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ON BORDISM THEORY OF MANIFOLDS WITH SINGULARITIES[1]

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Contents.

1. Introduction.

The objects of our study here will be manifolds with a certain type of singularities, or in another way, polyhedra with a special neighbourhood of the singularity subset. Here singularity subset means the subset where the polyhedron fails to be a (smooth) manifold.

These objects have been studied by Sullivan in [9] and [10]. Let us consider Sullivan's description of the objects. A polyhedron is of singularity type P_1 if it is "like" $S^n * P_1$, P_1 being a given closed manifold. So this has to be interpreted in a correct way, namely, that the polyhedron should have a decomposition of the form

$$\bar{A} = A \cup_{A(1)\times P_1} A(1) \times CP_1$$

$$\partial A = A(1) \times P_1$$

where A and $A(1)$ are manifolds. $\bar{A}$ should bound if there exists a $\bar{B}$ such that

$$\bar{B} = B \cup_{B(1)\times P_1} B(1) \times CP_1$$

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[1] This paper was written in the spring of 1970 and appeared under the same title as Preprint No. 31 (1969–70) at Matematisk Institut, Aarhus Universitet. (A discussion of the results appeared in [1]). For some time I planned to rewrite and extend the paper, but because of the great number of requests for it, I have decided to publish it in its more or less original form to avoid a further delay.

where

$$\partial B = A \cup B(1) \times P_1$$

and

$$\partial B(1) \cong A(1) .$$

Since the singularities have such a special structure we can remove the cones and work with the remaining manifolds with a special boundary structure – namely, A and B. In order to study bordism theory of these objects we define a new boundary operator

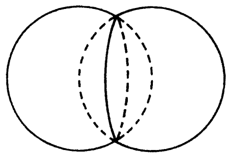
$$\delta B = A .$$

Obviously $\delta\delta B = \emptyset$ so this seems like a good starting point.

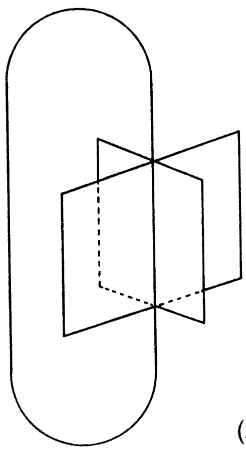
Before going further let us look at some examples which illustrate the situation in a good way.

EXAMPLE 1. Let us take two 2-spheres intersecting each other in R^3 as in the figure (i)

V:



(i)



(ii)

A neighbourhood of the singularity set can be written as $S^1 \times C\mathsf{Z}_4$ (figure (ii)). Remove this neighbourhood from V; the complement is then $D^2 \times \mathsf{Z}_4$ and we see that V can be written as

$$V = D^2 \times \mathsf{Z}_4 \cup_{S^1 \times \mathsf{Z}_4} S^1 \times C\mathsf{Z}_4$$

and is therefore a "manifold of singularity type Z_4" (or a Z_4-manifold).

EXAMPLE 2. An algebraic variety with a finite number of isolated singularities can be decomposed as follows

$$V = V_0 \cup CP_1 \cup \ldots \cup CP_k$$

where $V_0, P_1 \ldots P_n$ are smooth manifolds and

$$\partial V_0 = P_1 + P_2 + \ldots + P_n$$

(disjoint sum since the singularities are isolated) and the cones attached by the identity map. This decomposition follows from Milnor [8, Theorem 2.10, p. 18].

This gives also that the inverse image of a critical value of a Morse function will be decomposed as just described and therefore gives another example of "manifolds with cone-type singularities".

EXAMPLE 3. Consider S^n and identify q distinct points $x_1, \ldots, x_q$, the quotient space

$$V = S^n/\{x_1 = \ldots = x_q\}$$

can then be written as

$$V = V' \cup_{\partial V'} \mathrm{pt} \times C(S^{n-1} \times \mathsf{Z}_q), \quad \partial V' = S^{n-1} \times \mathsf{Z}_q ,$$

and is therefore of singularity type $S^{n-1} \times \mathsf{Z}_q$.

Let us reflect a bit more over the sentence: V is "like" $S^n * P_1$. This should mean that locally V would be homeomorphic to open subsets of R^n (since the cone on S^n is euclidean) or open subsets of $\mathsf{R}^{n_1} \times CP_1$ (n_1 depending on the dimension of P_1). We could therefore introduce a chart modelled on $\mathsf{R}^{n'}$ and $\mathsf{R}^{n_1} \times CP_1$ and give a rigorous definition of our objects similar to the way in which ordinary manifolds are defined. But of course we then have singularities in our objects!

Whether we keep the "bad" points or remove them, the case of one singularity $-P_1-$ is relatively easy to handle. The situation is, however, much more complicated for several singularities $(P_1, \ldots, P_n)$.

According to Sullivan, a polyhedron V is of singularity type $(P_1, \ldots, P_n)$ if it is "like"

$$S^{n'} * P_1 * P_2 * \ldots * P_n .$$

Again interpreted locally this means that locally we have homeomorphisms from V to open subsets of

$$\mathsf{R}^{m(0)}, \quad \mathsf{R}^{m(0,1)} \times CP_1, \quad \mathsf{R}^{m(0,1,2)} \times CP_1 \times CP_2, \ldots ,$$
$$\mathsf{R}^{m(0,1,\ldots n)} \times CP_1 \times CP_2 \times \ldots \times CP_n$$

where the $m(0,1 \ldots i)$'s depend on the dimensions of $P_1, \ldots, P_i$.

Once again we could use this information to define a suitable chart and copy the process for ordinary manifolds. But we want to adopt the philosophy that we remove the cones and hence the singularities and concentrate our interest on the structure of the remaining (smooth) manifold, its boundary and the "attachment data" for the cones.

As we have already pointed out it is quite easy to determine the structure of the "remaining boundary" in the case of one singularity, namely of the type:

$$\partial B = A \cup B(1) \times P_1 .$$

In the case of several singularities it is much more complicated when we are going to specify the "attachment data" and how they are related in order to reflect how the singularity sub-manifolds meet.

It seems natural to require a decomposition as follows

$$A = A(0) \cup A(1) \times P_1 \cup \ldots \cup A(n) \times P_n$$

but the important thing here is how the manifolds are glued together, our "attachment data". In order to specify this in an appropriate way we discover that we need that the manifolds $A(0), A(1), \ldots, A(n)$ should be decomposed in a similar way to what we want for A. Hence the $A(i)$'s give rise to new systems of manifolds and attachment data and we get some sort of a hierarchy of manifolds. But for dimensional reasons this process will stop after a finite number of steps. Therefore, in order to define our desired objects, we seem to be forced to study systems of manifolds as indicated (since we have adopted the philosophy of removing the cones!). One might think that this would be very complicated, but in fact it turns out to be quite convenient. The important thing is of course to find suitable definitions.

