are principal and M is an S^2 bundle over S^2 with structure group Spin 3. Thus the associated principal bundle is classified by

$$S^3 \to S^7 \to S^4$$

and hence $M = S^2 \times S^2$. If $M^* = D^2$ then the action is easily seen to admit a cross-section and it is the action of G in the first factor of the join $S^4 = S^2 \circ S^1$.

$G = T^2$ actions on simply connected 4-manifolds were classified in [7]. The only manifolds that occur are equivariant connected sums of S^4, CP^2, $\overline{CP}^2$, $S^2 \times S^2$.

THEOREM. *The degree of symmetry of a closed simply connected 4-manifold M is given as follows*:

M	S^4	CP^2	$S^2 \times S^2$, $\overline{CP}^2 \# CP^2$	Q_p, $p>1$	# of S^4, CP^2, $\overline{CP}^2$, $S^2 \times S^2$
$N(M)$	10	8	6	3	2

and for all other M, $N(M) \leqq 1$.

COROLLARY. *If Σ^4 is a counterexample to the Poincaré conjecture then the largest compact Lie group that can act effectively on Σ^4 is S^1.*

REMARK. A theorem of Atiyah and Hirzebruch [1] implies that there are smooth simply connected 4-manifolds with no smooth S^1-action.

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UNIVERSITY OF WISCONSIN, MADISON