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ON FORMAL GROUPS AND SINGULARITIES IN COMPLEX COBORDISM THEORY

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1. Introduction.

It is natural to ask if there is a tower of homology theories between complex bordism and ordinary homology theory. Or in other words if there is a tower of spectra factorizing the so-called Thom-map

$$MU \to K(\mathsf{Z}) .$$

As pointed out in [4] and [5], one way to do this is to introduce bordism theories based on manifolds with a suitable type of singularities. Introducing singularities corresponds to killing certain bordism classes. The main problem up to now with these theories has been that one has not been able to show whether they are multiplicative or not.

So one may ask if there are other ways to obtain towers of this type in which the spectra are multiplicative. We will here show that some towers can be obtained by extending the Quillen–Adams' splitting method based on formal groups in complex cobordism theory. The theories we get from this method are multiplicative by construction, but they have larger coefficients than Z.

But not all theories can be obtained by the idempotent method, namely those having just a finite number of "polynomial" generators in the coefficient-ring. We will see that we then get coefficients Q which of course is considerably less interesting.

2. Theories constructed by singularities.

We recall that

$$\pi_*(MU) = \mathsf{Z}[x_2,x_4,\ldots,x_{2n},\ldots] .$$

Let $\mathscr{S}$ be a collection of unitary manifolds and $D(\mathscr{S}) \subset \mathsf{Z}^+ = \{1,2,\ldots, n,\ldots\}$ such that

$$\mathscr{S} = \{S_i\}_{i \in D(\mathscr{S})}$$

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and passing to bordism classes we have that

$$[S_i] = x_{2i}\,.$$

By the methods of [4] we now introduce the collection $\mathscr{S}$ as singularities and obtain a geometrically defined homology theory whose representing spectrum we denote by $MU(\mathscr{S})$.

The results of [4] give

THEOREM 1. *For any set $\mathscr{S}$ as described there exists a spectrum $MU(\mathscr{S})$ and map*

$$MU \to MU(\mathscr{S})$$

such that

$$\pi_*(MU(\mathscr{S})) = \mathsf{Z}[x_{2i} \mid i \in \mathsf{Z}^+ - D(\mathscr{S})] \quad (\textit{additively})$$

and the map induces the canonical projection on the coefficients.

Let

$$\mathscr{S}_0, \mathscr{S}_1, \ldots, \mathscr{S}_n, \ldots$$

be a sequence of manifold collections satisfying

$$\emptyset = \mathscr{S}_0 \subset \mathscr{S}_1 \subset \ldots \subset \mathscr{S}_n \subset \ldots \subset \mathscr{S}_\infty = \bigcup_{i=1}^{\infty} \mathscr{S}_i$$

where $D(\mathscr{S}_\infty) = \mathsf{Z}^+$ such that $\mathscr{S}_\infty$ contains a manifold representative for every polynomial generator.

By application of Theorem 1 to all the $\mathscr{S}_n$'s we get a tower of spectra

$$\begin{array}{c} MU(\mathscr{S}_0) = MU \\ \downarrow \\ \vdots \\ \downarrow \\ MU(\mathscr{S}_n) \\ \downarrow \\ \vdots \\ \downarrow \\ MU(\mathscr{S}_\infty) \simeq K(\mathsf{Z}) \end{array}$$

and

$$\pi_*(MU(\mathscr{S}_n)) = \mathsf{Z}[x_{2i} \mid i \in \mathsf{Z}^+ - D(\mathscr{S}_n)]\,.$$

This is the general setting for constructing towers and we can now specialize the $\mathscr{S}_n$'s to obtain towers of particular interest.

For example if we choose $\mathscr{S}_n$ such that

$$\mathsf{Z}^+ - D(\mathscr{S}_n) = \{1, 2, \ldots, n\}$$

we get the tower considered in [4] and [5] with

$$MU(\mathscr{S}_n) = MU\langle n\rangle$$

and

$$\pi_*(MU\langle n\rangle) = \mathsf{Z}[x_2,\ldots,x_{2n}] .$$

We asked if the following two statements were equivalent

I. $MU_*(X) \to MU\langle n\rangle_*(X)$ is epic,
II. hom $\dim_{MU_*} MU_*(X) \leqq n+1$.

This question was motivated by work of Conner, Smith and Johnson (see [7], [9]).

