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COMPLETELY MONOTONIC FUNCTIONS ON n-DIMENSIONAL LATTICES*

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1. Introduction.

The Cartesian product L of n linearly ordered sets L_i is a partially ordered set with respect to the coordinatewise ordering. This partial ordering imposes a lattice structure on L and $(L;\wedge)$ is an idempotent semigroup. We investigate the nature of the completely monotonic (CM) functions on this semigroup and are able to give a sufficient condition for $f\colon L\to \mathsf{R}$ to be a CM-function. To do this we restrict our attention to a certain convex cone $C(L)$ of real valued functions, which satisfy two conditions (3.2 (i) and (ii)). We are able to identify the extreme points of a base of this cone as exponentials [2] and thus show that $C(L)$ is an extremal subcone of the cone of CM-functions, $C_\infty(L)$, on $(L;\wedge)$. This enables us in section 4 to show that a sufficient condition for $f\colon L\to \mathsf{R}$ to be a CM-function on $(L;\wedge)$ is that $\Delta_k f\geqq 0$, $0\leqq k\leqq n$ (for definition see section 2). We also decompose every $f\in X_\infty(L)$ into a certain type of convex sums.

2. Preliminaries.

If S denotes a commutative semigroup with identity e and if $f\colon S\to \mathsf{R}$, then the difference operators Δ_n, for n nonnegative integer, are defined inductively by $\Delta_0 f(x)=f(x)$ and

$$\Delta_n f(x_0;x_1,\ldots,x_n) = \Delta_{n-1}f(x_0;x_1,\ldots,x_{n-1})-\Delta_{n-1}f(x_0x_n;x_1,\ldots,x_{n-1}) .$$

The function f is said to be *completely monotonic* if $\Delta_n f(x_0;x_1,\ldots,x_n)\geqq 0$ for all choices of $x_0,x_1,\ldots,x_n\in S$ and all nonnegative integers n. Let $C_\infty(S)$ denote the family of all completely monotonic (CM) functions on S and

$$X_\infty(S) = \{f\in C_\infty(S) \mid f(e)=1\} .$$

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Then $C_\infty(S)$ is a convex cone with base [7] $X_\infty(S)$ in the linear space R^S of all real valued functions on S. If R^S is equipped with the topology of pointwise convergence, then the span $E_\infty(S)=C_\infty(S)-C_\infty(S)$ of $C_\infty(S)$ becomes a locally convex linear topological space and $X_\infty(S)$ is compact. It is known from [4] that $X_\infty(S)$ is an r-simplex, that is every $f\in X_\infty(S)$ admits a unique representing measure which is supported by the extreme points $(\operatorname{ext}X_\infty(S))$ of $X_\infty(S)$, and $\operatorname{ext}X_\infty(S)$ is closed.

3. An extremal subcone of $C_\infty(L)$.

In the following we consider the Cartesian product L of n linearly ordered sets L_i, each with a smallest element, o_i, and a largest element, e_i. We will leave out the indices when no misunderstanding may arise. Then $L=\prod_{i=1}^n L_i$ becomes a lattice if $x\vee y=(x_1\vee y_1,\ldots,x_n\vee y_n)$ where $x_i\vee y_i=x_i$ if $x_i\geqq y_i$ and $x_i\vee y_i=y_i$ if $x_i<y_i$ and $x\wedge y$ is defined similarly. Moreover $(L;\wedge)$ is an idempotent semigroup with identity $e=(e,\ldots e)$.

Lemma 3.1. *Let $n\geqq 2$. Given $x^1,\ldots,x^n\in L$ such that*

(i) $x^k=(x_1,\ldots,x_{k-1},x_k{}^k,x_{k+1},\ldots,x_n)$ *where* $x_k{}^k\leqq x_k$, $1\leqq k\leqq n$.

Then

$$\bigvee_{i=1}^n x^i = (x_1,\ldots,x_n),\qquad \bigwedge_{i=1}^n x^i = (x_1{}^1,\ldots,x_n{}^n)$$

and if $y\leqq\bigvee_{i=1}^n x^i$ and $y\nleq x^k$ for every k then $\bigwedge_{i=1}^n x^i\leqq y$.

Definition 3.2. If $n\geqq 2$ let $C(L)$ denote the set of real valued functions on L such that

(i) $\Delta_n f(\bigvee_{i=1}^n x^i;\, x^1,\ldots,x^n)\geqq 0$ whenever the collection $x^1,\ldots,x^n$ satisfies 3.1.(i).

