MATH. SCAND. 125 (2019), 227–238

A SHORT NOTE ON HELMHOLTZ DECOMPOSITIONS FOR BOUNDED DOMAINS IN $\mathbb{R}^3$

IMMANUEL ANJAM

Abstract

In this short note we consider several widely used L^2-orthogonal Helmholtz decompositions for bounded domains in $\mathbb{R}^3$. It is well known that one part of the decompositions is a subspace of the space of functions with zero mean. We refine this global property into a local equivalent: we show that functions from these spaces have zero mean in every part of specific decompositions of the domain.

An application of the zero mean properties is presented for convex domains. We introduce a specialized Poincaré-type inequality, and estimate the related unknown constant from above. The upper bound is derived using the upper bound for the Poincaré constant proven by Payne and Weinberger. This is then used to obtain a small improvement of upper bounds of two Maxwell-type constants originally proven by Pauly.

Although the two dimensional case is not considered, all derived results can be repeated in $\mathbb{R}^2$ by similar calculations.

1. Notation and Helmholtz decompositions

Let $\omega \subset \mathbb{R}^3$ be a bounded open set. The space of scalar- or vector-valued smooth functions with compact supports in ω is denoted by $\mathring{\mathsf{C}}^\infty(\omega)$. We denote by $|\cdot|_{\mathsf{L}^1(\omega)}$ the norm for functions in $\mathsf{L}^1(\omega)$, and by $\langle\cdot,\cdot\rangle_{\mathsf{L}^2(\omega)}$ and $|\cdot|_{\mathsf{L}^2(\omega)}$ the inner product and norm for functions in $\mathsf{L}^2(\omega)$. The space of scalar-valued functions in $\mathsf{L}^2(\omega)$ with zero mean is defined as

$$\mathsf{L}^2_0(\omega) := \left\{\varphi \in \mathsf{L}^2(\omega) \;\middle|\; \int_\omega \varphi \,\mathrm{d}x = 0\right\},$$

and, as usual, for a vector-valued function ϕ we write $\phi \in \mathsf{L}^2_0(\omega)$ if all its components belong to $\mathsf{L}^2_0(\omega)$.

Throughout this note Ω denotes a bounded domain in $\mathbb{R}^3$, and from now on, whenever $\omega = \Omega$, we sometimes omit the indication of the set in our notation.

Besides the gradient ∇ we will also need the divergence operator div and the rotation operator rot acting on vector-valued functions. For smooth functions

Received 21 December 2017. Accepted 24 April 2018.
DOI: https://doi.org/10.7146/math.scand.a-114908

they are defined as

$$\operatorname{div}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} := \partial_1\phi_1 + \partial_2\phi_2 + \partial_3\phi_3, \quad \operatorname{rot}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} := \begin{pmatrix}\partial_2\phi_3 - \partial_3\phi_2\\ \partial_3\phi_1 - \partial_1\phi_3\\ \partial_1\phi_2 - \partial_2\phi_1\end{pmatrix}.$$

We define the usual Sobolev spaces

$$\begin{aligned} \mathsf{H}^1 &:= \{\varphi \in \mathsf{L}^2 \mid \nabla\varphi \in \mathsf{L}^2\}, & \mathring{\mathsf{H}}^1 &:= \overline{\mathring{\mathsf{C}}^\infty}^{\mathsf{H}^1}, \\ \mathsf{D} &:= \{\phi \in \mathsf{L}^2 \mid \operatorname{div}\phi \in \mathsf{L}^2\}, & \mathring{\mathsf{D}} &:= \overline{\mathring{\mathsf{C}}^\infty}^{\mathsf{D}}, \\ \mathsf{R} &:= \{\phi \in \mathsf{L}^2 \mid \operatorname{rot}\phi \in \mathsf{L}^2\}, & \mathring{\mathsf{R}} &:= \overline{\mathring{\mathsf{C}}^\infty}^{\mathsf{R}}, \end{aligned}$$

which are Hilbert spaces. Note that on the former spaces the differential operators are now defined in the usual weak sense. The latter spaces, where the closures are taken with respect to graph norms, generalize the classical homogeneous scalar, normal, and tangential boundary conditions, respectively. The operators satisfy

$$\begin{aligned} &\forall\varphi \in \mathring{\mathsf{H}}^1\ \forall\phi \in \mathsf{D} && \langle\nabla\varphi, \phi\rangle_{\mathsf{L}^2} = -\langle\varphi, \operatorname{div}\phi\rangle_{\mathsf{L}^2}, \\ &\forall\varphi \in \mathsf{H}^1\ \forall\phi \in \mathring{\mathsf{D}} && \langle\nabla\varphi, \phi\rangle_{\mathsf{L}^2} = -\langle\varphi, \operatorname{div}\phi\rangle_{\mathsf{L}^2}, \\ &\forall\phi \in \mathring{\mathsf{R}}\ \ \forall\psi \in \mathsf{R} && \langle\operatorname{rot}\phi, \psi\rangle_{\mathsf{L}^2} = \langle\phi, \operatorname{rot}\psi\rangle_{\mathsf{L}^2}. \end{aligned}$$