2. Definitions.

We will now work in the category of smooth (unoriented) manifolds, and we allow the manifolds to have general corners. For manifolds with corners see for example [4] or [6]. Our constructions will also go through for manifolds with an additional G-structure; we will comment on this later on.

DEFINITION 2.1. V is a decomposed manifold iff there exist submanifolds

$$\partial_0 V, \partial_1 V, \ldots, \partial_n V$$

such that

$$\partial V = \partial_0 V \cup \partial_1 V \cup \ldots \cup \partial_n V$$

where union means identification along a common part of the boundary.

Each $\partial_i V$ is again a decomposed manifold by defining

$$\partial_j(\partial_i V) = \partial_j V \cap \partial_i V \quad \text{for } j \neq i$$
$$\partial_i(\partial_i V) = \emptyset$$

and then

$$\partial(\partial_i V) = \bigcup_{j=i}^{n} \partial_j(\partial_i V) .$$

Let us now fix a class of closed manifolds

$$S = \{P_0 = *, P_1, P_2, \ldots, P_n, \ldots\}$$

and put

$$S_n = \{P_0, P_1, \ldots, P_n\} .$$

These manifolds will later on play the role as singularity manifolds.

Definition 2.2. A manifold A is called an S_n-manifold (or a manifold of singularity type S_n) iff

i) $\forall\, \omega \subset \{0,1,\ldots,n\}$ there exists a decomposed manifold (in the sense of Definition 2.1) $A(\omega)$ such that

(a) $A(\emptyset) = A$,

(b) there exist isomorphisms

$$\beta(\omega,i)\colon \partial_i A(\omega) \cong A(\omega,i) \times P_i \quad \text{if } i \notin \omega ,$$
$$\partial_i A(\omega) = \emptyset \quad \text{if } i \in \omega ,$$

where (ω,i) means the subset ω union the element $i \in \{0,1,\ldots,n\}$.

ii) $\forall\, i,j \in \{0,1,\ldots,n\}$ the following diagram commutes

$$\begin{array}{ccccc}
\partial_j\partial_i A(\omega) & \xrightarrow{\beta(\omega,i)} & \partial_j A(\omega,i) \times P_i & \xrightarrow{\beta(\omega,i,j)\times \mathrm{id}} & A(\omega,i,j) \times P_j \times P_i \\
\| & & & & \\
\partial_j A(\omega) \cap \partial_i A(\omega) & & & & \Big\downarrow {\scriptstyle \mathrm{id}\times T} \\
\| & & & & \\
\partial_i\partial_j A(\omega) & \xrightarrow{\beta(\omega,j)} & \partial_i A(\omega,j) \times P_j & \xrightarrow{\beta(\omega,j,i)\times \mathrm{id}} & A(\omega,i,j) \times P_i \times P_j
\end{array}$$

where T is the twisting isomorphism.

The isomorphisms described here give information on how to glue the $A(\omega)$'s together in order to obtain a space with singularities as described in the Introduction.

So formally we write our objects as

$$A = \{A(\omega), \beta(\omega, i)\}.$$

But often we will just write A where no confusion can arise. We say that the dimension of an S_n-manifold A is the dimension of $A(\emptyset)$ which is the manifold whose boundary structure the other $A(\omega)$'s describe.

Let A be an S_n-manifold. We want to define a singular S_n-manifold in a pair (X, Y) of topological spaces.

DEFINITION 2.3. A singular S_n-manifold in (X, Y) is a sequence of pairs

$$(A(\omega), g(\omega)), \quad \omega \subset \{0, 1, \ldots, n\},$$

such that

i) A is an S_n-manifold

$$A = \{A(\omega), \beta(\omega, i)\},$$

ii) the $g(\omega)$'s are continuous maps such that the following diagram commutes

$$\begin{tikzcd}
A(\omega) \arrow[rr, "g(\omega)"] & & X & \supset Y \\
\cup & & & \\
\partial_i A(\omega) \arrow[d, "\beta(\omega,i)"'] & & & \\
A(\omega, i) \times P_i \arrow[rr, "\mathrm{pr}"] & & A(\omega, i) \arrow[uuu, "g(\omega,i)"] \arrow[uuur, "g(\omega,0)"'] &
\end{tikzcd}$$

DEFINITION 2.4. A morphism

$$f\colon (A, \beta) \to (B, \delta)$$

between two S_n-manifolds is a system of morphisms between manifolds

$$f = (f(\omega))$$

such that the following diagram commutes:

$$\begin{tikzcd}
A(\omega) \arrow[r, "f(\omega)"] & B(\omega) \\
\cup & \cup \\
\partial_i A(\omega) \arrow[r, "f(\omega)|"] \arrow[d, "\mathrm{pr}\cdot\beta(\omega,i)"'] & \partial_i B(\omega) \arrow[d, "\mathrm{pr}\cdot\delta(\omega,i)"] \\
A(\omega, i) \arrow[r, "f(\omega,i)"] & B(\omega, i)
\end{tikzcd}$$

Next we would like to have a bordism concept for our singular manifolds.

DEFINITION 2.5. Let $(A(\omega), g(\omega))$ be a singular S_n-manifold in (X, Y). It bords iff there exists a singular S_n-manifold $(B(\omega), h(\omega))$ (we do not require $h(\omega, 0)$ to factor through Y) such that

$$\partial_0 B(\omega) = B(\omega, 0) \supset A(\omega)$$

(as a sub-S_n-manifold of codimension zero),

$$h(\omega, 0) \mid A(\omega) = g(\omega)$$

and

$$h(\omega, 0)(B(\omega, 0) - A(\omega)^\circ) \subset Y ,$$

where $A(\omega)^\circ$ is the interior of $A(\omega)$.