(ii) $f(x)=0$ whenever for some k, $x_k=o$,

and let $X(L)$ denote the set $\{f\in C(L)\mid f(e)=1\}$.

If $n=1$ then $C(L)$ denotes the set of increasing functions such that $f(o)=0$.

Proposition 3.3. *The set $C(L)$ is a closed convex cone with compact base $X(L)$ in the space R^L equipped with the topology of pointwise convergence.*

Proof. We will show that $X(L)$ is a compact base. The function $\hat{e}$ as defined by $\hat{e}(f)=f(e)$ is a continuous linear functional,

$$H = \{f: L\to\mathsf{R}\mid \hat{e}(f)=1\}$$

is a hyperplane missing the origin and $X(L)=H\cap C(L)$. Let $f\in C(L)$, $f\neq 0$, and let $x\in L$ such that $f(x)\neq 0$.

By 3.2 (i) and (ii)

$$\Delta_n f(x;\,(o,x_2,\ldots,x_n),\ldots,(x_1,\ldots,x_{n-1}o)) = f(x) > 0\,.$$

Moreover $\Delta_1 f(y;\,o)\geqq 0$ for every $y\in L$. To show that $f(e)\geqq f(x)$ we observe that by 3.2 (i) and (ii)

$$\Delta_n f(e;\,(o,e,\ldots,e),\ldots,(e,\ldots e,o,e),(e,\ldots,e,x_n)) = f(e)-f(e,\ldots,e,x_n) \geqq 0\,.$$

Similarly we show that

$$f(e,\ldots,e,x_{n-k},\ldots,x_n) \geqq f(e,\ldots,e,x_{n-k-1},x_{n-k},\ldots,x_n)$$

and hence it follows that $f(e)\geqq f(x)$. More generally it follows that $\Delta_1 f(e;\,y)\geqq 0$ for all $y\in L$. Hence $g=f/f(e)\in X(L)$ and so $X(L)$ is a base. Since $0\leqq f(y)\leqq 1$ whenever $f\in X(L)$ and $X(L)=H\cap C(L)$ it follows from Tychonoffs theorem that $X(L)$ is a closed subset of a compact, and hence compact.

Lemma 3.4. *Let $n\geqq 2$. Given a collection $x^1,\ldots,x^n\in L$ which satisfies* 3.1 (i) *and let $x^{n+1}\leqq \bigvee_{i=1}^n x^i$. Then*

$$\Delta_{n+1} f\big(\textstyle\bigvee_{i=1}^n x^i;\,x^1,\ldots,x^n,x^{n+1}\big) \geqq 0$$

whenever $f\in C(L)$.

Proof. Consider first the case that for some integer $x^{n+1}<x^i$. Direct calculations then show that

$$\Delta_{n+1} f\big(\textstyle\bigvee_{i=1}^n x^i;\,\ldots,x^n,x^{n+1}\big) = \Delta_n f\big(\textstyle\bigvee_{i=1}^n x^i;\,x^1,\ldots,x^n\big) \geqq 0\,.$$

Suppose therefore that $x^{n+1}\nless x^k$ for each k. Then by Lemma 3.1 $\bigwedge_{i=1}^n x^i\leqq x^{n+1}$. Assume $x^k\leqq x^{n+1}$ for some k, say $n=k$. Then since $x^n\wedge x^{n+1}=x^n$,

$$\begin{aligned}
&\Delta_{n+1} f\big(\textstyle\bigvee_{i=1}^n x^i;\,x^1,\ldots,x^n,x^{n+1}\big)\\
&\quad= \Delta_n f\big(\textstyle\bigvee_{i=1}^n x^i;\,x^1,\ldots,x^n\big) - \Delta_n f(x^{n+1};\,x^1,\ldots,x^n)\\
&\quad= \Delta_{n-1} f\big(\textstyle\bigvee_{i=1}^n x^i;\,x^1,\ldots,x^{n-1}\big) - \Delta_{n-1} f(x^n;\,x^1,\ldots,x^{n-1})\\
&\qquad - \Delta_{n-1} f(x^{n+1};\,x^1,\ldots,x^{n-1}) + \Delta_{n-1} f(x^n;\,x^1,\ldots,x^{n-1})\\
&\quad= \Delta_n f\big(\textstyle\bigvee_{i=1}^n x^i;\,x^1,\ldots,x^{n-1},x^{n+1}\big).
\end{aligned}$$