But later on Johnson found a counter-example for $n=2$ showing that I. is not in general equivalent to II. Then Johnson and Wilson (see [10]) discovered that if the question is considered one prime at a time, the statements are equivalent. So for a fixed prime p one introduces Q_p-coefficients in the theories $MU(\mathscr{S}_n)$ where $\mathscr{S}_n$ is such that

$$\mathsf{Z}^+ - D(\mathscr{S}_n) = \{p-1,p^2-1,\ldots,p^n-1\} ,$$

and we obtain the so-called BP-tower since $MU(\mathscr{S}_\infty)\mathsf{Q}_p$ is the Brown Peterson spectrum for the prime p:

$$\begin{array}{c} BP \\ \downarrow \\ \vdots \\ \downarrow \\ BP\langle n\rangle \\ \downarrow \\ \vdots \\ \downarrow \\ BP\langle 0\rangle \cong K(\mathsf{Q}_p) . \end{array}$$

We recall that

$$\pi_*(BP\langle n\rangle) = \mathsf{Q}_p[x_{2(p-1)},\ldots,x_{2(p^n-1)}] .$$

We now quote the following interesting result of Johnson and Wilson [10]:

THEOREM 2. *For a finite* CW-*complex the following statements are equivalent*:

(i) hom $\dim_{MU\,\mathsf{Q}_{p*}} MU\,\mathsf{Q}_{p*}(X) \leqq n+1$.
(ii) hom $\dim_{BP_*} BP_*(X) \leqq n+1$.

(iii) *The map* $BP_*(X) \to BP\langle n\rangle_*(X)$ *is an epimorphism.*
(iv) *The map*

$$BP\langle n\rangle_* \otimes_{BP_*} BP_*(X) \to BP\langle n\rangle_*(X)$$

is an isomorphism.

We should also point out that in the cases where $\mathsf{Z}^+ - D(\mathscr{S}_n)$ only contains elements of the form $p^j - 1$ the cohomology of $MU(\mathscr{S}_n)$ has been calculated as a module over Steenrods algebra in [6]. We get

THEOREM 3.

$$H^*(BP\langle n\rangle; \mathsf{Z}_p) \cong \mathsf{A}/\mathsf{A}(Q_0, Q_1, \ldots, Q_n)$$

as left A*-modules where* A *is Steenrods algebra* mod p.

Other interesting results on the spectra $BP\langle n\rangle$ have been obtained by Wilson [15].

3. Theories constructed by formal groups.

Let us start by recalling from [2] that for any positive integer d one obtains a map of ring-spectra

$$e(d): MU\mathsf{Z}[d^{-1}] \to MU\mathsf{Z}[d^{-1}]$$

where $MU\mathsf{Z}[d^{-1}]$ is $MU \wedge$(Moore spectrum of $\mathsf{Z}[d^{-1}]$). The action on the coefficients is given by

$$e(d)[CP^n] = \begin{cases} [CP^n] & \text{if } n \not\equiv -1 \bmod d \\ 0 & \text{if } n \equiv -1 \bmod d \end{cases}$$

We also have that $e(d)^2 = e(d)$ and for any two integers d and δ, $e(d)$ and $e(\delta)$ commute.

The idea of the construction of $e(d)$ is as follows: We know from [2] that maps of ring-spectra

$$g: MUR \to MUR \quad (\mathsf{Z} \subset R \subset \mathsf{Q})$$

are in $1-1$ correspondence with power series of such that

$$g_*(x^{MU}) = f(x^{MU}) = \textstyle\sum_{i \geq 0} d_i (x^{MU})^{i+1}$$

where x^{MU} is the canonical class in $MUR^2(CP^\infty)$ and $d_i \in \pi_*(MU) \otimes R$ and we assume that $d_0 = 1$.