If Q is a closed ordinary manifold and A an S_n-manifold, we define their product to be the S_n-manifold given by

$$A \times Q = \{A(\omega) \times Q,\ \beta(\omega, i) \times \mathrm{id}\} .$$

That this is an S_n-manifold is obvious. Disjoint sum of two S_n-manifolds is defined naturally as

$$A + B = \{A(\omega) + B(\omega), \beta(\omega, i) + \delta(\omega, i)\}$$

and for singular S_n-manifolds in (X, Y)

$$(A(\omega), g(\omega)) + (B(\omega), h(\omega)) = (A(\omega) + B(\omega), g(\omega) + h(\omega)) .$$

DEFINITION 2.6. $(A(\omega), g(\omega))$ is bordant to $(B(\omega), h(\omega))$ iff

$$(A(\omega) + B(\omega), g(\omega) + h(\omega))$$

bords. We write this as

$$(A(\omega), g(\omega)) \sim (B(\omega), h(\omega)) .$$

3. Basic properties.

We begin by proving

LEMMA 3.1. *The relation of bordism* $(\sim)$ *between* S_n*-manifolds in* (X, Y) *is an equivalence relation.*

PROOF. i) *Symmetry* is obvious.

ii) *Reflexivity*

$$(A, g(\omega)) \sim (A, g(\omega)) .$$

We will organize $A \times I$ into an S_n-manifold in the following way. Define

$$(A \times I)(\omega) = A(\omega) \times I$$

and give it the structure of a decomposed manifold as follows:

$$\partial(A(\omega) \times I) = (A(\omega) + A(\omega)) \cup_\partial A(\omega) \times I$$
$$= (A(\omega) + A(\omega)) \cup_\partial \partial_0 A(\omega) \times I \cup \partial_1 A(\omega) \times I \cup \ldots \cup \partial_n A(\omega) \times I$$

and define

$$\partial_0(A(\omega) \times I) = (A(\omega) + A(\omega)) \cup_\partial \partial_0 A(\omega) \times I$$
$$\partial_i(A(\omega) \times I) = \partial_i A(\omega) \times I, \quad i = 1, \ldots, n .$$

This is done of course for all ω.

We define new isomorphisms $\beta'(\omega, i)$ as the composite:

$$T \circ \beta(\omega, i) \times id \colon \partial_i A(\omega) \times I \cong A(\omega, i) \times P_i \times I \cong A(\omega, i) \times I \times P_i .$$

The maps $g(\omega) \colon A(\omega) \to (X, Y)$ extend to the maps

$$g'(\omega) \colon A(\omega) \times I \to (X, Y)$$

defined by

$$g'(\omega)(-, t) = g(\omega)(-), \quad t \in I .$$

iii) *Transitivity.* We have the following situation

$$(A_1, g_1(\omega)) \sim (A_2, g_2(\omega)) ,$$
$$(A_2, g_2(\omega)) \sim (A_3, g_3(\omega)) .$$

The first bordism we denote by $(B, f(\omega))$ and the second by $(C, h(\omega))$. Therefore, for all ω

$$B(0, \omega) \supset A_1(\omega) + A_2(\omega) ,$$
$$C(0, \omega) \supset A_2(\omega) + A_3(\omega) .$$

Define $D(\omega)$ as follows:

$$D(\omega) = (B(\omega) + C(\omega))/\sim ,$$

where $\sim$ means that we identify common elements (up to isomorphism) of $A_2(\omega)$ in $B(\omega)$ and $C(\omega)$. Consider

$$\partial(B(\omega) + C(\omega)) = \partial B(\omega) + \partial C(\omega)$$
$$= \partial_0 B(\omega) \cup \ldots \cup \partial_n B(\omega) + \partial_0 C(\omega) \cup \ldots \cup \partial_n C(\omega) .$$

Here

$$[A_1(\omega) + A_2(\omega)] \subset \partial_0 B(\omega) \quad \text{and} \quad [A_2(\omega) + A_3(\omega)] \subset \partial_0 C(\omega) ,$$

so we give $D(\omega)$ the following decomposition

$$\partial D(\omega) = [\partial_0 B(\omega) \cup_{A_2(\omega)} \partial_0 C(\omega)] \cup \ldots \cup [\partial_n B(\omega) \cup_{A_2(\omega)} \partial_n C(\omega)]$$

defining

$$\partial_i D(\omega) = \partial_i B(\omega) \cup_{A_2(\omega)} \partial_i C(\omega), \quad i=0,1,\ldots,n .$$

We also have isomorphisms

$$\partial_i D(\omega) = [\partial_i B(\omega) \cup_{A_2(\omega)} \partial_i C(\omega)] \cong [B(\omega,i) \cup_{A_2(\omega,\, i)} C(\omega,i)] \times P_i$$

induced by the corresponding isomorphisms for $B(\omega)$ and $C(\omega)$, so

$$D(\omega,i) = B(\omega,i) \cup_{A_2(\omega,\, i)} C(\omega,i) .$$

Clearly

$$A_1(\omega)+A_3(\omega) \subset \partial_0 D(\omega)$$

and the maps $f(\omega)+g(\omega)$ induce maps $k(\omega)\colon D(\omega) \to (X, Y)$ such that $(D, k(\omega))$ serves as the desired bordism between $A_1(\omega)$ and $A_3(\omega)$.

We have defined addition between singular S_n-manifolds and this is clearly compatible with our bordism relation, and for a fixed dimension m of our singular manifolds (this means $\dim A(\emptyset)=m$) we get an abelian group

$$M(S_n)_m(X, Y)$$

and hence a graded abelian group $M(S_n)_*(X, Y)$.

Let us denote the usual bordism ring of closed, smooth manifolds by M_*. We have defined a product between singular S_n-manifolds and closed, smooth manifolds, and this gives us now $M(S_n)_*(X, Y)$ as an M_*-module. We should also remark that in the absolute groups $(Y=\emptyset)$

$$M(S_n)_*(X) ,$$

$(A, g(\omega))$ bords via $(B, f(\omega))$ iff $B(0,\omega)=A(\omega)$, not just $B(0,\omega)\supset A(\omega)$ (see Definition 2.5). Further for the singular S_n-manifolds we have $\partial_0 A(\omega)=\emptyset$ since $g(\omega,0)$ factors through Y, but now $Y=\emptyset$.

Now we want to study the groups $M(S_n)_*(X)$, and the first question we want to answer is: How are $M(S_n)_*(X)$ and $M(S_{n+1})_*(X)$ related?

Before stating our theorem we have to define certain homomorphisms between these groups. Let

$$\beta\colon\ M(S_n)_*(X) \to M(S_n)_*(X)$$

be defined by:

$$\beta([A, g(\omega)]_n) = [A \times P_{n+1}, g(\omega)\circ \mathrm{pr}]_n$$

where $[\ ]_n$ means equivalence class based on S_n. β is obviously well-defined and is of degree $+\dim P_{n+1}$.

Further we shall define a map

$$\gamma\colon\ M(S_n)_*(X) \to M(S_{n+1})_*(X)\ .$$

A singular S_n-manifold $(A,g(\omega))$ can always be considered as an S_{n+1}-manifold by extending all decompositions by a trivial component as follows

$$\partial A(\omega) = \partial_0 A(\omega) \cup \ldots \cup \partial_n A(\omega) \cup \emptyset$$

such that $\partial_{n+1}A(\omega) \cong \emptyset \times P_{n+1}$. We define

$$\gamma([A,g(\omega)]_n) = [A,g(\omega)]_{n+1}$$

which is obviously well-defined and γ is of degree 0.

Finally

$$\delta\colon\ M(S_{n+1})_*(X) \to M(S_n)_*(X)$$

is defined by

$$\delta([A(\omega),g(\omega)]_{n+1}) = [A(\omega,n+1),g(\omega,n+1)]_n\ .$$

We have to prove that δ is well-defined, namely $\delta(0)=0$.