Hence

$$\Delta_{n+1}f\left(\bigvee_{i=1}^{n} x^i; x^1,\ldots,x^n,x^{n+1}\right) = \Delta_n f\left(\bigvee_{i=1}^{n} x^i; x^1,\ldots,x^{k-1},x^{k+1},\ldots,x^{n+1}\right) \tag{1}$$

whenever for some k, $x^k \leqq x^{n+1}$. The collection $x^1,\ldots,x^{k-1},x^{n+1},x^{k+1},\ldots,x^n$ satisfies 3.1 (i) and hence by 3.2 (i) and (1) it follows that

$$\Delta_{n+1}f\left(\bigvee_{i=1}^{n} x^i; x^1,\ldots,x^n,x^{n+1}\right) \geqq 0 \tag{2}$$

whenever $x^k \leqq x^{n+1} \leqq \bigvee_{i=1}^{n} x^i$ for some k. We define

$$z^0 = \bigvee_{i=1}^{n} x^i, \quad z^1 = (x_1,\ldots,x_{n-1},x_n{}^{n+1}),\ldots,$$
$$z^k = (x_1,\ldots,x_{n-k},x_{n-k+1}^{n+1},\ldots,x_n{}^{n+1}),\ldots, \quad z^n = x^{n+1}.$$

Then $x^n \leqq z^1 \leqq \bigvee_{i=1}^{n} x^i$ and hence by (2)

$$\Delta_{n+1}f\left(\bigvee_{i=1}^{n} x^i; x^1,\ldots,x^n,z^1\right) \geqq 0$$

that is

$$\Delta_n f\left(\bigvee_{i=1}^{n} x^i; x^1,\ldots,x^n\right) \geqq \Delta_n f(z^1; x^1,\ldots,x^n) = \Delta_n f(z^1; z^1 \wedge z^1,\ldots,x^n \wedge z^1).$$

The collection $x^1 \wedge z^1,\ldots,x^n \wedge z^1$ satisfies 3.1 (i) and

$$x^{n-1} \wedge z^1 \leqq z^2 \leqq \bigvee_{i=1}^{n} (x^i \wedge z^1) = z^1$$

and hence by (2)

$$\Delta_{n+1}f(z^1; x^1 \wedge z^1,\ldots,x^n \wedge z^1,z^2) \geqq 0$$

that is

$$\Delta_n f(z^1; x^1 \wedge z^1,\ldots,x^n \wedge z^1) \geqq \Delta_n f(z^2; x^1 \wedge z^2,\ldots,x^n \wedge z^2).$$

Similarly we show that

$$\Delta_n f(z^k; x^1 \wedge z^k,\ldots,x^n \wedge z^k) \geqq \Delta_n f(z^{k+1}; x^1 \wedge z^{k+1},\ldots,x^n \wedge z^{k+1}), \quad 0 \leqq k \leqq n-1. \tag{3}$$

These inequalities (3) give us

$$\Delta_n f\left(\bigvee_{i=1}^{n} x^i; x^1,\ldots,x^n\right) \geqq \Delta_n f(z^1; x^1 \wedge z^1,\ldots,x^n \wedge z^1) \geqq \ldots \geqq \Delta_n f(z^n; x^1 \wedge z^n,\ldots,x^n \wedge z^n) = \Delta_n f(x^{n+1}; x^1,\ldots,x^n)$$

that is

$$\Delta_{n+1}f\left(\bigvee_{i=1}^{n} x^i; x^1,\ldots,x^n,x^{n+1}\right) \geqq 0.$$

LEMMA 3.5. *If $f \in C(L)$ and g is defined by $g(y)=f(x \wedge y)$, x a fixed element of L, then $f-g \in C(L)$, that is, $f \geqq g$ in the ordering induced by the cone $C(L)$.*

PROOF. Since the case $n=1$ is trivially established let $n \geqq 2$. Given a collection $x^1,\ldots,x^n \in L$ which satisfies 3.1 (i) and $x^{n+1}=(\bigvee_{i=1}^n x^i)\wedge x$, where x is a fixed element of L. Since $\bigvee_{i=1}^n (x^i \wedge x)=x^{n+1}$ and $x^i \wedge x = x^i \wedge x^{n+1}$ it follows that

$$\begin{aligned}
&\Delta_n(f-g)\left(\bigvee_{i=1}^n x^i; x^1,\ldots,x^n\right)\\
&\quad = \Delta_n f\left(\bigvee_{i=1}^n x^i; x^1,\ldots,x^n\right) - \Delta_n f(x^{n+1}; x^1 \wedge x^{n+1},\ldots,x^n \wedge x^{n+1})\\
&\quad = \Delta_{n+1} f\left(\bigvee_{i=1}^n x^i; x^1,\ldots,x^n,x^{n+1}\right)
\end{aligned}$$