Note, that even though it is not indicated in the notation, we have two of each differential operator, one acting on a space without a boundary condition, and one acting on a space with a boundary condition. We also define

$$\begin{aligned} \mathsf{D}_0 &:= \{\phi \in \mathsf{D} \mid \operatorname{div}\phi = 0\}, & \mathring{\mathsf{D}}_0 &:= \{\phi \in \mathring{\mathsf{D}} \mid \operatorname{div}\phi = 0\}, \\ \mathsf{R}_0 &:= \{\phi \in \mathsf{R} \mid \operatorname{rot}\phi = 0\}, & \mathring{\mathsf{R}}_0 &:= \{\phi \in \mathring{\mathsf{R}} \mid \operatorname{rot}\phi = 0\}. \end{aligned}$$

By the projection theorem we obtain the L^2-orthogonal Helmholtz decompositions

$$\mathsf{L}^2 = \overline{\nabla\mathring{\mathsf{H}}^1} \oplus \mathsf{D}_0 = \mathring{\mathsf{R}}_0 \oplus \overline{\operatorname{rot}\mathsf{R}} = \overline{\nabla\mathring{\mathsf{H}}^1} \oplus \mathscr{H}_{\mathrm{D}} \oplus \overline{\operatorname{rot}\mathsf{R}}, \quad \mathscr{H}_{\mathrm{D}} := \mathsf{D}_0 \cap \mathring{\mathsf{R}}_0, \tag{1}$$

$$\mathsf{L}^2 = \overline{\nabla\mathsf{H}^1} \oplus \mathring{\mathsf{D}}_0 = \mathsf{R}_0 \oplus \overline{\operatorname{rot}\mathring{\mathsf{R}}} = \overline{\nabla\mathsf{H}^1} \oplus \mathscr{H}_{\mathrm{N}} \oplus \overline{\operatorname{rot}\mathring{\mathsf{R}}}, \quad \mathscr{H}_{\mathrm{N}} := \mathring{\mathsf{D}}_0 \cap \mathsf{R}_0, \tag{2}$$

where $\mathscr{H}_{\mathrm{D}}$ and $\mathscr{H}_{\mathrm{N}}$ are the spaces of Dirichlet and Neumann fields, respectively. In particular, we have the decompositions

$$\mathring{\mathsf{D}}_0 = \mathscr{H}_{\mathrm{N}} \oplus \overline{\operatorname{rot}\mathring{\mathsf{R}}}, \quad \mathring{\mathsf{R}}_0 = \overline{\nabla\mathring{\mathsf{H}}^1} \oplus \mathscr{H}_{\mathrm{D}}. \tag{3}$$

It is easy to see that functions from $\mathring{\mathsf{D}}_0$ have zero mean globally, i.e., they belong to L^2_0:

$$\forall \phi \in \mathring{\mathsf{D}}_0 \quad \int_\Omega \phi_i \, \mathrm{d}x = \langle \phi, \nabla x_i \rangle_{\mathsf{L}^2} = -\langle \operatorname{div} \phi, x_i \rangle_{\mathsf{L}^2} = 0. \tag{4}$$

We also observe that $\mathring{\mathsf{R}}_0$ belongs to L^2_0: for $v_1(x) := (0, 0, x_2)$, $v_2(x) := (x_3, 0, 0)$, and $v_3(x) := (0, x_1, 0)$ we have

$$\forall \phi \in \mathring{\mathsf{R}}_0 \quad \int_\Omega \phi_i \, \mathrm{d}x = \langle \phi, \operatorname{rot} v_i \rangle_{\mathsf{L}^2} = \langle \operatorname{rot} \phi, v_i \rangle_{\mathsf{L}^2} = 0.$$

In this note we show that functions from the above two spaces satisfy local zero mean properties with respect to certain decompositions of Ω.

For our considerations, it is not needed to assume any regularity of the domain. However, we mention that if Ω is Lipschitz, then Rellich's selection theorem and Weck's selection theorem [12] hold. This means that the closure bars in (1)–(3) can be skipped, and both $\mathcal{H}_\mathrm{D}$ and $\mathcal{H}_\mathrm{N}$ are finite dimensional. Furthermore, if the domain is topologically equivalent to a ball, then $\mathcal{H}_\mathrm{D} = \mathcal{H}_\mathrm{N} = \{0\}$. For more information on Helmholtz decompositions we refer to [5] and [9], which contains a concise exposition of Helmholtz decompositions in a general Hilbert space setting.

This note is organized as follows. Section 2 contains additional notation related to decompositions of the domain. Our main results, Theorems 1 and 2, and the local zero mean properties of Corollaries 3 and 4, are in Section 3. In Section 4 we use these results to derive, in the case of convex domains, slightly improved upper bounds of certain Maxwell-type constants related to the theory of electromagnetism.

2. Decompositions of the domain

Our calculations are invariant with respect to translations of the domain, so without loss of generality we assume Ω to be contained in the rectangular cuboid

$$I := (0, \ell_1) \times (0, \ell_2) \times (0, \ell_3), \quad 0 < \ell_1, \ell_2, \ell_3 < \infty.$$

We assume Ω is translated such that I is as small as possible. Note that the calculations of the following section no longer hold if the domain is rotated.

