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VANDERMONDE DETERMINANTAL IDEALS

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Abstract

We show that the ideal generated by maximal minors (i.e., $k+1$-minors) of a $(k+1)\times n$ Vandermonde matrix is radical and Cohen-Macaulay. Note that this ideal is generated by all Specht polynomials with shape $(n-k, 1, \dots, 1)$.

1. Introduction

Let n, k be integers with $n > k \geq 1$. Consider the polynomial ring $R = K[x_1, \dots, x_n]$ over a field K, and the following non-square Vandermonde matrix

$$M_{n,k} := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^k & x_2^k & \cdots & x_n^k \end{pmatrix}.$$

Let $I_{n,k}^{\mathrm{Vd}}$ denote the ideal of R generated by all maximal minors (i.e., $k+1$ minors) of $M_{n,k}$.

The purpose of this paper is to prove the following.

THEOREM 1.1. *$R/I_{n,k}^{\mathrm{Vd}}$ is a reduced Cohen-Macaulay ring with* $\dim R/I_{n,k}^{\mathrm{Vd}} = k$ *and* $\deg R/I_{n,k}^{\mathrm{Vd}} = S(n,k)$*, where $S(n,k)$ stands for the Stirling number of the second kind.*

The present paper can be seen as the precursor of our ongoing project [6] on *Specht ideals*. For a partition λ of n, we can consider the ideal

$$I_\lambda^{\mathrm{Sp}} = (\Delta_T \mid T \text{ is a Young tableau of shape } \lambda)$$

of R, where $\Delta_T \in R$ denotes the *Specht polynomial* corresponding to T (see [2]). Then we have $I_{n,k}^{\mathrm{Vd}} = I_{(n-k,1,\dots,1)}^{\mathrm{Sp}}$. Note that the K-vector subspace

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$\langle \Delta_T \mid T$ is a Young tableau of shape $\lambda\rangle$ of R is the *Specht module* associated with λ as an S_n-module. The Specht modules are often constructed in different manner (e.g., using Young tabloids), and play crucial role in the representation theory of symmetric groups (see, for example [5]). General Specht ideals are much more delicate than the Vandermont case $I_{n,k}^{\mathrm{Vd}}$. For example, $R/I_\lambda^{\mathrm{Sp}}$ is not even pure dimensional for many λ, and the Cohen-Macaulayness of $R/I_\lambda^{\mathrm{Sp}}$ may depend on $\mathrm{char}(K)$ for some fixed λ. In [6], we will use the representation theory of symmetric groups.

It is noteworthy that Fröberg and Shapiro [1] also studied some variants of $R/I_\lambda^{\mathrm{Vd}}$ in a different context.

2. Results and proofs

Extending the base field, we may assume that K is algebraically closed. Theoretically, this assumption is not necessary in the following argument, but it makes the expositions more readable.

For an ideal $I \subset R$, set $V(I) := \{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R, \mathfrak{p} \supset I\}$ as usual. For $\mathbf{a} = (a_1, \dots, a_n) \in K^n$, let $\mathfrak{m}_\mathbf{a}$ denote the maximal ideal $(x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$ of R. By abuse of notation, we just write $\mathbf{a} \in V(I)$ to mean $\mathfrak{m}_\mathbf{a} \in V(I)$. Clearly, $\mathbf{a} \in V(I)$ if and only if $f(\mathbf{a}) = 0$ for all $f \in I$.

PROPOSITION 2.1. *We have* $\dim R/I_{n,k}^{\mathrm{Vd}} = k$ *and*

$$\deg\left(R\Big/\sqrt{I_{n,k}^{\mathrm{Vd}}}\right) = S(n,k),$$

where $S(n,k)$ *stands for the Stirling number of the second kind, that is, the number of ways to partition the set* $\{1, 2, \dots, n\}$ *into k non-empty subsets.*

PROOF. For $\mathbf{a} = (a_1, \dots, a_n) \in K^n$, $\mathbf{a} \in V(I_{n,k}^{\mathrm{Vd}})$ if and only if

$$\mathrm{rank}(M_{n,k}(\mathbf{a})) \le k,$$

where $M_{n,k}(\mathbf{a})$ is the matrix given by putting $x_i = a_i$ for each i in $M_{n,k}$. The latter condition is equivalent to that $\#\{a_1, \dots, a_n\} \le k$. This is also equivalent to that there is a partition $\Pi = \{F_1, \dots, F_k\}$ of the set $[n] := \{1, 2, \dots, n\}$ such that $a_i = a_j$ for all $i, j \in F_\ell$ ($\ell = 1, 2, \dots, k$). For the above partition Π, let P_Π denote the prime ideal

$$(x_i - x_j \mid i, j \in F_\ell \text{ for } \ell = 1, \dots, k)$$

of R. Since

$$\sqrt{I_{n,k}^{\mathrm{Vd}}} = \bigcap_{\substack{\Pi:\ \text{partition of } [n] \\ \#\Pi = k}} P_\Pi,$$

$\dim R/P_\Pi = k$ for all Π, and $\deg R/P_\Pi = 1$, we are done.