Assume $[A(\omega),g(\omega)]_{n+1}=0$, then there exists $(B(\omega),f(\omega))$ such that

$$B(0,\omega) = A(\omega)$$

as S_{n+1}-manifolds $f(0,\omega)=g(\omega)$ but then we must have

$$B(0,\omega,n+1) = A(\omega,n+1)$$

as S_n-manifolds, hence $[A(\omega,n+1),g(\omega,n+1)]_n = 0$.

We can now state our theorem on the relation between $M(S_n)_*(X)$ and $M(S_{n+1})_*(X)$.

THEOREM 3.2. *The following sequence is exact*

$$\ldots \to M(S_n)_*(X) \xrightarrow{\beta} M(S_n)_*(X) \xrightarrow{\gamma} M(S_{n+1})_*(X) \xrightarrow{\delta} \ldots$$

where β, γ and δ are the homomorphisms just defined.

PROOF. i) $\gamma\beta=0$.

$$\begin{aligned}\gamma\beta([A(\omega),g(\omega)]_n &= \gamma([A(\omega)\times P_{n+1},g(\omega)\circ \mathrm{pr}]_n)\\ &= [A(\omega)\times P_{n+1},g(\omega)\circ \mathrm{pr}]_{n+1} = 0\end{aligned}$$

because we can define a bordism to zero $-$ $(B(\omega),f(\omega))$ where

$$B(\omega) = A(\omega)\times P_{n+1}\times I$$

with the following decomposition

$$\partial B(\omega) = A(\omega) \times P_{n+1} \cup \partial_1 A(\omega) \times P_{n+1} \times I \cup \ldots \cup \partial_n A(\omega) \times \times P_{n+1} \times I \cup A(\omega) \times P_{n+1}$$

and

$$\partial_0 B(\omega) = A(\omega) \times P_{n+1}$$

$$\partial_i B(\omega) = \partial_i A(\omega) \times P_{n+1} \times I, \quad i = 1, \ldots, n ,$$

$$\partial_{n+1} B(\omega) = A(\omega) \times P_{n+1} .$$

It is clearly an S_{n+1}-manifold, and we define for $t \in I$ $f(\omega)(-,t) = g(\omega) \circ \mathrm{pr}(-)$ as the corresponding system of maps.

ii) $\ker\gamma \subset \operatorname{im}\beta$. Assume

$$\gamma([A(\omega),g(\omega)]_n) = [A(\omega),g(\omega)]_{n+1} = 0 .$$

Hence there exists $(B(\omega),f(\omega))$, a singular S_{n+1}-manifold such that

$$B(0,\omega) = A(\omega)$$

as S_{n+1}-manifolds. $A(\omega)$ has been given the structure of an S_{n+1}-manifold in "the trivial" way as mentioned before. We have

$$\partial B(\omega) = \partial_0 B(\omega) \cup \partial_1 B(\omega) \cup \ldots \cup \partial_{n+1} B(\omega) .$$

Here

$$\partial_0 B(\omega) = B(0,\omega) = A(\omega)$$

$$\partial_0 \partial_{n+1} B(\omega) = \partial_{n+1} \partial_0 B(\omega) \cong A(\omega, n+1) \times P_{n+1} = \emptyset ,$$

since we consider the S_n-manifold $A(\omega)$ as an S_{n+1}-manifold. So we can write

$$\partial B(\omega) = (A(\omega) + B(\omega,n+1) \times P_{n+1}) \cup \partial_1 B(\omega) \cup \ldots \cup \partial_n B(\omega) ,$$

putting

$$\partial_0 B(\omega) = (A(\omega) + B(\omega,n+1) \times P_{n+1}) .$$

This shows that

$$\begin{aligned} [A(\omega),g(\omega)]_n &= [B(\omega,n+1) \times P_{n+1}, f(\omega,n+1) \circ \mathrm{pr}]_n \\ &= \beta([B(\omega,n+1), f(\omega,n+1)]) . \end{aligned}$$

iii) $\delta\gamma = 0$.

$$\begin{aligned} \delta\gamma([A(\omega),g(\omega)]_n) &= \delta([A(\omega),g(\omega)]_{n+1}) \\ &= [A(\omega,n+1), g(\omega,n+1)]_n = 0 , \end{aligned}$$

since $A(\omega)$ is an S_n-manifold and serves as a bordism to zero.

iv) $\ker\delta \subset \operatorname{im}\gamma$. Assume

$$\delta([A(\omega),g(\omega)]_{n+1}) = [A(\omega,n+1),g(\omega,n+1)]_n = 0 .$$

Hence there exists an S_n-manifold $(B(\omega),f(\omega))$ such that

$$B(0,\omega) = A(\omega,n+1)$$

and $f(0,\omega)=g(\omega,n+1)$. We have

$$\partial A(\omega) = \partial_1 A(\omega) \cup \ldots \cup \partial_n A(\omega) \cup \partial_{n+1} A(\omega)$$

and

$$\partial_{n+1} A(\omega) \cong A(\omega,n+1) \times P_{n+1} = B(0,\omega) \times P_{n+1} ,$$

$$\partial_0 B(\omega) \cong B(0,\omega) .$$

Form now

$$C(\omega) = [A(\omega) \cup_{A(\omega,n+1)\times P_{n+1}} B(\omega) \times P_{n+1}]$$

where union means identification of $A(\omega,n+1)\times P_{n+1}$ via the above isomorphisms. We give $C(\omega)$ the following decomposition where $A = A(\omega,n+1)\times P_{n+1}$:

$$\partial C(\omega) = (\partial_1 A(\omega) \cup_{\partial_1 A} \partial_1 B(\omega) \times P_{n+1}) \cup \ldots \cup (\partial_n A(\omega) \cup_{\partial_n A} \partial_n B(\omega) \times P_{n+1})$$

by defining

$$\partial_i C(\omega) = (\partial_i A(\omega) \cup_{\partial_i A} \partial_i B(\omega) \times P_{n+1}), \quad i=1,\ldots,n .$$

we organize $C(\omega)$ into an S_n-manifold and $g(\omega)$ and $f(\omega)\circ\mathrm{pr}$ give rise to the following singular maps from $C(\omega)$:

$$h(\omega) = g(\omega) \cup_A f(\omega)\circ\mathrm{pr} .$$

Consider $C(\omega)\times I$; then we put

$$\begin{aligned}
\partial(C(\omega)\times I) &= (C(\omega)+C(\omega)) \cup_\partial \partial C(\omega) \times I \\
&= (C(\omega)+A(\omega) \cup_A B(\omega)\times P_{n+1}) \cup_\partial C(\omega)\times I \\
&= (C(\omega)+A(\omega)) \cup_\partial C(\omega)\times I \cup B(\omega)\times P_{n+1} \\
&= (C(\omega)+A(\omega)) \cup \partial_1 C(\omega)\times I \cup \ldots \cup \partial_n C(\omega)\times I \cup B(\omega)\times P_{n+1} .
\end{aligned}$$

