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THE DUAL BALL OF A LINDENSTRAUSS SPACE

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1. Introduction.

We call a Banach space a *Lindenstrauss space* if its dual is an $L^1(\mu)$ space for some measure μ. This class was introduced by Grothendieck and was studied extensively by Lindenstrauss [14]. If the unit ball of a Lindenstrauss space X contains an extreme point, then X is (isometric to) the space of continuous affine functions on a Choquet simplex. The unit ball of the dual X^* of a Lindenstrauss space will be called an *L-ball.* Lazar [13] has proved a characterization of L-balls similar to the Choquet–Meyer uniqueness theorem for simplexes, which is the basis of much what we will do.

Lazar and Lindenstrauss [12] have generalized the Edwards extension theorem on Choquet simplexes to L-balls. In section 2, we prove that the result actually characterizes L-balls. We also obtain some other equivalent conditions and collaries.

Namioka and Phelps [16] proved that a compact convex set in a locally convex space is a Choquet simplex if and only if there exists an affine map φ from K to the set of probability measures on K such that for $x \in K$, the resultant of $\varphi(x)$ is x. (Fakhoury [5] has given a simpler proof.) This suggests the problem of whether there exists a similar criterion for a compact absolutely convex set K to be an L-ball. In [3], we prove that this is the case. (This result has been obtained independently by Lacey [11].)

In [4], we present some characterizations of those L-balls K for which the extreme points $\partial_e K$ union $\{0\}$ is closed.

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2. Extension theorems for L-ball.

Throughout this paper, we shall use the following notations. Suppose K is a compact absolutely convex (symmetric about 0) subset of a locally

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convex space. We let $C(K)$ denote the space of real valued continuous functions on K. A real valued function f on K is called *affine* (*convex*) if

$$f(\lambda x+(1-\lambda)y) = (\leqq)\, \lambda f(x)+(1-\lambda)f(y), \quad x,y \in K,\ \lambda \in [0,1].$$

We let $A(K)$ $(Q(K))$ denote the set of continuous affine (convex) functions on K and we let $A_0(K)$ be the set of all functions f in $A(K)$ such that $f(0)=0$. We call such functions *affine symmetric*. A function f is called *concave* if $-f$ is convex. Suppose f is a bounded real valued function on K, we define the *upper envelope* $\hat{f}$ of f by

$$\hat{f}(x) = \inf\{a(x) : a \in A(K), a \geqq f\}, \quad x \in K.$$

Note that $\hat{f}$ is an upper semicontinuous concave function on K. For any real valued function on K we define the functions σf, $\operatorname{odd} f$ and $\operatorname{even} f$ as follows:

$$\begin{aligned} \sigma f(x) &= f(-x), \quad x \in K, \\ \operatorname{odd} f &= \tfrac{1}{2}(f-\sigma f), \\ \operatorname{even} f &= \tfrac{1}{2}(f+\sigma f). \end{aligned}$$

A function is called *odd* if $f=\operatorname{odd} f$.

We will let $M_1(K)$ be the set of regular Borel measures on K with norm at most 1 and let $P(K)$ be the set of probability measures on K. For $\mu \in M_1(K)$, we use $\operatorname{supp}\mu$ to denote the support of μ and μ_A for the restriction of μ to a Borel subset A in K. The measure μ is called a *boundary measure* on K if

$$\mu(\hat{f}-f) = 0, \quad f \in C(K).$$

The *resultant* of μ is the point $r(\mu) \in K$ satisfying

$$f(r(\mu)) = \mu(f) \ (= \textstyle\int f d\mu), \quad f \in A_0(K).$$

The point $r(\mu)$ is also called the *barycenter* of μ. For a finite signed Borel measure on K, we define $\sigma\mu$ and $\operatorname{odd}\mu$ as follows

$$\begin{aligned} \sigma\mu(f) &= \mu(\sigma f), \quad f \in C(K), \\ \operatorname{odd}\mu &= \tfrac{1}{2}(\mu-\sigma\mu). \end{aligned}$$

A measure μ is called *odd* if $\mu=\operatorname{odd}\mu$.