In what follows we will often need two or three distinct indices from the index set $\{1, 2, 3\}$. To this end, we define $\{1, 2, 3\}_p$ to denote the set of all p-permutations of the set $\{1, 2, 3\}$, where p is either 2 or 3.

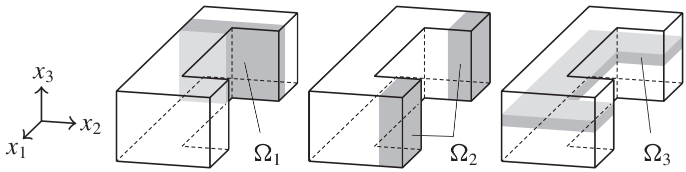

FIGURE 1. Examples of Ω_i.

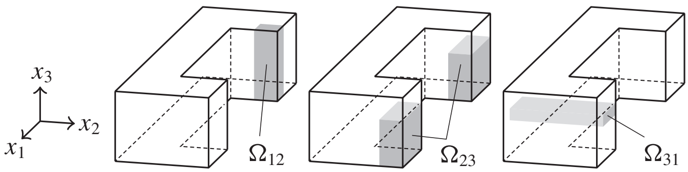

FIGURE 2. Examples of Ω_{ij}. For illustrative purposes the Ω_{ij} are chosen such that they belong to the Ω_i of Figure 1.

For $0 \leq \alpha_i < \beta_i \leq \ell_i$, $i \in \{1, 2, 3\}$, we define

$$I_i := \{x \in I \mid \alpha_i < x_i < \beta_i\}, \qquad I_{ij} := I_i \cap I_j,$$
$$\Omega_i := \{x \in \Omega \mid \alpha_i < x_i < \beta_i\}, \quad \Omega_{ij} := \Omega_i \cap \Omega_j,$$

where in the latter definitions $(i, j) \in \{1, 2, 3\}_2$. Note that $\Omega_i \subset I_i$ and $\Omega_{ij} \subset I_{ij}$ hold. Examples of these subsets are illustrated in Figures 1 and 2. It is clear that Ω can be decomposed in such pieces in a way that the pieces are nonintersecting, and that the union of their closures equals $\overline{\Omega}$. Note also that if Ω_i and Ω_{ij} appear in the same relation, they are always related to each other, i.e., in particular $\Omega_{ij} \subset \Omega_i$ holds.

3. Local zero mean properties

In order to prove the local zero mean properties, we show that the mean value of functions from $\mathring{\mathsf{D}}$ and $\mathring{\mathsf{R}}$ can be locally estimated from below and above by L^1-norms of their divergence and rotation, respectively.

THEOREM 1. *For any $\phi \in \mathring{\mathsf{D}}(\Omega)$, the estimate*

$$\forall i \in \{1, 2, 3\} \quad \left| \int_{\Omega_i} \phi_i \, \mathrm{d}x \right| \leq (\beta_i - \alpha_i) |\operatorname{div} \phi|_{\mathsf{L}^1(\Omega)}$$

holds for an arbitrary Ω_i.

PROOF. For any $\phi \in \mathring{\mathsf{C}}^\infty(\Omega)$ its zero extension $\hat{\phi}: I \to \mathbb{R}^3$ belongs to $\mathring{\mathsf{C}}^\infty(I)$. By the Fundamental Theorem of Calculus the components of this extension can be represented as

$$\hat{\phi}_1(x_1, x_2, x_3) = \int_0^{x_1} \partial_a \hat{\phi}_1(a, x_2, x_3)\, \mathrm{d}a,$$

$$\hat{\phi}_2(x_1, x_2, x_3) = \int_0^{x_2} \partial_b \hat{\phi}_2(x_1, b, x_3)\, \mathrm{d}b,$$

$$\hat{\phi}_3(x_1, x_2, x_3) = \int_0^{x_3} \partial_c \hat{\phi}_3(x_1, x_2, c)\, \mathrm{d}c.$$

Using the above representations we write

$$\begin{aligned} &\pm \int_0^{x_3}\int_0^{x_2} \hat{\phi}_1(x_1, b, c)\, \mathrm{d}(bc) \pm \int_0^{x_3}\int_0^{x_1} \hat{\phi}_2(a, x_2, c)\, \mathrm{d}(ac) \\ &\quad \pm \int_0^{x_2}\int_0^{x_1} \hat{\phi}_3(a, b, x_3)\, \mathrm{d}(ab) \\ &\quad = \pm \int_0^{x_3}\int_0^{x_2}\int_0^{x_1} \partial_a\hat{\phi}_1(a,b,c) + \partial_b\hat{\phi}_2(a,b,c) + \partial_c\hat{\phi}_3(a,b,c)\, \mathrm{d}(abc) \\ &\quad \le |\operatorname{div}\hat{\phi}|_{\mathsf{L}^1(I)}. \end{aligned} \tag{5}$$

By choosing $x_2 = \ell_2$ and $x_3 = \ell_3$, the two last terms on the left-hand side vanish, and we obtain

$$\pm \int_0^{\ell_3}\int_0^{\ell_2} \hat{\phi}_1(x_1, b, c)\, \mathrm{d}(bc) \le |\operatorname{div}\hat{\phi}|_{\mathsf{L}^1(I)}.$$