So what we have to do is to pick the right power series. The power series Adams picks is obtained by a modification of the logarithm of the universal group law μ in complex cobordism. For if

$$\log x^{MU} = \sum_{i \geq 0} m_i (x^{MU})^{i+1}$$

with

$$m_i = [CP^i]/(i+1) \in \pi_*(MU) \otimes \mathbf{Q}$$

Adams defines

$$\operatorname{mog} x^{MU} = \sum_{i \geq 0} (g_* m_i)(x^{MU})^{i+1}$$

from which it follows that

$$f^{-1}z = \exp \operatorname{mog} z .$$

Then he takes

$$\operatorname{mog} z = \log z - d^{-1}[\log \xi_1 z + \ldots + \log \xi_d z]$$

giving

$$f^{-1}z = z -_\mu (d^{-1})_\mu (\xi_1 z +_\mu \ldots +_\mu \xi_d z)$$

where $\xi_1, \ldots, \xi_d$ are the dth complex roots of unity. It is easily seen that $f^{-1}z$ and hence fz in have coefficients in

$$\pi_*(MU\,\mathbf{Z}[d^{-1}]) = \pi_*(MU) \otimes \mathbf{Z}[d^{-1}]$$

and the corresponding $e(d)$ will have the required properties.

Let $D \subset \mathbf{Z}^+$ be a multiplicatively closed subset, and order its elements into a sequence

$$D = \{d_1, d_2, \ldots, d_k, \ldots\} .$$

We will now define an idempotent $e(D)$ – depending on D,

$$e(D): MUR \to MUR$$

where

$$R = D^{-1}\mathbf{Z} = \mathbf{Z}[d_1^{-1}, d_2^{-1}, \ldots, d_k^{-1}, \ldots] .$$

We simply put

$$e(D) = \prod_{d_k \in D} e(d_k)$$

where $\prod$ means composition, and this is well-defined since the product is convergent in the complete and Hausdorff skeleton filtration topology of

$$MUR^*(MUR) .$$

Analogous to the case of $e(d)$ we read off $e(D)$'s effect on the coefficients to be

$$e(D)[CP^n] = \begin{cases} [CP^n] & \text{if } n \not\equiv -1 \bmod D . \\ 0 & \text{if } n \equiv -1 \bmod D . \end{cases}$$

and clearly

$$e(D)^2 = e(D) .$$

For any complex X consider

$$e(D)^*: MUD^{-1}\mathbf{Z}^*(X) \to MUD^{-1}\mathbf{Z}^*(X)$$

$\mathrm{Im}\, e(D)^*$ is a new cohomology theory and by Brown's representation theorem it is representable by a spectrum $MU(D)$ and since $e(D)$ is a map of ring spectra it will be multiplicative. We also have canonical maps of ring spectra making the following diagram commutative

$$\begin{array}{ccc} MUD^{-1}\mathbf{Z} & \xrightarrow{e(D)} & MUD^{-1}\mathbf{Z} \\ \pi_D \downarrow & & \uparrow i_D \\ & MU(D) & \end{array}$$

such that

$$\pi_D \circ i_D = \mathrm{id}: MU(D) \to MU(D) .$$

It remains to calculate $\pi_*(MU(D))$. By passing to the indecomposable quotient and studying the effect of $e(D)$ there — as in [1] — we get that $\pi_*(MU(D))$ is a polynomial algebra over $D^{-1}\mathbf{Z}$. We only have to find the dimensions of the generators. Since the CP^n's represent polynomial generators in $\pi_*(MU) \otimes \mathbf{Q}$, $e(D)$'s effect on the CP^n's gives that

$$\pi_*(MU(D)) \otimes \mathbf{Q} = \mathbf{Q}[u_{2(i-1)} \mid i \in \mathbf{Z}^+ - (D)]$$

where $(D) = \bigcup_{d_i \in D} (d_i)$, (d_i) is the ideal generated by d_i, and hence there exist w_n's in $\pi_*(MU(D))$ such that

$$\pi_*(MU(D)) = D^{-1}\mathbf{Z}[w_{2(i-1)} \mid i \in \mathbf{Z}^+ - (D)] .$$

Analogous to the situation in [1] we have that $\pi_*(MUD^{-1}\mathbf{Z})$ is a free module over $\pi_*(MU(D))$, and representing the basis elements by maps

$$f_i: S^{n(i)} \to MUD^{-1}\mathbf{Z}$$

we then consider the map

$$g: \bigvee_i S^{n(i)} \wedge MU(D) \to MUD^{-1}\mathbf{Z}$$

whose ith component is given by the composite map

$$S^{n(i)} \wedge MU(D) \xrightarrow{f_i \wedge i_D} MUD^{-1}\mathbf{Z} \wedge MUD^{-1}\mathbf{Z} \xrightarrow{m} MUD^{-1}\mathbf{Z} .$$