DEFINITION 2.1. A Banach space X is called a *Lindenstrauss space* if X^* is isometric to an $L^1(\mu)$ space for some measure μ. A compact absolutely convex subset K in a locally convex space is called an *L-ball* if K is affinely homeomorphic to the unit ball of the dual of a Lindenstrauss space in its weak* topology.

Much of what we do is based on the following theorem of Lazar [13].

THEOREM 2.2. *Let K be a compact absolutely convex subset of a locally convex space; then the following are equivalent:*

(i) *K is an L-ball.*

(ii) *For any continuous convex function f on K,* $\mathrm{odd}\hat{f}$ *is an affine function.*

(iii) *If μ is a boundary measure on K which represents the point x in K and if f is a continuous convex function on K, then*

$$\mathrm{odd}\hat{f}(x) = \mu(\mathrm{odd} f)$$

(iv) *If μ_1 and μ_2 are boundary measures on K having the same barycenter, then*

$$\mathrm{odd}\,\mu_1 = \mathrm{odd}\,\mu_2$$

(v) *For each continuous convex function f on K,*

$$\begin{aligned}\hat{f}(0) &= \tfrac{1}{2}\sup\{f(x)+f(-x) :\ x \in K\}\\ &= \sup\{\mathrm{even} f(x) :\ x \in K\}\,.\end{aligned}$$

Suppose K is a compact absolutely convex subset of a locally convex space, a subset H of K is called a *biface* if $H = \mathrm{conv}(F \cup -F)$ where F is a face in K. It was proved in [12] that if K is an L-ball and H is a closed biface of K, then H is also an L-ball. Our main object in this section is to prove the following theorem.

THEOREM 2.3. *Let K be a compact absolutely convex subset of a locally convex space, the following properties are equivalent:*

(i) *K is an L-ball.*

(ii) *Suppose that H is a closed biface H of K and that h is a continuous affine symmetric function on H. If f is a lower semicontinuous concave function on K such that*

$$\mathrm{even} f \geqq 0 \ \textit{on}\ K \ \textit{and}\ f \geqq h \ \textit{on}\ H\,.$$

then there exists an $\bar{h} \in A_0(K)$ such that $\bar{h}$ extends h and $f \geqq \bar{h}$ on K.

(iii) *For any lower semicontinuous concave function f on K such that* $\mathrm{even} f \geqq 0$, *there exists a continuous affine symmetric function h such that $h \leqq f$.*

(iv) *Let f be a lower semicontinuous concave function and g is an upper semicontinuous convex function on K such that*

$$\sup\{\mathrm{even}\, g(x) : x \in K\} \leqq \inf\{\mathrm{even}\, f(x) : x \in K\}$$

and

$$g \leqq f .$$

Then there exists a continuous affine function h on K such that $g \leqq h \leqq f$.

REMARK. The fact that (i) implies (ii) has been proved by Lazar–Lindenstrauss [12]. Also, (ii) implies (iii) has been obtained both by Fahkoury [6] and the author independently.

PROOF. (i) implies (ii). Cf. [12].

(ii) implies (iii). Let x_0 be an extreme point of K, let $H=[-x_0,x_0]$ and let h_1 be a continuous affine symmetric function on H such that $h_1 \leqq f$ on H. Condition (ii) implies that there exists a continuous affine extension h on K such that $h \leqq f$.

(iii) implies (iv). Without loss of generality, we assume that

$$\mathrm{even}\, g \leqq 0 \leqq \mathrm{even}\, f .$$

Let $k(x)=\min\{f(x), -g(-x)\}$, $x \in K$; then k is a lower semicontinuous concave function on K and it is easily checked that even $k \geqq 0$. By hypothesis, there exists a continuous affine symmetric function h on K such that $h \leqq k$ and therefore $g \leqq h \leqq f$.