By integrating with respect to x_1 over (α_1, β_1) we obtain

$$\pm \int_{I_1} \hat{\phi}_1\, \mathrm{d}x \le (\beta_1 - \alpha_1)|\operatorname{div}\hat{\phi}|_{\mathsf{L}^1(I)} \implies \pm \int_{\Omega_1} \phi_1\, \mathrm{d}x \le (\beta_1 - \alpha_1)|\operatorname{div}\phi|_{\mathsf{L}^1(\Omega)},$$

since the integrals are nonzero only in Ω. By density the latter inequality above holds for any $\phi \in \mathring{\mathsf{D}}(\Omega)$, and we have proven the assertion for $i = 1$. To prove the cases $i = 2$ and $i = 3$, one chooses $x_1 = \ell_1$, $x_3 = \ell_3$ and $x_1 = \ell_1$, $x_2 = \ell_2$ in (5), respectively, and proceeds in a similar manner.

THEOREM 2. *For any $\phi \in \mathring{\mathsf{R}}(\Omega)$, the estimate*

$$\forall (i, j, k) \in \{1, 2, 3\}_3 \quad \left|\int_{\Omega_{jk}} \phi_i\, \mathrm{d}x\right| \le (\beta_j - \alpha_j)|(\operatorname{rot}\phi)_k|_{\mathsf{L}^1(\Omega_k)}$$

holds for an arbitrary Ω_{jk}.

PROOF. For any $\phi \in \mathring{\mathsf{C}}^\infty(\Omega)$ its zero extension $\hat{\phi}\colon I \to \mathbb{R}^3$ belongs to $\mathring{\mathsf{C}}^\infty(I)$. By the Fundamental Theorem of Calculus the components of this extension can be represented as

$$\hat{\phi}_2(x_1, x_2, x_3) = \int_0^{x_1} \partial_a \hat{\phi}_2(a, x_2, x_3)\,\mathrm{d}a,$$
$$\hat{\phi}_1(x_1, x_2, x_3) = \int_0^{x_2} \partial_b \hat{\phi}_1(x_1, b, x_3)\,\mathrm{d}b.$$

Using the above representations we write

$$\begin{aligned} &\pm \int_0^{x_2} \hat{\phi}_2(x_1, b, x_3)\,\mathrm{d}b \mp \int_0^{x_1} \hat{\phi}_1(a, x_2, x_3)\,\mathrm{d}a \\ &\quad = \pm \int_0^{x_2}\int_0^{x_1} \partial_a \hat{\phi}_2(a, b, x_3) - \partial_b \hat{\phi}_1(a, b, x_3)\,\mathrm{d}(ab) \\ &\quad \le \int_0^{\ell_2}\int_0^{\ell_1} |\partial_a \hat{\phi}_2(a, b, x_3) - \partial_b \hat{\phi}_1(a, b, x_3)|\,\mathrm{d}(ab). \end{aligned} \tag{6}$$

By choosing $x_1 = \ell_1$ in (6) and integrating with respect to x_3 over (α_3, β_3), we obtain

$$\pm \int_{\alpha_3}^{\beta_3}\int_0^{\ell_1} \hat{\phi}_1(a, x_2, x_3)\,\mathrm{d}(ax_3) \le |(\operatorname{rot}\hat{\phi})_3|_{\mathsf{L}^1(I_3)}.$$

By integrating with respect to x_2 over (α_2, β_2) we obtain

$$\pm \int_{I_{23}} \hat{\phi}_1\,\mathrm{d}x \le (\beta_2 - \alpha_2)|(\operatorname{rot}\hat{\phi})_3|_{\mathsf{L}^1(I_3)},$$

which implies

$$\pm \int_{\Omega_{23}} \phi_1\,\mathrm{d}x \le (\beta_2 - \alpha_2)|(\operatorname{rot}\phi)_3|_{\mathsf{L}^1(\Omega_3)}, \tag{7}$$

since the integrals are nonzero only in Ω. On the other hand, by choosing $x_2 = \ell_2$ in (6) and integrating with respect to x_3 over (α_3, β_3), we obtain

$$\pm \int_{\alpha_3}^{\beta_3}\int_0^{\ell_2} \hat{\phi}_2(x_1, b, x_3)\,\mathrm{d}(bx_3) \le |(\operatorname{rot}\hat{\phi})_3|_{\mathsf{L}^1(I_3)}.$$