Clearly g induces an isomorphism of homotopy groups, so we have a splitting

$$MUD^{-1}\mathbf{Z} \simeq \bigvee_i S^{n(i)} MU(D) .$$

Let us collect the results in:

THEOREM 4. *For any multiplicatively closed subset D of Z^+ there exists a ring spectrum $MU(D)$ with the following properties*

(i) $$MUD^{-1}\mathsf{Z} \cong \bigvee_i S^{n(i)}MU(D).$$

(ii) *The inclusion*

$$i_D\colon MU(D) \to MUD^{-1}\mathsf{Z}$$

and the projection

$$\pi_D\colon MUD^{-1}\mathsf{Z} \to MU(D)$$

are canonical.

(iii) $$\pi_*(MU(D)) = D^{-1}\mathsf{Z}[w_{2(i-1)} \mid i \in \mathsf{Z}^+ - (D)]\,.$$

for suitable w's in $\pi_*(MU(D))$.

(iv) *If $D_1 \subset D_2 \subset \mathsf{Z}^+$ and D_1, D_2 and $D_2 - D_1$ are multiplicatively closed then there are canonical maps*

$$MU(D_1)R \underset{i(D_1,\,D_2)}{\overset{\pi(D_1,\,D_2)}{\rightleftarrows}} MU(D_2)$$

such that $\pi(D_1,D_2)\circ i(D_1,D_2) = \mathrm{id}_{MU(D_2)}$ where $R = D_2{}^{-1}\mathsf{Z}$. Furthermore we have

$$MU(D_1)R \cong \bigvee_i S^{n(i)}MU(D_2)\,.$$

In fact by composing $i(D_1,D_2)$ by the inclusion map

$$MU(D_1) \to MU(D_1)R$$

we have a canonical map

$$MU(D_1) \to MU(D_2)\,.$$

PROOF. The only remaining point is (iv) which follows from the fact that

$$e(D_2) = e(D_2 - D_1)\circ e(D_1)\,.$$

And the wedge-splitting comes from an argument analogous to the proof of (i).

We have determined $\pi_*(MU(D))$ so it is natural to ask for the cohomological structure $H^*(MU(D); \mathsf{Z}_p)$ for some prime p.

Let $R = D^{-1}\mathsf{Z}$. The universal coefficient formulae (R is not a finitely generated abelian group!) give

$$H^*(MUR; \mathsf{Z}_p) = R \otimes_{\mathsf{Z}} \mathsf{Z}_p \otimes H^*(MU; \mathsf{Z}),$$

and clearly $H^*(MU(D); \mathsf{Z}_p)$ sits as direct summand in $H^*(MUR; \mathsf{Z}_p)$ which is zero if p is invertible in R or in other words $p \in D$.

We know from Milnor [12] that $H^*(MU; \mathsf{Z}_p)$ is a free $\mathsf{A}/(Q_0)$-module where A is the mod p Steenrod algebra. So from these facts we conclude:

THEOREM 5. *$H^*(MU(D); \mathsf{Z}_p)$ is a free $\mathsf{A}/(Q_0)$-module if $p \notin D$ and it is trivial if $p \in D$.*

REMARK. The number of summands of $\mathsf{A}/(Q_0)$ is a decreasing function of the number of elements in D.

Let us now specialize D. For a fixed prime p let $D = D(p)$ so that

$$\mathsf{Z}^+ - D(p) = \{p, p^2, \ldots, p^n, \ldots\} .$$

Then we see that

$$MU(D(p)) = BP$$

the Brown-Peterson spectrum for the prime p as given by the Quillen–Adams construction (see [2] and [14]).

Furthermore let P_∞ be the set of all primes ordered into a sequence

$$P_\infty = \{p_1, p_2, \ldots, p_n, \ldots\} ,$$

let

$$P_n = \{p_1, p_2, \ldots, p_n\} ,$$
$$P^n = \{p_{n+1}, p_{n+2}, \ldots\} ,$$

and let $(P_n)^-$, $(P^n)^-$ denote the multiplicative closure of the sets.