(iv) implies (i). We will apply Theorem 2.2 (v) $\Rightarrow$ (i) by showing that for each continuous convex function g on K,

$$\hat{g}(0) = \sup\{\mathrm{even}\, g(x) : x \in K\} .$$

Indeed, let $a=\hat{g}(0)$ and let $b=\sup\{\mathrm{even}\, g(x) : x \in K\}$. Since

$$\hat{g}(0) = \sup\{\mu(g) : \mu \in P(K),\ \mu \text{ represents } 0\} ,$$

it is obvious that $a \geqq b$. Next, we define the function f by

$$f = -\sigma g + (a+b);$$

then f is continuous and concave. Since

$$a+b \geqq 2b \geqq 2 \text{ even } g = g + \sigma g ,$$

we have

$$-\sigma g + (a+b) \geqq g ,$$

that is,

$$f \geqq g .$$

For any x,y in K, we have

$$\begin{aligned} 2\,\mathrm{even}f(x) &= [-\sigma g(x)+(a+b)]+[-g(x)+(a+b)] \\ &= -2\,\mathrm{even}\,g(x)+2(a+b) \geqq a+b \geqq g(y)+g(-y) \\ &= 2\,\mathrm{even}\,g(y)\,, \end{aligned}$$

that is,

$$\sup\{\mathrm{even}\,g(x) : x \in K\} \leqq \inf\{\mathrm{even}f(x) : x \in K\}\,.$$

Thus by (iv), there exists a continuous affine function h on K such that $g \leqq h \leqq f$. Since $a = \hat{g}(0)$, and

$$\hat{g}(0) = \inf\{k(0) : k \in A(K),\ k \geqq g\}$$

we have

$$\begin{aligned} a \leqq h(0) &\leqq \inf\{\mathrm{even}f(x) : x \in K\} \\ &= \inf\{-\mathrm{even}\,g(x) : x \in K\}+(a+b) \\ &= -\sup\{\mathrm{even}\,g(x) : x \in K\}+(a+b) \\ &= -b+(a+b) = a\,, \end{aligned}$$

hence $h(0)=a$. Define $h' = -\sigma h+(a+b)$, so that $h'(0)=b$. Since

$$h \leqq f \quad \text{implies} \quad g \leqq -\sigma h+(a+b) = h'\,,$$

we have

$$a = \hat{g}(0) = \inf\{k(x) : k \in A(K),\ k \geqq g\} \leqq b\,.$$

This shows that $a=b$ and theorem 2.2 (v) applies.

COROLLARY 2.4. *Let F be a proper closed face of an L-ball K. Suppose f is a lower semicontinuous concave function and g is an upper semicontinuous convex function on F such that $g \leqq f$. Then there exists a continuous affine function h on K such that $g \leqq h \leqq f$ on F.*

PROOF. Since f is lower semicontinuous and F is compact, it attains its minimum. We choose a lower bound a of f less than zero. Similarly, g attains its maximum and we let $b>0$ be an upper bound of g. Define f' and g' on K as

$$f'(x) = \begin{cases} f(x) \wedge b, & x \in F \\ b & x \notin F\,. \end{cases}$$

$$g'(x) = \begin{cases} g(x) \vee a, & x \in F \\ a & x \notin F\,. \end{cases}$$

It is easily seen that f' is a lower semicontinuous concave function and g' is an upper semicontinuous convex function. Furthermore, for any $x,y \in K$, even $f'(x)$ equals either b or $\frac{1}{2}(b+f(-x))$ or $\frac{1}{2}(f(x)+b)$ and $\mathrm{even}\,g'(y)$ equals either a or $\frac{1}{2}(a+g(-y))$ or $\frac{1}{2}(g(y)+a)$. In any case,

$$\text{even}\, g'(y) \leqq \text{even}\, f'(x)\ ,$$

that is,

$$\sup\{g'(x)+g'(-x) : x \in K\} \leqq \inf\{f'(x)+f'(-x) : x \in K\}\ .$$

By the above theorem, we can find an affine continuous function h on K such that $g' \leqq h \leqq f'$ hence $g \leqq h \leqq f$ on F.

As a direct consequence of the above corollary, we obtain the following result.