By integrating with respect to x_1 over (α_1, β_1) we obtain

$$\pm \int_{I_{13}} \hat{\phi}_2\,\mathrm{d}x \le (\beta_1 - \alpha_1)|(\operatorname{rot}\hat{\phi})_3|_{\mathsf{L}^1(I_3)},$$

which implies

$$\pm \int_{\Omega_{13}} \phi_2 \,\mathrm{d}x \leq (\beta_1 - \alpha_1)|(\operatorname{rot}\phi)_3|_{\mathsf{L}^1(\Omega_3)}, \tag{8}$$

since the integrals are nonzero only in Ω. By density (7) and (8) hold for any $\phi \in \mathring{\mathsf{R}}(\Omega)$, and we have proven two of the six estimates of the assertion. The remaining estimates are proven in a similar manner by repeating the proof using the representations

$$\hat{\phi}_1(x_1, x_2, x_3) = \int_0^{x_3} \partial_c \hat{\phi}_1(x_1, x_2, c)\,\mathrm{d}c,$$
$$\hat{\phi}_3(x_1, x_2, x_3) = \int_0^{x_1} \partial_a \hat{\phi}_3(a, x_2, x_3)\,\mathrm{d}a,$$

and

$$\hat{\phi}_3(x_1, x_2, x_3) = \int_0^{x_2} \partial_b \hat{\phi}_3(x_1, b, x_3)\,\mathrm{d}b,$$
$$\hat{\phi}_2(x_1, x_2, x_3) = \int_0^{x_3} \partial_c \hat{\phi}_2(x_1, x_2, c)\,\mathrm{d}c.$$

The following two corollaries are directly implied by Theorems 1 and 2.

COROLLARY 3. *Let* $\phi \in \mathring{\mathsf{D}}_0(\Omega)$. *Then* ϕ_i *belongs to* $\mathsf{L}^2_0(\Omega_i)$ *for any* Ω_i, *where* $i \in \{1, 2, 3\}$.

COROLLARY 4. *Let* $\phi \in \mathring{\mathsf{R}}_0(\Omega)$. *Then* ϕ_i *belongs to* $\mathsf{L}^2_0(\Omega_{jk})$ *for any* Ω_{jk}, *where* $(i, j, k) \in \{1, 2, 3\}_3$.

REMARK 5. Theorem 2 allows for more general statements about $\mathring{\mathsf{R}}$ than Corollary 4:

(i) It is easy to see that Corollary 4 holds not only for $\mathring{\mathsf{R}}_0$ but even for

$$\left\{\psi \in \mathring{\mathsf{R}} \mid (\operatorname{rot}\psi)_i = (\operatorname{rot}\psi)_j = 0,\ (i, j) \in \{1, 2, 3\}_2\right\}.$$

(ii) Even if only one component of the rotation of $\phi \in \mathring{\mathsf{R}}$ vanishes on a subset of Ω, in certain cases we might still be able to obtain information about where ϕ has zero mean. If, for example, $(\operatorname{rot}\phi)_3 = 0$ in $\omega \subset \Omega$ which is a Ω_3-set, then Theorem 2 implies that $\phi_1, \phi_2 \in \mathsf{L}^2_0(\omega)$.

4. An application for convex domains

In this section we assume the domain Ω to be convex. Then Ω is Lipschitz [4], and Rellich's selection theorem and Weck's selection theorem [12] hold, i.e., all spaces in (1)–(3) are closed. Furthermore, the Dirichlet and Neumann fields are absent, i.e., the Helmholtz decompositions (1)–(3) become

$$\mathsf{L}^2 = \nabla \mathring{\mathsf{H}}^1 \oplus \mathsf{D}_0 = \mathring{\mathsf{R}}_0 \oplus \operatorname{rot} \mathsf{R}, \quad \nabla \mathring{\mathsf{H}}^1 = \mathring{\mathsf{R}}_0, \quad \mathsf{D}_0 = \operatorname{rot} \mathsf{R}, \tag{9}$$

$$\mathsf{L}^2 = \nabla \mathsf{H}^1 \oplus \mathring{\mathsf{D}}_0 = \mathsf{R}_0 \oplus \operatorname{rot} \mathring{\mathsf{R}}, \quad \nabla \mathsf{H}^1 = \mathsf{R}_0, \quad \mathring{\mathsf{D}}_0 = \operatorname{rot} \mathring{\mathsf{R}}. \tag{10}$$

In the following, we consider the inequalities

$$\forall \varphi \in \mathsf{H}^1 \cap \mathsf{L}^2_0 \qquad |\varphi|_{\mathsf{L}^2} \le c_{\mathrm{p}} |\nabla \varphi|_{\mathsf{L}^2},$$

$$\forall \phi \in \mathring{\mathsf{R}} \cap \mathsf{D}_0 \qquad |\phi|_{\mathsf{L}^2} \le c_{\mathrm{m},1} |\operatorname{rot} \phi|_{\mathsf{L}^2}, \tag{11}$$

$$\forall \phi \in \mathsf{R} \cap \mathring{\mathsf{D}}_0 \qquad |\phi|_{\mathsf{L}^2} \le c_{\mathrm{m},2} |\operatorname{rot} \phi|_{\mathsf{L}^2}, \tag{12}$$

where the first is the Poincaré inequality, and the latter Maxwell-type inequalities. The Poincaré constant $c_{\mathrm{p}} > 0$ and Maxwell constants $c_{\mathrm{m},1}, c_{\mathrm{m},2} > 0$ are under the assumptions finite. In what follows, we assume we have chosen the best, i.e., the smallest possible constants in these inequalities. Note that these constants are related to eigenvalues of the Laplace and rot rot operators.