Hence by Lemma 3.4, $f-g \in C(L)$. To see that $g \in C(L)$ observe that if the collection $\{x^i\}_{i=1}^n$ satisfies 3.1 (i) then so does $\{x^i \wedge x\}_{i=1}^n$. Hence g satisfies 3.2 (i) since

$$\begin{aligned}
&\Delta_n f\left(\bigvee_{i=1}^n (x^i \wedge x); x^1 \wedge x,\ldots,x^n \wedge x\right)\\
&\qquad\qquad = \Delta_n g\left(\bigvee_{i=1}^n x^i; x^1,\ldots,x^n\right).
\end{aligned}$$

It therefore follows that $f \geqq g$.

Recall that $C_\infty(L)$ denotes the cone of CM-functions on $(L;\wedge)$ and $X_\infty(L)$ is a compact base of $C_\infty(L)$ which is an r-simplex.

THEOREM 3.6. *$C(L)$ is an extremal subcone of $C_\infty(L)$ and $X(L)$ is a closed face of $X_\infty(L)$, hence an r-simplex.*

PROOF. Let $f \in \operatorname{ext} X(L)$. By Lemma 3.5 then $f \geqq g$ where $g(y)=f(x\wedge y)$ and x is a fixed element of L. Direct calculations show that $g \in C(L)$. Since f is an extreme point, g therefore, must be a multiple of f, that is there exists an $\alpha>0$ such that $g=\alpha f$. Evaluating g at e gives $f(x)=\alpha$.

Hence $f(x\wedge y)=f(x)f(y)$ which implies that f is an exponential. Thus from [2] it follows that $\operatorname{ext} X(L) \subset \operatorname{ext} X_\infty(L)$ which implies that

$$\operatorname{co}(\operatorname{ext} X(L)) \subset \operatorname{co}(\operatorname{ext} X_\infty(L)).$$

By the Krein–Milman theorem [3]

$$X(L) = \overline{\operatorname{co}}(\operatorname{ext} X(L)), \qquad X_\infty(L) = \overline{\operatorname{co}}(\operatorname{ext} X_\infty(L))$$

which means that $X(L) \subset X_\infty(L)$ and $C(L) \subset C_\infty(L)$. Routine checking shows that $X(L)$ is a closed face of $X_\infty(L)$. Since $X(L)$ is compact and convex it follows from the Krein–Milman theorem [7] in the integral representation form that there exists a representing measure μ_f supported by $\operatorname{ext} X(L)$. But $X(L)$ is a closed face of the r-simplex $X_\infty(L)$ so that μ_f is unique. Hence $X(L)$ is an r-simplex.

REMARK. If $L_i = [a_i, b_i]$ is an interval of the extended real numbers then the collection of cumulative distribution functions [8] is a subset of $X(L)$, that is, any cumulative distribution functions is a completely monotonic function with respect to that semigroup operation. An example, due to Munroe [6] shows that the boundary condition 3.2 (ii) is essential.

4. A decomposition of C_∞ (*L*).

PROPOSITION 4.1. *Let*

$$C_n(L) = \{f : L \to \mathsf{R} \mid \Delta_k f(x^0; x^1, \ldots, x^k) \geqq 0,\ 0 \leqq k \leqq n,\ x^i \in L\}\,.$$

Then $C_n(L)$ is a closed convex cone with base

$$X_n(L) = \{f \in C_n(L) \mid f(e) = 1\}\,.$$

We omit the proof which is similar to that of 3.3.