PROPOSITION 2.5. *Let F be a closed face of an L-ball K. If h is a continuous affine function on F, then h has a continuous affine norm preserving extension $\bar{h}$ on K.*

PROOF. We define the functions f', g' on K such that

$$f'(x) = \begin{cases} \|h\|, & x \notin F \\ h(x), & x \in F\ , \end{cases}$$

$$g'(x) = \begin{cases} -\|h\|, & x \notin F \\ h(x), & x \in F\ . \end{cases}$$

It follows from the proof of the above corollary that there exists a continuous affine function $\bar{h}$ on K such that $g' \leqq \bar{h} \leqq f'$, hence $\bar{h}$ is a norm preserving extension of h.

We conclude this section by giving a necessary and sufficient condition for a subset of an L-ball to be a peak face. The analogous result for Choquet simplexes is well-known.

LEMMA 2.6. *Let F be a proper closed face of an L-ball K. Suppose h is a continuous affine function on K such that $h \geqq 0$ on F. Then there exists a continuous affine function k on K such that $k \geqq 0$, $k \geqq h$ and $k/F = h/F$.*

PROOF. We define f, g on K by

$$f(x) = \begin{cases} h(x), & x \in F \\ 2\|h\|, & x \notin F\ , \end{cases}$$

$$g(x) = h(x) \vee 0, \quad x \in K\ .$$

It is clear that f is a lower semicontinuous concave function and g is an upper semicontinuous convex function such that $g \leqq f$. Also, we observe that $\operatorname{even} f$ equals either $2\|h\|$, $\frac{1}{2}(h(y)+2\|h\|)$, or $\frac{1}{2}(h(-y)+2\|h\|)$, while $\operatorname{even} g$ equals either 0, $\frac{1}{2}h(x)$, $\frac{1}{2}h(-x)$ or $\operatorname{even} h$. In any case, we have

$$\operatorname{even} g(x) \leqq \operatorname{even} f(y) \quad \text{for any } x,y \in K .$$

By Theorem 2.3(iv), we have a continuous affine function k on K such that $g \leqq k \leqq f$. Note that $h/F = k/F$ and $k \geqq g = h \vee 0$, hence k is the required function.

PROPOSITION 2.7. *Let K be an L-ball and F be a subset of K. Then F is a G_δ closed face of K if and only if there exists a continuous affine function h on K such that $h(x)=0$ for $x \in F$ and $h(x)>0$ for $x \in K \setminus F$.*

PROOF. We clearly only need to prove the sufficiency. For each x which is not in F, there exists a continuous affine function h on K such that $h(x)<0$ and $h \geqq 0$ on F. By the above lemma, there exists a nonnegative continuous affine function k on K such that $h/F = k/F$ and $k \geqq h$. Hence we have shown that for each x not in F, there exists a continuous affine function (that is, $k-h$) which is identically zero on F, positive at x and nonnegative on K.

Now since F is a G_δ set, it is the intersection of a sequence of open sets $\{U_n\}_{n=1}^{\infty}$. For each n, and for each $x \in K \setminus U_n$, we can find a continuous affine function h_x such that $h_x = 0$ on F and $h_x(x) > 0$. By compactness of $K \setminus U_n$, we can find $x_1,\ldots,x_k \in U_n$ such that

$$\bigcup_{i=1}^{k} \{x \in K : \ h_{x_i}(x) > 0\} \supseteqq K \setminus U_n .$$

Let $h_n = \sum_{i=1}^{k} h_{x_i}$, and let

$$h = \sum_{n=1}^{\infty} 2^{-n} h_n / \sup |h_n(K)|$$

then h is a continuous affine function on K such that $h=0$ on F and $h>0$ on $K \setminus F$.

3. Another characterization of L-balls.

In [16], Namioka and Phelps first studied "simplex-like" compact convex sets, that is, those compact convex sets K which admits an affine map φ from K to the set of probability measures such that $r(\varphi(x))=x$. They proved that those sets are just the Choquet simplexes. An elegant proof has been given by Fakhoury [5], who also proved the analogous result for L-balls under certain restrictions [8]. Lacey [11] obtains the same result as here for $\mathscr{L}_\infty$ spaces.