The proofs of finiteness of the above constants are based on indirect arguments, and give no hints as to their magnitude. However, in some situations explicit knowledge of these constants is needed: they appear, e.g., in functional type a posteriori error estimates for partial differential equations [11]. For convex domains there is a constructive method for obtaining an upper bound of c_{p} due to Payne and Weinberger [10] (see also [2]). The bound is

$$c_{\mathrm{p}} \le \frac{d}{\pi}, \tag{13}$$

where $d = \operatorname{diam} \Omega$ is the diameter of Ω. In [6], [7], [8] Pauly has shown that for convex domains $c_{\mathrm{m},1} = c_{\mathrm{m},2} \le c_{\mathrm{p}}$, so together with (13) we have

$$c_{\mathrm{m},1} = c_{\mathrm{m},2} \le c_{\mathrm{p}} \le \frac{d}{\pi}. \tag{14}$$

Using Corollary 3 this upper bound can be slightly improved. However, for the sake of completeness, we first show that the Maxwell constants are indeed equal.

LEMMA 6. $c_{\mathrm{m},1} = c_{\mathrm{m},2}$.

PROOF. Let $\phi \in \mathring{\mathsf{R}} \cap \mathsf{D}_0$. From (9)–(10) we deduce $\mathsf{D}_0 = \mathrm{rot}\, \mathsf{R} = \mathrm{rot}(\mathsf{R} \cap \mathring{\mathsf{D}}_0)$. Thus there exists a vector potential $\Phi \in \mathsf{R} \cap \mathring{\mathsf{D}}_0$ such that $\mathrm{rot}\, \Phi = \phi$. Using (12) we obtain

$$|\phi|_{\mathsf{L}^2}^2 = \langle \phi, \mathrm{rot}\, \Phi \rangle_{\mathsf{L}^2} = \langle \mathrm{rot}\, \phi, \Phi \rangle_{\mathsf{L}^2} \le |\mathrm{rot}\, \phi|_{\mathsf{L}^2} |\Phi|_{\mathsf{L}^2} \le c_{\mathrm{m},2} |\mathrm{rot}\, \phi|_{\mathsf{L}^2} |\mathrm{rot}\, \Phi|_{\mathsf{L}^2},$$

which implies $|\phi|_{\mathsf{L}^2} \le c_{\mathrm{m},2} |\mathrm{rot}\, \phi|_{\mathsf{L}^2}$. In view of (11) we see that $c_{\mathrm{m},1} \le c_{\mathrm{m},2}$. On the other hand, let $\phi \in \mathsf{R} \cap \mathring{\mathsf{D}}_0$. From (9)–(10) we deduce $\mathring{\mathsf{D}}_0 = \mathrm{rot}\, \mathring{\mathsf{R}} = \mathrm{rot}(\mathring{\mathsf{R}} \cap \mathsf{D}_0)$. Thus there exists a vector potential $\Phi \in \mathring{\mathsf{R}} \cap \mathsf{D}_0$ such that $\mathrm{rot}\, \Phi = \phi$. Using (11) we obtain

$$|\phi|_{\mathsf{L}^2}^2 = \langle \phi, \mathrm{rot}\, \Phi \rangle_{\mathsf{L}^2} = \langle \mathrm{rot}\, \phi, \Phi \rangle_{\mathsf{L}^2} \le |\mathrm{rot}\, \phi|_{\mathsf{L}^2} |\Phi|_{\mathsf{L}^2} \le c_{\mathrm{m},1} |\mathrm{rot}\, \phi|_{\mathsf{L}^2} |\mathrm{rot}\, \Phi|_{\mathsf{L}^2},$$

which implies $|\phi|_{\mathsf{L}^2} \le c_{\mathrm{m},1} |\mathrm{rot}\, \phi|_{\mathsf{L}^2}$. In view of (12) we see that $c_{\mathrm{m},2} \le c_{\mathrm{m},1}$, and the assertion is proven.

Note that the above proof is not restricted to convex domains. It holds true whenever Weck's selection theorem [12] holds, provided that the Dirichlet and Neumann fields are excluded from the considered functions.

For improving (14) we will need the following specialized Poincaré inequality.

LEMMA 7. *Let Ω be convex and $\varphi \in \mathsf{H}^1(\Omega)$ be scalar-valued. In the following $(i, j, k) \in \{1, 2, 3\}_3$. Assume $\varphi \in \mathsf{L}_0^2(\Omega_i)$ for an arbitrary Ω_i. Then we have*

$$|\varphi|_{\mathsf{L}^2(\Omega)} \le c_{\mathrm{p},i} |\nabla \varphi|_{\mathsf{L}^2(\Omega)}, \quad c_{\mathrm{p},i} \le c_{\mathrm{p}}, \quad c_{\mathrm{p},i} \le \frac{d_{jk}}{\pi},$$

where d_{jk} is the diameter of the two-dimensional projection of Ω into the (e_j, e_k)-plane. Here e_j and e_k denote the j-th and k-th Euclidean orthonormal basis vectors (see Figure 3).