Define:

$$\partial_0(C(\omega)\times I) = C(\omega)+A(\omega) ,$$

$$\partial_i(C(\omega)\times I) = \partial_i C(\omega)\times I; \quad i=1,\ldots,n ,$$

$$\partial_{n+1}(C(\omega)\times I) = B(\omega)\times P_{n+1} .$$

This decomposition organizes $C(\omega)\times I$ into an S_n-manifold, and we define singular maps by

$$h'(\omega)(-,t) = h(\omega)(-), \quad t\in I .$$

Hence $(C(\omega)\times I, h'(\omega))$ serves as a bordism and gives

$$\gamma([C(\omega),h(\omega)]_n) = [C(\omega),h(\omega)]_{n+1} = [A(\omega),g(\omega)]_{n+1} .$$

v) $\beta\delta=0$.

$$\begin{aligned}\beta\delta([A(\omega),g(\omega)]_{n+1}) &= \beta([A(\omega,n+1),g(\omega,n+1)]_n)\\ &= [A(\omega,n+1)\times P_{n+1}, g(\omega,n+1)\circ\mathrm{pr}]_n = 0\end{aligned}$$

because we can give $A(\omega)$ the following decomposition:

$$\partial A(\omega) = A(\omega,n+1)\times P_{n+1}\cup\partial_1 A(\omega)\cup\ldots\cup\partial_n A(\omega)$$

with

$$\partial_0 A(\omega) = A(\omega,n+1)\times P_{n+1}$$

and $\partial_i A(\omega)$, $i=1\ldots n$, as before, and hence $(A(\omega),g(\omega)\circ\mathrm{pr})$ serves as a bordism to zero.

vi) $\ker\beta\subset\operatorname{im}\delta$. Assume

$$\beta([A(\omega),g(\omega)]_n) = [A(\omega)\times P_{n+1}, g(\omega)\circ\mathrm{pr}]_n = 0 ,$$

that is, there exists $(B(\omega),f(\omega))$ such that $B(0,\omega)=A(\omega)\times P_{n+1}$ and $f(0,\omega)=g(\omega)\circ\mathrm{pr}$. $B(\omega)$ is decomposed as follows:

$$\partial B(\omega) = \partial_0 B(\omega)\cup\partial_1 B(\omega)\cup\ldots\cup\partial_n B(\omega)$$

with

$$\partial_0 B(\omega) = A(\omega)\times P_{n+1} .$$

Therefore, consider $B(\omega)$ as an S_{n+1}-manifold with the following decomposition

$$\partial B'(\omega) = \partial_1 B'(\omega)\cup\ldots\cup\partial_n B'(\omega)\cup\partial_0 B'(\omega)$$

where we define

$$\partial_i B'(\omega) = \partial_i B(\omega), \quad i=1,\ldots,n ,$$

$$\partial_{n+1} B'(\omega) = \partial_0 B(\omega) .$$

Put

$$f(0,\omega) = f'(n+1,\omega)$$

and

$$f(\omega) = f'(\omega), \quad 0\notin\omega .$$

Thus we see that

$$\delta([B'(\omega),f'(\omega)]_{n+1}) = [A(\omega),g(\omega)]_n .$$

REMARK. A relative version of Theorem 3.2 for a pair (X, Y) can be obtained in a similar way.

Our next step in the study of the groups $M(S_n)_*(X)$ is the following

THEOREM 3.3. *The groups $M(S_n)_*(\text{-})$ form a generalized homology theory.*

PROOF. i) *The homotopy axiom.* Assume $\varphi_0, \varphi_1 \colon (X, Y) \to (X_1, Y_1)$ and $\varphi_0 \sim \varphi_1$ via a homotpy

$$h\colon (X \times I, Y \times I) \to (X_1, Y_1)\,.$$

We want to show that $\varphi_{0*} = \varphi_{1*}$ where $_*$ denotes the induced homomorphism in $M(S_n)_*(-,-)$, which is clearly well-defined.

Let $(A(\omega), g(\omega))$ be a singular S_n-manifold in (X, Y). Define

$$\theta(\omega)\colon\ A(\omega) \times I \to X_1$$

by

$$\theta(\omega)(x,t) = h(g(\omega)(x),t), \quad t \in I\,.$$

Then $(A(\omega) \times I, \theta(\omega))$ will be a singular S_n-manifold in (X_1, Y_1) where $A(\omega) \times I$ is decomposed as usual

$$\partial(A(\omega) \times I) = (A(\omega) + A(\omega)) \cup_\partial \partial A(\omega) \times I$$

$$= [(A(\omega) + A(\omega)) \cup \partial_0 A(\omega) \times I] \cup \partial_1 A(\omega) \times I \cup \ldots \cup \partial_n A(\omega) \times I\,.$$

So we put

$$\partial_0(A(\omega) \times I) = [(A(\omega) + A(\omega)) \cup \partial_0 A(\omega) \times I]$$

$$\partial_i(A(\omega) \times I) = \partial_i A(\omega) \times I, \quad i = 1, \ldots, n\,.$$

Clearly

$$A(\omega) + A(\omega) \subset \partial_0(A(\omega) \times I)$$

and $\theta(\omega)(\partial_0 A(\omega)) \subset Y_1$. Furthermore

$$\theta(\omega)(x, 0) = \varphi_0(g(\omega)(x))\,,$$

$$\theta(\omega)(x, 1) = \varphi_1(g(\omega)(x))\,.$$

Therefore we get

$$[A(\omega), \varphi_0 \circ g(\omega)]_n = [A(\omega), \varphi_1 \circ g(\omega)]_n$$

and hence $\varphi_{0*} = \varphi_{1*}$.

ii) *The excision axiom.* We use induction over the number of singularity types: $M(S_0)_*(-)$ is just ordinary bordism theory and here we have excision (see [5]), so we assume inductively that we also have excision in $M(S_n)_*(-)$ and want to prove it for $M(S_{n+1})_*(-)$.