DEFINITION 3.1. Let K be a compact absolutely convex subset of a locally convex space. A map φ from K to $M_1(K)$ is called an *affine selection* on K if φ is an affine function and $r(\varphi(x))=x$, $x \in K$. An affine selection is called *odd* if $\varphi(x)=\text{odd}\,\varphi(x)$, $x \in K$.

DEFINITION 3.2. A Banach space X is called a $\mathscr{P}_\lambda$ *space*, $\lambda \in \mathbb{R}$ if for any bounded linear operator T from X to a Banach space Y and any Banach space Z containing X, the operator T admits a linear extension T from Z to Y such that $\|\tilde{T}\| \leqq \lambda\|T\|$.

It is known that if Z is a $\mathscr{P}_\lambda$ space, X is a subspace of Z and there exists a projection of norm one form Z onto X, then X is also a $\mathscr{P}_\lambda$ space [2, p. 94]. In [9], Grothendieck proved the following theorem:

THEOREM 3.3. *A Banach space X is a Lindenstrauss space if and only if X^{**} is a $\mathscr{P}_1$ space.*

Let K be a compact absolutely convex set, which, as is known, is affinely homeomorphic to the unit ball of $A_0(K)^*$ under the weak*-topology. We will consider the natural injection $i: A_0(K) \to C(K)$ and the adjoint maps

$$i^*: C(K)^* \to A_0(K)^* \quad \text{and} \quad i^{**}: A_0(K)^{**} \to C(K)^{**}.$$

THEOREM 3.4. *A compact absolutely convex subset K is an L-ball if and only if there exists an affine selection φ on K to $M_1(K)$.*

Moreover, if such a selection exists, then there is a unique odd affine selection φ and $\varphi(x)$ is a boundary measure for each x in K.

PROOF. *Necessity.* Define $\varphi: K \to M_1(K)$ as $\varphi(x)=\text{odd}\,\mu_x$ where μ_x is a boundary probability measure representing x. Theorem 2.2(iv) shows that this is a well defined map. Let f be a continuous convex function; by Theorem 2.2(ii), we know that odd$\hat{f}$ is an affine function, and by (iii), $(\varphi(x))(f)=\text{odd}\hat{f}(x)$. Hence we have

$$[\varphi(\lambda x+(1-\lambda)y)](f) = [\lambda\varphi(x)+(1-\lambda)\varphi(y)](f)$$

where $0 \leqq \lambda \leqq 1$, $x,y \in K$. The same equality holds for continuous concave function, hence holds for all functions in $C(K)$. Thus

$$\varphi(\lambda x+(1-\lambda)y) = \lambda\varphi(x)+(1-\lambda)\varphi(y), \quad 0 \leqq \lambda \leqq 1,\ x,y \in K.$$

Sufficiency. If φ is an affine selection, then $x \to \frac{1}{2}(\varphi(x)-\varphi(-x))$ is also an affine selection which is 0 at 0. We assume that φ has this property.

Define

$$\bar{\varphi}\colon A_0(K)^* \to C(K)^*$$

by

$$\bar{\varphi}(\alpha x) = \alpha\varphi(x) \quad \text{where } \alpha \in \mathsf{R},\ x \in K\ .$$

The map $\bar{\varphi}$ is easily seen to be linear, and it has norm 1 since it carries the unit ball K of $A_0(K)^*$ into the unit ball $M_1(K)$ of $C(K)^*$. Its adjoint $\bar{\varphi}^*$ maps $C(K)^{**}$ into $A_0(K)^{**}$ and has norm 1.

Consider $A_0(K)^*$ as a subspace of $C(K)^*$ under the injection $\bar{\varphi}$, the hypothesis on φ shows that i^* is a projection from $C(K)^*$ onto $A_0(K)^*$, hence if we let $A_0(K)^{**}$ to be a subspace of $C(K)^{**}$ under the injection i^{**}, it is easily checked that $\bar{\varphi}^*$ is a projection of norm one from $C(K)^{**}$ onto $A_0(K)^{**}$. By the remark on Definition 3.2 and by Theorem 3.3, we conclude that $A_0(K)$ is a Lindenstrauss space. Thus, K is an L-ball.