PROOF. Let $i = 1$. Under the assumptions there exists a decomposition of Ω into nonintersecting convex Ω_1-sets $\Omega_{1,n}$, $n = 1, \dots, N$ such that

$$\overline{\Omega} = \bigcup_{n=1}^{N} \overline{\Omega}_{1,n}, \quad \varphi \in \mathsf{L}_0^2(\Omega_{1,n}), \qquad n = 1, \dots, N,$$

where each $\Omega_{1,n}$ has width ℓ_1/N in the direction of the x_1-coordinate, and

$$\forall n \in \{1, \dots, N\} \quad \mathrm{diam}\, \Omega_{1,n} \le \sqrt{d_{23}^2 + \frac{\ell_1^2}{N^2}}$$

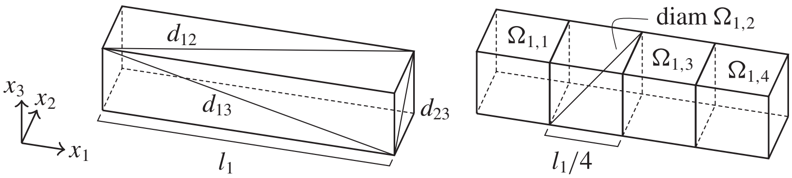

FIGURE 3. Diameters of a rectangular cuboid Ω and its decomposition into Ω_1-sets.

holds (see Figure 3). For each subdomain we can apply (13) to obtain

$$\forall n \in \{1, \ldots, N\} \quad |\varphi|_{\mathsf{L}^2(\Omega_{1,n})} \leq \frac{\operatorname{diam} \Omega_{1,n}}{\pi} |\nabla \varphi|_{\mathsf{L}^2(\Omega_{1,n})},$$

which implies

$$\begin{aligned} |\varphi|_{\mathsf{L}^2(\Omega)} &\leq \frac{1}{\pi} \max_{n \in \{1,\ldots,N\}} \operatorname{diam} \Omega_{1,n} |\nabla \varphi|_{\mathsf{L}^2(\Omega)} \\ &\leq \frac{1}{\pi} \sqrt{d_{23}^2 + \frac{\ell_1^2}{N^2}} |\nabla \varphi|_{\mathsf{L}^2(\Omega)} \xrightarrow{N \to \infty} \frac{d_{23}}{\pi} |\nabla \varphi|_{\mathsf{L}^2(\Omega)}. \end{aligned}$$

The cases $i = 2$ and $i = 3$ are proven in a similar way.

As in [6], [7], [8], we will rely on the following essential regularity result proven in [1, Theorem 2.17].

LEMMA 8. *Let Ω be convex and $\phi \in \mathring{\mathsf{R}} \cap \mathsf{D}$ or $\phi \in \mathsf{R} \cap \mathring{\mathsf{D}}$. Then $\phi \in \mathsf{H}^1$ and*

$$|\nabla \phi|_{\mathsf{L}^2}^2 \leq |\operatorname{div} \phi|_{\mathsf{L}^2}^2 + |\operatorname{rot} \phi|_{\mathsf{L}^2}^2.$$

We can now state the improved bound.

THEOREM 9. *Let Ω be convex. Then we have the estimate*

$$c_{\mathrm{m},1} = c_{\mathrm{m},2} \leq \max\{c_{\mathrm{p},1}, c_{\mathrm{p},2}, c_{\mathrm{p},3}\} \leq \frac{\max\{d_{23}, d_{13}, d_{12}\}}{\pi}.$$

PROOF. Let $\phi \in \mathsf{R} \cap \mathring{\mathsf{D}}_0$. Then $\phi \in \mathsf{H}^1$ by Lemma 8 and $\phi \in \mathsf{L}_0^2$ by Corollary 3. More specifically, Corollary 3 shows that the specialized Poincaré inequality of Lemma 7 can be applied to each component of ϕ, and we directly get

$$\begin{aligned} |\phi|_{\mathsf{L}^2}^2 = |\phi_1|_{\mathsf{L}^2}^2 + |\phi_2|_{\mathsf{L}^2}^2 + |\phi_3|_{\mathsf{L}^2}^2 &\leq c_{\mathrm{p},1}^2 |\nabla \phi_1|_{\mathsf{L}^2}^2 + c_{\mathrm{p},2}^2 |\nabla \phi_2|_{\mathsf{L}^2}^2 + c_{\mathrm{p},3}^2 |\nabla \phi_3|_{\mathsf{L}^2}^2 \\ &\leq \max\{c_{\mathrm{p},1}^2, c_{\mathrm{p},2}^2, c_{\mathrm{p},3}^2\} |\nabla \phi|_{\mathsf{L}^2}^2 \leq \max\{c_{\mathrm{p},1}^2, c_{\mathrm{p},2}^2, c_{\mathrm{p},3}^2\} |\operatorname{rot} \phi|_{\mathsf{L}^2}^2, \end{aligned}$$

where in the last step we used Lemma 8. In view of (12) we obtain the estimate $c_{m,2} \le \max\{c_{p,1}, c_{p,2}, c_{p,3}\}$. Together with Lemmas 6 and 7 we have the assertion.

REMARK 10. If we had used in the above proof the global zero mean property (4) and the Payne-Weinberger estimate (13) (instead of Corollary 3 and Lemma 7, respectively), we would have arrived at (14). Note that Pauly's proof of (14) does not use knowledge of (4), but is rather based on finding suitable potential functions.