Applying elementary column operations to $M_{n,k}$, we get the following matrix

$$M'_{n,k} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ x_1 & x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ x_1^2 & x_2^2 - x_1^2 & x_3^2 - x_1^2 & \cdots & x_n^2 - x_1^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^k & x_2^k - x_1^k & x_3^k - x_1^k & \cdots & x_n^k - x_1^k \end{pmatrix}.$$

Consider its $k \times (n-1)$ submatrix

$$N_{n,k} := \begin{pmatrix} x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ x_2^2 - x_1^2 & x_3^2 - x_1^2 & \cdots & x_n^2 - x_1^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_2^k - x_1^k & x_3^k - x_1^k & \cdots & x_n^k - x_1^k \end{pmatrix}.$$

Clearly, $I_{n,k}^{\mathrm{Vd}}$ is generated by all maximal minors (i.e., k-minors) of $N_{n,k}$.

THEOREM 2.2. *$R/I_{n,k}^{\mathrm{Vd}}$ is Cohen-Macaulay. Moreover, its minimal graded free resolution is given by the Eagon-Northcott complex* (*see, for example* [4]) *associated to the matrix $N_{n,k}$.*

PROOF. Since $\mathrm{ht}(I_{n,k}^{\mathrm{Vd}}) = \dim R - \dim R/I_{n,k}^{\mathrm{Vd}} = n - k = (n-1) - k + 1$, $I_{n,k}^{\mathrm{Vd}}$ is a *standard determinantal ideal* in the sense of [4]. Hence the assertion is immediate from well-known properties of this notion (cf. §1.2 of [4]).

When we construct the Eagon-Northcott resolution of $I_{n,k}^{\mathrm{Vd}}$, we use a symmetric power $\mathrm{Sym}_i\, V$ of a k-dimensional vector space V with a basis $e_1, \ldots, e_k$ such that $\deg e_i = i$ for each i. Set

$$p_{i,j}^m := \#\{(a_1, \ldots, a_m) \in \mathbb{N}^m \mid a_1 + a_2 + \cdots + a_m = i,\ a_1 + 2a_2 + \cdots + ma_m = j\}$$

For simplicity, set $p_{0,j}^m := \delta_{0,j}$. The following facts are easy to see:

(1) $p_{i,j}^m \neq 0$ if and only if $i \leq j \leq im$;

(2) $\displaystyle\sum_j p_{i,j}^m = \binom{m+i-1}{i}$.

For the vector space V discussed above, the dimension of the degree j part of $\mathrm{Sym}_i\, V$ is $p_{i,j}^k$.

COROLLARY 2.3. *For $i \geq 1$, we have*

$$\beta_{i,j}(R/I_{n,k}^{\mathrm{Vd}}) = p_{i-1,j-\frac{1}{2}k(k+1)}^{k} \times \binom{n-1}{k+i-1}.$$

PROOF. Since the minimal free resolution of $R/I_{n,k}^{\mathrm{Vd}}$ is given by the Eagon-Northcott complex, we have

$$\begin{aligned}\beta_{i,j}(R/I_{n,k}^{\mathrm{Vd}}) &= \left(\dim_K\left[(\mathrm{Sym}_{i-1} V)\otimes_K \bigwedge^{k} V\right]_j\right) \times \left(\dim_K \bigwedge^{k+i-1} W\right)\\ &= \left(\dim_K[\,\mathrm{Sym}_{i-1} V\,]_{j-\frac{1}{2}k(k+1)}\right) \times \left(\dim_K \bigwedge^{k+i-1} W\right)\\ &= p_{i-1,j-\frac{1}{2}k(k+1)}^{k} \times \binom{n-1}{k+i-1},\end{aligned}$$

where V is the K-vector space considered above, and W is a K-vector space of dimension $n-1$.

EXAMPLE 2.4. Since $p_{i,j}^2 = 0$ or 1 for all i, j, we have $\beta_{i,j}(R/I_{n,2}^{\mathrm{Vd}}) = 0$ or $\binom{n-1}{i+1}$ for all $i \geq 1$. For example, the Betti table of $R/I_{6,2}^{\mathrm{Vd}}$ is the following.

```
total: 1 10 20 15  4
    0: 1  .  .  .  .
    1: .  .  .  .  .
    2: . 10 10  5  1
    3: .  . 10  5  1
    4: .  .  .  5  1
    5: .  .  .  .  1
```

The following are the Betti tables of $R/I_{6,3}^{\mathrm{Vd}}$ and $R/I_{7,3}^{\mathrm{Vd}}$, respectively.

```
total: 1 10 15  6
    0: 1  .  .  .
    1: .  .  .  .
    2: .  .  .  .
    3: .  .  .  .
    4: .  .  .  .
    5: . 10  5  1
    6: .  .  5  1
    7: .  .  5  2
    8: .  .  .  1
    9: .  .  .  1
```

```
total: 1 20 45 36 10
    0: 1  .  .  .  .
    1: .  .  .  .  .
    2: .  .  .  .  .
    3: .  .  .  .  .
    4: .  .  .  .  .
    5: . 20 15  6  1
    6: .  . 15  6  1
    7: .  . 15 12  2
    8: .  .  .  6  2
    9: .  .  .  6  2
   10: .  .  .  .  1
   11: .  .  .  .  1
```