So let $\bar{U} \subset \operatorname{int} Y$ and $i: (X-U, Y-U) \to (X, Y)$ be the inclusion, then we use a relative version of Theorem 3.2 in order to get the following commutative diagram where the rows are exact ($p = \dim P_{n+1}$):

$$\begin{array}{ccccccc}
\dots \xrightarrow{\delta} & M(S_n)_{k-p}(X,Y) & \xrightarrow{\beta} & M(S_n)_k(X,Y) & \xrightarrow{\gamma} & M(S_{n+1})_k(X,Y) & \xrightarrow{\delta} \dots \\
& \cong \Big\uparrow i_* & & \cong \Big\uparrow i_* & & \Big\uparrow i_* & \\
\dots \xrightarrow{\delta} & M(S_n)_{k-p}(X-U,Y-U) & \xrightarrow{\beta} & M(S_n)_k(X-U,Y-U) & \xrightarrow{\gamma} & M(S_{n+1})_k(X-U,Y-U) & \xrightarrow{\delta} \dots
\end{array}$$

and then the five-lemma gives

$$M(S_{n+1})_k(X-U, Y-U) \approx M(S_{n+1})_k(X, Y)$$

which is the desired result.

iii) *The exactness axiom.* Given

$$Y \xrightarrow{i} X \xrightarrow{j} (X, Y)\,,$$

we are going to prove that the following sequence is exact:

$$\dots \to M(S_n)_*(Y) \xrightarrow{i_*} M(S_n)_*(X) \xrightarrow{j_*} M(S_n)_*(X,Y) \xrightarrow{\partial} \dots$$

where i^* and j^* are the induced homomorphisms of i and j. ∂ is defined as follows (here $[-] = [-]_n$).

Let $(A(\omega), g(\omega))$ be a singular S_n-manifold in (X, Y), then

$$\partial([A(\omega), g(\omega)]) = [A(0,\omega), g(0,\omega)]$$

and it is well-defined since if $(A(\omega), g(\omega))$ bords there exists $(B(\omega), h(\omega))$ such that

$$B(0,\omega) \supset A(\omega) \quad \text{and} \quad h(0,\omega)\,|\,A(\omega) = g(\omega)\,,$$
$$h(0,\omega)(B(0,\omega) - A(\omega)^\circ) \subset Y\,.$$

Therefore $\big((B(0,\omega) - A(\omega)^\circ), h(0,\omega)|\big)$ serves as a bordism to zero for $(A(0,\omega), g(0,\omega))$.

We are now ready to start proving exactness.

i) $j_* i_* = 0$.

$$j_* i_*([A(\omega), g(\omega)]) = j_*([A(\omega), i \circ g(\omega)]) = [A(\omega), j \circ i \circ g(\omega)] = 0$$

as a singular S_n-manifold in (X, Y) since

$$\partial(A(\omega) \times I) = [A(\omega) + A(\omega)] \cup \partial_1 A(\omega) \times I \cup \dots \cup \partial_n A(\omega) \times I$$

and we put

$$\partial_0 = A(\omega) + A(\omega)$$
$$\partial_i = \partial_i A(\omega) \times I, \quad i = 1, \dots, n\,.$$

Define $g'(\omega)(x,t)=g(\omega)(x)$, for $t \in I$. Then

$$g'(\omega)\big(A(\omega)\times 1\big) \subset Y$$

so $\big(A(\omega)\times I, g'(\omega)\big)$ serves as a bordism to zero.

ii) $\partial j_* = 0$.

$$\partial j_*([A(\omega),g(\omega)]) = \partial([A(\omega), j\circ g(\omega)]) = [\emptyset,\emptyset] = 0$$

since $\partial_0 A(\omega)=\emptyset$.

iii) $i_*\partial = 0$.

$$i_*\partial([A(\omega),g(\omega)]) = i_*([A(0,\omega),g(0,\omega)]) = [A(0,\omega), i\circ g(0,\omega)] = 0$$

since the original $\big(A(\omega),g(\omega)\big)$ serves as a bordism to zero.

iv) $\ker j_* \subset \operatorname{im} i_*$. Assume

$$j_*([A(\omega),g(\omega)]) = 0 .$$

Then there exists $(B(\omega),h(\omega))$ such that

$$B(0,\omega) \supset A(\omega)$$

(as sub-S_n-manifold) and such that

$$h(0,\omega) = j\circ g(\omega) ,$$
$$h(0,\omega)\big(B(0,\omega)-A(\omega)^\circ\big) \subset Y .$$

Then $\big(B(0,\omega)-A(\omega)^\circ, i\circ h(0,\omega)\big)$ will be a singular S_n-manifold in X and

$$\partial B(\omega) = \big(A(\omega)+\big(B(0,\omega)-A(\omega)^\circ\big)\big) \cup \partial_1 B(\omega) \cup \ldots \cup \partial_n B(\omega)$$

with

$$\partial_0 B(\omega) = [A(\omega)+\big(B(0,\omega)-A(\omega)^\circ\big)] .$$

Therefore we get

$$[A(\omega),g(\omega)] = [\big(B(0,\omega)-A(\omega)^\circ\big), i\circ h(0,\omega)]$$
$$= i_*[\big(B(0,\omega)-A(\omega)^\circ\big), h(0,\omega)] .$$

v) $\ker i_* \subset \operatorname{im} \partial$. Assume

$$i_*([A(\omega),g(\omega)]) = 0 .$$

Hence there exists $\big(B(\omega),h(\omega)\big)$ — a singular S_n-manifold in X — such that

$$B(0,\omega) = A(\omega) \quad \text{and} \quad h(0,\omega) = i\circ g(\omega) .$$

Obviously

$$\partial([B(\omega),h(\omega)]) = [A(\omega),g(\omega)] ,$$

since $h(0,\omega)$ as a map into Y equals $g(\omega)$.

vi) $\ker\partial\subset\operatorname{im} j_*$. Let $(A(\omega),g(\omega))$ be a singular S_n-manifold in (X,Y) and

$$\partial([A(\omega),g(\omega)]) = 0$$

then there exists a singular S_n-manifold $(B(\omega),g(\omega))$ in Y such that

$$B(0,\omega) = A(0,\omega) \quad \text{and} \quad f(0,\omega) = g(0,\omega)\,.$$

Define now

$$V(\omega) = \{B(\omega)+A(\omega)\}/B(0,\omega) \equiv A(0,\omega)$$

and maps

$$F(\omega)\colon\ V(\omega)\to X$$

induced by $f(\omega)+g(\omega)$ (this is well-defined since $f(0,\omega)=g(0,\omega)$ on the part we have made identifications).

We give $V(\omega)$ the following natural decomposition

$$\partial V(\omega) = (\partial_1 B(\omega)\cup_{\partial_0}\partial_1 A(\omega))\cup\ldots\cup(\partial_n B(\omega)\cup_{\partial_0}\partial_n A(\omega))$$

where

$$\partial_1 = \partial_1 B(\omega)\cup_{\partial_0}\partial_1 A(\omega)\,,$$
$$\ldots$$
$$\partial_n = \partial_n B(\omega)\cup_{\partial_0}\partial_n A(\omega)$$

and where $\cup_{\partial_0}$ means the identification described above. We see that $(V(\omega),f(\omega))$ is a singular S_n-manifold in X.