DEFINITION 4.2. Fix an index $j \in \{0, 1, \ldots, n\}$ and $x \in L$. For each $i \leqq \binom{n}{j}$ let $x_{i,j}$ be that member of L whose coordinate values agree with the coordinate values of x in j given coordinates and are zero elsewhere. The selection of the j coordinates where agreement occurs, is the same for all x, dependent on i and distinct for distinct i. Thus i ranges over the set $1, 2, \ldots, \binom{n}{j}$. Let

$$G_{1,0} = \{f \in X_n(L) \mid f(o) = 0\}\,.$$

For each positive integer $p \leqq \binom{n}{j}$, $(j = 0, 1, \ldots, n)$, let

$$H_{p,j} = \bigcap_{i=1}^{p} \{f \in X_n(L) \mid f(x_{i,j}) = 0,\ \forall x \in L\}$$

and

$$G_{l,k} = \left(\bigcap_{j=1}^{k-1} H_{\binom{n}{j},j}\right) \cap H_{l,k}, \quad 1 \leqq k \leqq n,\ 1 \leqq l \leqq \binom{n}{k}\,.$$

Let

$$G_{0,k} = G_{\binom{n}{k-1},\,k-1}, \quad 1 \leqq k \leqq n\,,$$

$$F_{l,k} = \{f \in X_n(L) \mid f(x) = f(x_{l,k})\} \cap G_{l-1,k}$$

and $F_{1,0} = \{\text{identically 1-function}\}$.

Some properties of the sets $G_{l,k}$ and $F_{l,k}$ are as follows:

(a) $G_{l,k}, F_{l,k} \subset G_{l-1,k}$.
(b) If $x_{l,k}$ has more than $n-k$ zero coordinate values and $f \in F_{l,k}$ then $f(x)=0$.
(c) $F_{l,k} \cap G_{l,k} = \emptyset$.
(d) $F_{l,k} \cap F_{l',k'} = \emptyset$ if $(l,k) \neq (l',k')$.
(e) $G_{1,n} = \emptyset$.

LEMMA 4.3. *Let $\{G_{l,k}\}$ and $\{F_{l,k}\}$ be collections of subsets of $X_n(L)$ according to 4.2. Then $F_{l,k}$ and $G_{l,k}$ are closed, convex and extremal with respect to $G_{l-1,k}$.*

Moreover the sets $G_{l,k}$ and $F_{l,k}$ are complemented in $L_{l-1,k}$ when $0 \leqq k \leqq n$ and $1 \leqq l \leqq \binom{n}{k}$.

PROOF. The sets $F_{l,k}$ and $G_{l,k}$ are trivially closed and convex. That $G_{l,k}$ and $F_{l,k}$ are extremal in $G_{l-1,k}$ follows from the nonnegativity of $\Delta_0 f$ and $\Delta_1 f$ respectively. Direct calculations show that if

$$f \in G_{l-1,k} - G_{l,k} \cup F_{l,k}$$

then $0 < f(e_{l,k}) < 1$ because $\Delta_0 f \geqq 0$, $\Delta_1 f \geqq 0$ and $\Delta_2 f \geqq 0$. Hence

$$f(x) = \bigl(1-f(e_{l,k})\bigr) \frac{f(x)-f(x_{l,k})}{1-f(e_{l,k})} + f(e_{l,k}) \frac{f(x_{l,k})}{f(e_{l,k})}$$

which is the desired convex combination.

REMARK. We only used the properties $\Delta_0 f \geqq 0$, $\Delta_1 f \geqq 0$ and $\Delta_2 f \geqq 0$ in order to prove the above lemma.

Fix $(l,k) \in \{0,1,\ldots,\binom{n}{k}\} \times \{1,2,\ldots,n\}$ according to 4.2. If $\{m_1,\ldots,m_k\}$ is the collection of specified k indices for which the coordinate values of $x_{l,k}$ agree with x for every $x \in L$, then denote by $L_{l,k}$ the Cartesian product $\prod_{i=1}^{k} L_{m_i}$ with the usual ordering. The projection map $\Pi \colon L \to L_{l,k}$, defined as

$$\Pi(x) = (x_{m_1},\ldots,x_{m_k}) ,$$

is order preserving and surjective. For each $f \colon L \to \mathsf{R}$, define $\tilde{f} \colon L_{l,k} \to \mathsf{R}$ as

$$\tilde{f}[\Pi(x)] = f(x_{l,k}) .$$

LEMMA 4.4. *Let* $f: L \to \mathsf{R}$ *and let* $\tilde{f}$ *be defined as above, then*

(a) $\Delta_m f \geqq 0 \Rightarrow \Delta_m \tilde{f} \geqq 0 \quad (m=0,1,\ldots)$.
(b) *If* $f(x)=f(x_{l,k})$ *for every* $x \in L$ *then* $\Delta_m \tilde{f} \geqq 0 \Rightarrow \Delta_m f \geqq 0$.
(c) $F_{l,k} \subset X_\infty(L)$ *and* $F_{l,k}$ *and* $X(L_{l,k})$ *are affinely isomorphic under the map* $f \to \tilde{f}$.