We now apply Theorem 4 and obtain spectra $MU(P^n)^-$ and these are obviously generalizations of the Brown–Peterson spectrum to a *collection of primes* P_n instead of just a single prime, so we will denote them by $B(P_n)$. And from Theorem 4 it also follows that

$$\pi_*(B(P_n)) = \left(\bigcap\nolimits_{p \in P_n} \mathsf{Q}_p\right)[w_{2(i-1)} \mid i \in (P_n)^- = \mathsf{Z}^+ - (P^n)^-]$$

for suitable w's.

Furthermore we deduce

$$B(P_{n+1})\mathsf{Z}[1/p_{n+1}] \sim \bigvee\nolimits_i S^{n(i)} B(P_n) .$$

In fact we have a tower of ring-spectra

$$\begin{array}{l} B(P_\infty) = MU \\ \downarrow \\ \vdots \\ \downarrow \\ B(P_{n+1}) \\ \downarrow \\ B(P_n) \\ \downarrow \\ \vdots \\ B(P_1) = BP \text{ for the prime } p_1 \end{array}$$

and we see that this tower extends naturally the tower of $BP\langle n\rangle$'s considered in section 2.

Localization at a collection of primes instead of just a single prime has been emphasized in [13] and it seems natural that the $B(P_n)$'s will play the same role with respect to P_n-torsion as $BP_{(p_1)}$ does for p_1-torsion.

The two towers we have mentioned are very different in many ways. The $B(P_n)$'s are ring-spectra by construction, for the $BP\langle n\rangle$'s the question is open. Also the cohomological structure is rather different.

One could possibly ask if spectra of the $BP\langle n\rangle$-type could be obtained by the splitting method used to obtain the $B(P_n)$. The answer is no, because we would then have to take

$$D = \{p_1^{n+1}, p_2, p_3, \ldots\}^-$$

such that

$$\mathsf{Z}^+ - (D) = \{p_1, p_1^2, \ldots, p_1^n\},$$

but then

$$D^{-1}\mathsf{Z} = \mathsf{Q}$$

so we would be reduced to the case of Eilenberg–MacLane spectra.

We could also see this in a different way. If spectra of the type $BP\langle n\rangle$ should split off from MUR (for some R) the corresponding projection map would be epic for any space contradicting the results mentioned on homological dimension. So at the moment the only way to construct the $BP\langle n\rangle$'s is by using bordism theory of manifolds with singularities.

4. Remarks.

The idempotent splittings here have been constructed in a different way from Adams' in [1]. There he first splits K-theory with coefficients

R (or the classifying space BUR) and then lifts the splitting to cobordism. So it seems natural to ask when splittings of the type considered here, obtained in cobordism by formal group laws, or other methods, can be pushed down to K-theory via the Thom-isomorphism to obtain a splitting of BUR for suitable $R=D^{-1}\mathbf{Z}$. Would then the component we split off — say $(BUR)^{\sim}$ — serve as some sort of classifying space for $MU(D)$ such that $MU(D)$ could be viewed as the Thom-spectrum of the R-localized universal bundle pulled back to $(BUR)^{\sim}$? If so it would mean that $\pi_*(MU(D))$ classifies U-manifolds with a lifting of the R-localized normal bundle to $(BUR)^{\sim}$. It would certainly be interesting to have a geometric interpretation of these spectra — say BP.

As suggested for the splitting in [1] one should also have Conner and Floyd type theorems.

For the theories $MU(D)$ it is important to have polynomial generators in $\pi_*(MU(D))$ with "nice" properties. In the case of BP it has been shown by Liulevicius [11] and Hazewinkel [8] that

$$\pi_*(BP) = \mathbf{Q}_p[v_1,\ldots v_n,\ldots]$$

where $\deg v_n = 2(p^n-1)$ and such that the image of v_n by the Hurewicz homomorphism can be given explicitly. The v_n's can be considered as elements in $\pi_*(MU\mathbf{Q}_p)$. Liulevicius then conjectured that v_n's actually lie in $\pi_*(MU)$. This has been proved by Alexander [3]. It seems natural to guess that there are polynomial generators with the same properties for the spectra $MU(D)$.

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