Consider $V(\omega)\times I$ as an S_n-manifold as usual

$$\begin{aligned}\partial(V(\omega)\times I) &= (V(\omega)+V(\omega))\cup\partial_1 V(\omega)\times I\cup\ldots\cup\partial_n V(\omega)\times I\\ &= (V(\omega)+A(\omega)\cup_{\partial_0}B(\omega))\cup\partial_1 V(\omega)\times I\cup\ldots\cup\partial_n V(\omega)\times I\end{aligned}$$

and maps $F'(\omega)(-,t)=F(\omega)(-)$, $t\in I$. Since $g(\omega)\colon B(\omega)\to Y$ this now shows that

$$\begin{aligned}j_*([V(\omega),F(\omega)]) &= [V(\omega),j\circ F(\omega)]\\ &= [A(\omega),g(\omega)]\,.\end{aligned}$$

This completes the proof of the exactness axiom and also that the groups $M(S_n)_*(-)$ form a generalized homology theory.

Remark. By going through the preceding ideas and constructions we see that they carry over to the case when the manifolds that we consider have a certain G-structure, and we get homology theories which we will denote by $MG(S_n)_*(-)$, f. ex. $G=\mathrm{O}$, U, Sp.

4. The homology theories $MG(S_n)_*(-)$.

We will now let $MG(S_n)_*(-)$ denote the reduced homology theory and we would like to know its coefficients $MG(S_n)_*(S^0)$ in terms of $MG_*(S^0)$. We proceed inductively by means of Theorem 3.2.

Assume that P_1 is such that the map

$$MG_*(S^0) \xrightarrow{\times [P_1]} MG_*(S^0)$$

is a monomorphism, then from Theorem 3.2 we get that

$$MG(S_1)_*(S^0) \cong MG_*(S^0)/([P_1]) .$$

(This is satisfied if for example $MG_*(S^0)$ is an integral domain.) Let us assume inductively that

$$MG(S_{n-1})_*(S^0) \cong MG_*(S^0)/([P_1],\ldots,[P_{n-1}]) .$$

Then, if the map

$$MG(S_{n-1})_*(S^0) \xrightarrow{\times [P_n]} MG(S_{n-1})_*(S^0)$$

is a monomorphism, we have

THEOREM 4.1. *Under the conditions just stated we get*

$$MG(S_n)_*(S^0) \cong MG_*(S^0)/([P_1],\ldots,[P_n]) .$$

PROOF. This follows immediately from our assumptions and Theorem 3.2.

From a well-known representation theorem we know that the homology theory $MG(S_n)_*(-)$ — defined on the category of finite CW-complexes, to which we now will restrict our attention — has a representing spectrum which we will denote by $MG(S_n)$ (MG is the representing spectrum for $MG_*(-)$.

From the exact sequence in Theorem 3.2 we get a direct system of homology theories

$$MG_*(-) \to MG(S_1)_*(-) \to \ldots \to MG(S_n)_*(-) \to \ldots .$$

It is an interesting question whether the spectrum $MG(S_n)$ can be obtained as some sort of Thom-spectrum in the ordinary way. The direct limit is also a homology theory which we will denote by

$$\varinjlim_n MG(S_n)_*(-) = MG(S)_*(-)$$

and the representing spectrum by $MG(S)$.

It seems natural to ask for a connection between for example the two spectra $MG(S_n)$ and $MG(S_{n+1})$. In the case of one singularity this can be answered nicely as follows.

$S_1 = \{*, P_1\}$ and we have that

$$[P_1] \in MG_*(S^0) \cong \pi_*(MG)$$

(Thom's theorem), so P_1 has a representative map up to homotopy

$$\varphi \colon S^n \to MG \quad (n = \dim P) .$$

We assume that MG is a multiplicative spectrum. Consider then the diagram

$$\begin{tikzcd} S^n \wedge MG \arrow[r, "\times [P_1]"] \arrow[dr, "\varphi \wedge 1"'] & MG \\ & MG \wedge MG \arrow[u, "\mu"'] \end{tikzcd}$$

the composite map which we have denoted by $\times [P_1]$ induces multiplication by $[P_1]$ in homotopy. Let us take the cofibre of this map (or otherwise: extend it to an exact triangle)

$$S^n \wedge MG \xrightarrow{\times [P_1]} MG \to X(P_1) .$$

The spectrum $X(P_1)$ has then the following property

THEOREM 4.2. *$X(P_1)$ is homotopy equivalent to $MG(P_1)$ under condition* A *stated in the proof.*

PROOF. (The idea of the proof was kindly pointed out to me by J. F. Adams.) Let $MG = \varinjlim_\alpha MG_\alpha$ where the MG_α's are finite complexes. Therefore they also have Spanier–Whitehead duals DMG_α. Theorem 3.2 gives an exact sequence for all α

$$\ldots \to MG_*(DMG_\alpha) \xrightarrow{\times [P_1]} MG_*(DMG_\alpha) \to MG(P)_*(D\,MG_\alpha) \to \ldots .$$

By duality this exact sequence can be written as

$$\ldots \to MG^*(MG_\alpha) \xrightarrow{\times [P_1]} MG^*(MG_\alpha) \to MG(P)^*(MG_\alpha) \to \ldots .$$

Apply the functor $\varprojlim_\alpha$ to this sequence

$$(*) \qquad \ldots \to \varprojlim_\alpha MG^*(MG_\alpha) \xrightarrow{\times [P_1]} \varprojlim_\alpha MG^*(MG_\alpha) \\ \to \varprojlim_\alpha MG(P)^*(MG_\alpha) \to \ldots$$

Clearly the composition of two successive maps here is zero (functorial property of $\varprojlim$) and that is what we need, so we do not worry about exactness. But we have to assume the following about MG:

A. $\{MG_\alpha\}$ can be chosen such that

$$\varprojlim_\alpha{}^1 MG^*(MG_\alpha) = 0 = \varprojlim_\alpha{}^1 MG(P_1)^*(MG_\alpha)$$

and multiplication by $[P]$ is injective.

(This is satisfied for example for MU and any P_1).

Therefore we have according to Milnor's lemma

$$\varprojlim_\alpha MG^*(MG_\alpha) = MG^*(MG) = \{MG, MG\}_*$$

and

$$\varprojlim MG(P_1)^*(MG_\alpha) = MG(P_1)^*(MG) = \{MG, MG(P_1)\}_* \, .$$

So we can write the sequence $(*)$ as

$$\ldots \to MG^*(MG) \xrightarrow{\times [P]} MG^*(MG) \xrightarrow{\gamma} MG(P_1)^*(MG) \to \ldots ,$$

or as

$$\ldots \to \{MG, MG\}_* \xrightarrow{\beta} \{MG, MG\}_* \xrightarrow{\gamma} \{MG, MG(P_1)\}_* \to \ldots ,$$

where we have put $\beta = \times [P_1]$.