PROOF. (c) The map $f \to \tilde{f}$ restricted to $F_{l,k}$ is a bijection. Let $f \in F_{l,k}$. Since $\Delta_k f \geqq 0$ it follows from (a) that $\Delta_k \tilde{f} \geqq 0$. Clearly $\tilde{f}$ satisfies 3.2 (ii) and hence by Theorem 3.6, f is a CM-function on $(L_{l,k}; \wedge)$. By (b), therefore, f is a CM-function.

THEOREM 4.5. *The collection of completely monotonic functions on* $(L; \wedge)$ *is the cone* $C_n(L)$. *Moreover for given collections* $\{G_{l,k}\}$ *and* $\{F_{l,k}\}$ *which satisfy* 4.2, *each* $f \in X_\infty(L)$ *can be written, uniquely, as a convex sum of the form*

(i) $$f = \textstyle\sum_{k=0}^{n} \left(\sum_{l=1}^{\binom{n}{k}} \alpha_{l,k} f_{l,k}\right)$$

where $f_{l,k} \in F_{l,k}$. *Thus the r-simplex* $X_\infty(L)$ *can be written as a direct convex sum of the closed pairwise disjoint faces* $\{F_{l,k}\}$.

PROOF. We only need to show that $X_n(L) \subset X_\infty(L)$. Let $f \in X_n(L)$. Since $G_{1,0}$ and $F_{1,0}$ are complemented in $X_n(L)$ by Lemma 4.3

$$f = (1-\alpha)g + \alpha f_{1,0}$$

where $0 \leqq \alpha \leqq 1$, $g \in G_{1,0}$ and $f_{1,0} \in F_{1,0}$. By the same lemma

$$g = (1-\beta)h + \beta f_{1,1}$$

where $h \in G_{1,1}$ and $f_{1,1} \in F_{1,1}$ and so on. The process stops after a finite number of steps since $G_{1,n} = \emptyset$ and by repeated substitution we obtain (i).

By Lemma 4.4 (c), each function $f_{l,k}$ is a CM-function on $(L; \wedge)$ and hence $f \in X_\infty(L)$. Thus the r-simplex $X_\infty(L)$ can be written as a direct convex sum of the collection of closed pairwise disjoint faces $\{F_{l,k}\}$. From Alfsen [1], for given $\{F_{l,k}\}$ and $f \in X_\infty(L)$ it follows that the convex sum (i) is unique.

REMARK. If $L=[0,1]\times[0,1]$ and $f(x_1,x_2)=\chi_A$ where $A=L-\{(0,0)\}$ then $\Delta_i f \geqq 0$, $i=0,1$ while

$$\Delta_2 f((1,1),(1,0),(0,1)) = -1 < 0 .$$

Thus $f \notin C_\infty(L)$. From the example it follows that at least for $n=2$ Theorem 4.5 is "the best possible". The above theorem lead to a natural decomposition of the representing measure, μ_f, supported by the filter-space

$$\mathscr{F}(L) = \{F \in \mathscr{F}(L) \mid F \text{ filter on } (L; \wedge)\}$$

[5], for given $f \in X_\infty(L)$. For given $\{F_{l,k}\}$ according to 4.2 let

$$\mathscr{F}_{l,k}(L) = \{F \in \mathscr{F}(L) \mid \chi_F \in F_{l,k}\}$$

(χ_F is the characteristic function of F). Then the collection $\{\mathscr{F}_{l,k}(L)\}$ consists of pairwise disjoint closed subsets of $\mathscr{F}(L)$. If $\mu_{l,k}$ is the representing measure of $f_{l,k}$ then the measures $\{\mu_{l,k}\}$ are mutually singular and hence

COROLLARY 4.6. *Given* $\{F_{l,k}\}$ *according to* 4.2. *If* $f \in X_\infty(L)$ *is decomposed according to* 4.5 (i) *and if* μ_f *is* f's *representing measure then*

$$\text{(i)} \qquad \mu_f = \sum_{k=0}^{n} \left(\sum_{l=1}^{\binom{n}{k}} \alpha_{l,k}\mu_{l,k}\right)$$

where $\mu_{l,k}$ *is* $f_{l,k}$'s *representing measure. For given* $\{F_{l,k}\}$ *the measures* $\{\mu_{l,k}\}$ *are mutually singular and the convex sum is unique.*

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