THEOREM 2.5. *We have*

$$\deg R/I^{\mathrm{Vd}}_{n,k} = S(n,k).$$

PROOF. Since $I^{\mathrm{Vd}}_{n,1}$ is an ideal generated by linear forms, we have $\deg R/I^{\mathrm{Vd}}_{n,1} = 1 = S(n,1)$ for all $n \geq 2$. Since $I^{\mathrm{Vd}}_{n,n-1}$ is a principal ideal generated by a polynomial of degree $\binom{n}{2}$, we have $\deg R/I^{\mathrm{Vd}}_{n,n-1} = \binom{n}{2} = S(n,n-1)$. It is well-known that the Stirling numbers of the second kind satisfy the recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

So it suffices to show that $\deg R/I^{\mathrm{Vd}}_{n,k}$ also satisfies the corresponding relation

$$\deg R/I^{\mathrm{Vd}}_{n,k} = \deg R'/I^{\mathrm{Vd}}_{n-1,k-1} + k\big(\deg R'/I^{\mathrm{Vd}}_{n-1,k}\big) \tag{2.1}$$

for $n-1 > k$, where R' is the polynomial ring $K[x_1, \dots, x_{n-1}]$.

From now on, we assume that $n-1 > k$. Note that the matrices

$$N_{n-1,k-1} := \begin{pmatrix} x_2 - x_1 & x_3 - x_1 & \cdots & x_{n-1} - x_1 \\ x_2^2 - x_1^2 & x_3^2 - x_1^2 & \cdots & x_{n-1}^2 - x_1^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{k-1} - x_1^{k-1} & x_3^{k-1} - x_1^{k-1} & \cdots & x_{n-1}^{k-1} - x_1^{k-1} \end{pmatrix}$$

and

$$N_{n-1,k} := \begin{pmatrix} x_2 - x_1 & x_3 - x_1 & \cdots & x_{n-1} - x_1 \\ x_2^2 - x_1^2 & x_3^2 - x_1^2 & \cdots & x_{n-1}^2 - x_1^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{k} - x_1^{k} & x_3^{k} - x_1^{k} & \cdots & x_{n-1}^{k} - x_1^{k} \end{pmatrix}$$

can be regarded as submatrices of $N_{n,k}$. Let J_1 and J_2 be the ideals (of R) generated by all maximal minors of $N_{n-1,k-1}$ and $N_{n-1,k}$, respectively. By [3, Lemma 2.3(2)], we have

$$\deg R/I^{\mathrm{Vd}}_{n,k} = \deg R/J_1 + k(\deg R/J_2).$$

On the other hand, we have $R/J_1 \cong (R'/I^{\mathrm{Vd}}_{n-1,k-1})[x_n]$, and hence $\deg R/J_1 = \deg R'/I^{\mathrm{Vd}}_{n-1,k-1}$. Similarly, $\deg R/J_2 = \deg R'/I^{\mathrm{Vd}}_{n-1,k}$. Now (2.1) is clear.

REMARK 2.6. In the first version of this paper, the key formula (2.1) was shown by a direct computation from Corollary 2.3. More precisely, the equations

$$\beta_{1,j}(R/I^{\mathrm{Vd}}_{n,k}) = \beta_{1,j-k}(R'/I^{\mathrm{Vd}}_{n-1,k-1}) + \beta_{1,j}(R'/I^{\mathrm{Vd}}_{n-1,k})$$

and

$$\beta_{i,j}(R/I_{n,k}^{\mathrm{Vd}}) = \beta_{i,j-k}(R'/I_{n-1,k-1}^{\mathrm{Vd}}) + \beta_{i,j}(R'/I_{n-1,k}^{\mathrm{Vd}}) + \beta_{i-1,j-k}(R'/I_{n-1,k}^{\mathrm{Vd}})$$

hold for $i \geq 2$. One can check this in Example 2.4. Anyway, we see that these equations imply (2.1).

Now we know that (2.1) is a direct consequence of [3, Lemma 2.3(2)]. It is noteworthy that this lemma is a result of Gorenstein liaison theory.

COROLLARY 2.7. *$R/I_{n,k}^{\mathrm{Vd}}$ is reduced.*

PROOF. Since $A := R/I_{n,k}^{\mathrm{Vd}}$ is Cohen-Macaulay, any non-zero ideal $I \subset A$ satisfies $\dim I = \dim A$ as an A-module. Hence if A is not reduced, then $\deg A > \deg A/\sqrt{(0)}$. However, it contradicts the fact that

$$\deg\bigl(R/I_{n,k}^{\mathrm{Vd}}\bigr) = S(n,k) = \deg\Bigl(R\Big/\sqrt{I_{n,k}^{\mathrm{Vd}}}\Bigr).$$

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