EXAMPLE 11. (i) Let $\Omega = (0, 1)^3$. Then $d = \sqrt{3}$ and $d_{23} = d_{13} = d_{12} = \sqrt{2}$. The bounds of (14) and Theorem 9 then give

$$c_{m,1} = c_{m,2} \le \frac{\sqrt{3}}{\pi}, \quad c_{m,1} = c_{m,2} \le \frac{\sqrt{2}}{\pi},$$

respectively.

(ii) Let $\Omega = B(0, 1)$, i.e., the unit ball in $\mathbb{R}^3$. Then $d = d_{23} = d_{13} = d_{12} = 2$, and the bound in Theorem 9 offers no improvement over the bound (14).

REMARK 12. In [7] it was proven that for convex domains Ω the two Maxwell constants in the inequalities

$$\forall \phi \in \mathring{\mathsf{R}} \cap \mathsf{D} \quad |\phi|^2_{\mathsf{L}^2} \le c^2_{m,t}\big(|\operatorname{div}\phi|^2_{\mathsf{L}^2} + |\operatorname{rot}\phi|^2_{\mathsf{L}^2}\big),$$
$$\forall \phi \in \mathsf{R} \cap \mathring{\mathsf{D}} \quad |\phi|^2_{\mathsf{L}^2} \le c^2_{m,n}\big(|\operatorname{div}\phi|^2_{\mathsf{L}^2} + |\operatorname{rot}\phi|^2_{\mathsf{L}^2}\big),$$

satisfy $c_{m,t} \le c_{m,n} = c_p$, and it was conjectured that $c_{m,t} < c_{m,n}$ holds. By using Theorem 9 instead of [7, Lemma 4] in the proof of [7, Theorem 6], we obtain

$$\forall \phi \in \mathring{\mathsf{R}} \cap \mathsf{D} \quad |\phi|^2_{\mathsf{L}^2} \le c^2_f|\operatorname{div}\phi|^2_{\mathsf{L}^2} + \max\{c_{p,1}, c_{p,2}, c_{p,3}\}^2|\operatorname{rot}\phi|^2_{\mathsf{L}^2},$$
$$\forall \phi \in \mathsf{R} \cap \mathring{\mathsf{D}} \quad |\phi|^2_{\mathsf{L}^2} \le c^2_p\big(|\operatorname{div}\phi|^2_{\mathsf{L}^2} + |\operatorname{rot}\phi|^2_{\mathsf{L}^2}\big),$$

where c_f is the constant in the Friedrichs' inequality $|\varphi|_{\mathsf{L}^2} \le c_f|\nabla\varphi|_{\mathsf{L}^2}$ which holds for all scalar valued functions $\varphi \in \mathring{\mathsf{H}}^1$. It is well known that $c_f < c_p$ (see, e.g., [3]). Thus, if one can prove that $\max\{c_{p,1}, c_{p,2}, c_{p,3}\} < c_p$, then the conjecture $c_{m,t} < c_{m,n}$ follows.

Note also that weighted L^2-orthogonal Helmholtz decompositions were used in [7]. In this note unweighted decompositions were used only for simplicity.

ACKNOWLEDGEMENTS. The author is grateful to Dirk Pauly, Michael Schomburg, and Frank Osterbrink for inspiring scientific discussions.

REFERENCES

1. Amrouche, C., Bernardi, C., Dauge, M., and Girault, V., *Vector potentials in three-dimensional non-smooth domains*, Math. Methods Appl. Sci. 21 (1998), no. 9, 823–864.
2. Bebendorf, M., *A note on the Poincaré inequality for convex domains*, Z. Anal. Anwendungen 22 (2003), no. 4, 751–756.
3. Filonov, N., *On an inequality for the eigenvalues of the Dirichlet and Neumann problems for the Laplace operator*, St. Petersburg Math. J. 16 (2005), no. 2, 413–416.
4. Grisvard, P., *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics, vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.
5. Leis, R., *Initial boundary value problems in mathematical physics*, B. G. Teubner, Stuttgart, 1986.
6. Pauly, D., *On constants in Maxwell inequalities for bounded and convex domains*, J. Math. Sci. (N.Y.) 210 (2015), no. 6, 787–792.
7. Pauly, D., *On Maxwell's and Poincaré's constants*, Discrete Contin. Dyn. Syst. Ser. S 8 (2015), no. 3, 607–618.
8. Pauly, D., *On the Maxwell constants in 3D*, Math. Methods Appl. Sci. 40 (2017), no. 2, 435–447.
9. Pauly, D., *Solution theory, variational formulations, and functional a posteriori error estimates for general first order systems with applications to electro-magneto-statics and more*, Numer. Funct. Anal. Optim. online (2019), 97 pp.
10. Payne, L. E., and Weinberger, H. F., *An optimal Poincaré inequality for convex domains*, Arch. Rational Mech. Anal. 5 (1960), 286–292.
11. Repin, S., *A posteriori estimates for partial differential equations*, Radon Series on Computational and Applied Mathematics, vol. 4, Walter de Gruyter, Berlin, 2008.
12. Weck, N., *Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries*, J. Math. Anal. Appl. 46 (1974), 410–437.

TAITONIEKANTIE 9 F AS. 401
40740 JYVÄSKYLÄ
FINLAND
E-mail: immanuel.anjam@gmail.com