Let 1_{MG} be the element in $\{MG, MG\}_*$ represented by the identity map $MG \to MG$. Take

$$\gamma(1_{MG}) \in \{MG, MG(P_1)\}_*$$

and represent it by a map

$$f\colon\; MG \to MG(P_1) \, .$$

We now combine this map with the exact triangle we have (instead of $S^n \wedge MG$ we write just MG by a degree shift)

$$\begin{tikzcd} MG \arrow[r, "{\times [P_1]}"] \arrow[dr, "\sim 0"'] & MG \arrow[r] \arrow[d, "f"] & X(P_1) \arrow[dl, dashed, "g"] \\ & MG(P_1) & \end{tikzcd}$$

But we know that $\gamma\beta = 0$ which implies $f \circ (\times [P_1]) = 0$, hence there exists a g making the above triangle commutative; this follows at once from the exact sequence

$$\ldots \leftarrow \{MG, MG(P_1)\}_* \xleftarrow{\times [P_1]} \{MG, MG(P_1)\}_* \leftarrow \{X(P_1), MG(P_1)\}_* \leftarrow \ldots ,$$

which we obtain by applying the functor $\{-, MG(P_1)\}_*$ to the exact triangle we have considered.

Clearly we now have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \pi_*(MG) & \xrightarrow{\times [P_1]} & \pi_*(MG) & \to & \pi_*(MG(P_1)) & \to & 0 \\ \| & & \| & & \| & & \uparrow g_* & & \| \\ 0 & \to & \pi_*(MG) & \xrightarrow{\times [P_1]} & \pi_*(MG) & \to & \pi_*(X(P_1)) & \to & 0 \end{array}$$

The five-lemma then gives that g_* is an isomorphism. Hence g is a homotopy equivalence between $MG(P_1)$ and $X(P_1)$ and for all complexes Y we have that

$$X(P_1)_*(Y) \cong MG(P_1)_*(Y) .$$

We have already shown that we have a pairing

$$MG_* \otimes MG(S_n)_* \to MG(S_n)_* ,$$

and we conclude that $MG(S_n)$ is a module spectrum over MG.

Consider the exact sequence in Theorem 3.2.

$$\ldots \to MG(S_n)_*(X) \xrightarrow{\times [P_{n+1}]} MG(S_n)_*(X) \to MG(S_{n+1})_*(X) \to \ldots$$

We remark that such a sequence always gives rise to an exact couple and hence a spectral sequence. This can be considered as a generalization of the so-called Bockstein spectral sequence which we get in the special case of one singularity $P_1 = \mathsf{Z}_p$ (see for example [7]).

5. The unitary case.

It is well-known that $MU_*(S^0) = \mathsf{Z}[x_2, x_4, \ldots, x_{2n}, \ldots]$. We will be interested in the case when $S = \{P_0, P_1 \ldots P_n \ldots\}$ is such that P_n is a manifold representing the polynomial generator x_{2n}. As an immediate consequence we then get

$$MU(S)_k(S^0) = \begin{cases} 0 & \text{if } k \neq 0 , \\ \mathsf{Z} & \text{if } k = 0 . \end{cases}$$

since $MU_*(S^0)$ has no divisors of zero. Hence from the Eilenberg–Steenrod uniqueness theorem we get

Corollary 5.1. $MU(S)_*(-) = H_*(-, \mathsf{Z})$.

Therefore, ordinary homology can be considered as some sort of bordism theory with singularities!

We defined

$$S_n = \{P_0, \ldots, P_n\}.$$

Let us define

$$\tilde{S}_n = \{P_{n+1} \ldots\}$$

and

$$S^n{}_m = \{P_{n+1}, \ldots, P_m\}$$

and denote $MU(\tilde{S}_n)$ by $MU\langle n\rangle$.

Obviously we get

COROLLARY 5.2. $MU\langle n\rangle_*(S^0) \cong \mathbf{Z}[x_2, \ldots, x_{2n}]$.

We obtain a natural transformation

$$t^n(-)\colon\ MU\langle n\rangle_*(-) \to MU\langle n-1\rangle_*(-)$$

as follows:

Consider the exact sequence

$$\ldots \to MU(S_m{}^n)_*(-) \xrightarrow{\times [P_1]} MU(S_{mu})_*(-) \overset{\gamma_{m,n}}{\to} MU(S_m{}^{n-1})_*(-) \to \ldots$$

By passing to the direct limit we get

$$\begin{array}{ccc} \varinjlim_m MU(S_m{}^n)_*(-) & \xrightarrow[=t^n]{\varinjlim \gamma_{m,n}} & \varinjlim_m MU(S_m{}^{n-1})_*(-) \\ \| & & \| \\ MU\langle n\rangle_*(-) & \xrightarrow[t^n]{} & MU\langle n-1\rangle_*(-) \end{array}$$

So we get a "tower" of homology theories and natural transformations (I am grateful to Larry Smith for drawing my attention to these theories)

$$\begin{array}{c} MU\langle\infty\rangle_*(-) = MU^*(-) \\ \downarrow \\ \vdots \\ \downarrow \\ MU\langle n\rangle_*(-) \\ \downarrow \\ \vdots \\ \downarrow \\ MU\langle 1\rangle_*(-) \\ \Big\downarrow \\ MU\langle 0\rangle_*(-) = H_*(-, \mathbf{Z})\,. \end{array}$$

6. Addendum.

We can certainly vary the singularity classes S_n a lot and obtain many different spectra $MU(S_n)$. For a general formulation, see [3]. For example, by using as singularities manifolds representing generators in dimensions $\neq 2(p^n-1)$ for a given prime, we get a tower of "integral" Brown–Peterson spectra. After localizing with respect to p we get BP and $BP\langle n\rangle$ (the analogues of $MU\langle n\rangle$) and they give interesting relations with homological dimension of bordism modules. See [3] and the references there. Also the cohomology of these spectra have been calculated as modules over Steenrods algebra; see [2]. However, one major problem still seems to be to decide when these spectra are multiplicative. From our point of view this should be based on an analysis of product of manifolds with singularities.

I also feel that much of what has been done here should be done in a broader context including more refined singularities (for example, algebraic) and that the theory of stratified sets would seem appropriate here. And this would also be interesting in connection with the question of realizing ordinary homology classes.

It should also be pointed out that since $MG(S_n)$ is a module-spectrum over MG the general machinery of Adams' Seattle notes (in *Category theory, homology theory and their applications* III, edited by P. Hilton, Lecture Notes in Mathematics 99, Springer-Verlag, Berlin · Heidelberg · New York, 1969) applies to give generalized universal coefficient spectral sequences and these deserve to be studied in detail.

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