The last assertion follows from [7, Lemma 5].

COROLLARY 3.5. (Namioka–Phelps-Fakhoury.) *A compact convex set K is a Choquet simplex if and only if there exists an affine selection φ from K to the set of probability measures on K.*

Moreover, if such a selection exists, then it is unique and $\varphi(x)$ is a boundary measure for each x in K.

PROOF. Let φ be the selection from K to the set of probability measures $P(K)$. Embed K into $A(K)^*$ and let $H = \operatorname{conv}(K \cup -K)$; then H is a compact absolutely convex set. Define $\bar{\varphi}\colon H \to M_1(H)$ by

$$\bar{\varphi}(\lambda x - (1-\lambda)y) = \lambda\varphi(x) - (1-\lambda)\varphi(y)\ ,$$

where $\lambda \in [0,1]$. This map is well defined and affine. By Theorem 3.4, we conclude that H is an L-ball. Since K is a maximal face and is compact, it follows that K is a Choquet simplex.

To prove the last assertion, we see that $\bar{\varphi}$ is odd. If there exists another selection ψ, then $\bar{\psi}$ will be odd and $\bar{\varphi} = \bar{\psi}$, this implies $\varphi = \psi$. That each $\varphi(x)$ is a boundary measure follows easily from Theorem 3.4.

4. L-balls with $\partial_e K \cup \{0\}$ closed.

Our aim in this section is to give some characterizations of L-balls for which union of $\{0\}$ and the set of extreme points $\partial_e K$ is closed. Such sets can be considered as generalizations of the Bauer simplexes. We will first prove several lemmas.

LEMMA 4.1. *Let K be a compact absolutely convex set and consider K as the unit ball of $A_0(K)^*$. Then*

(i) *Suppose x,y are in K, that $\|x\|=1$ and that $y \in [0, -x]$. Then for any $1>\varepsilon>0$, there exists a function f in $A(K)$ which satisfies $\|f\|=1, f(x)>1-\varepsilon$ and $f(y)=0$.*

(ii) *Suppose that x is in K, that $\|x\|=1$ and that there exist y,z in K with $x=\frac{1}{2}(y+z)$. Then for $1>\varepsilon>0$, there exists a function f in $A_0(K)$ satisfying $\|f\|=1$ and $f(y), f(z), f(x)>1-\varepsilon$.*

PROOF. (i) For $1>\varepsilon>0$, take $\varepsilon'=\varepsilon(1-\varepsilon)^{-1}$, then $(1+\varepsilon')x \notin K$. By the separation theorem, there exists a function f_1 in $A_0(K)$ such that $\|f_1\|=1$ and

$$\sup\{f_1(z) : z \in K\} = 1 < f_1((1+\varepsilon')x) .$$

Hence we have $f_1(x)>(1+\varepsilon')^{-1}=1-\varepsilon>0$. Since $y \in [0, -x]$, we have $f_1(y)<0$. Let $-c=f_1(y)$ and let

$$f = (1+c)^{-1}(f_1+c) .$$

Then $\|f\|=1, f(y)=0$ and $f(x)>(1+c)^{-1}(1-\varepsilon+c)>1-\varepsilon$.

We omit the proof of (ii) since it is similar to (i).

LEMMA 4.2. *If f_1,f_2 are two continuous convex functions on a compact convex set K such that $f_1=f_2$ on $\partial_e K$, then $\hat{f}_1=\hat{f}_2$.*

PROOF. We need only show that if f is a continuous convex function on K and if for each x in K, we let

$$h(x) = \inf\{a(x) : a \in A(K), a \geqq f \text{ on } \partial_e K\} ,$$

then $\hat{f}=h$. It is easily seen that $h \leqq \hat{f}$. To show the other inequality, let $a \in A(K)$ be such that $a \geqq h$ on $\partial_e K$ so that $a-f$ is a continuous concave function and $a-f \geqq 0$ on $\partial_e K$. By the minimum principle, $a-f$ attains its minimum on the extreme points of K, so $a-f \geqq 0$ on K. It follows that $h \geqq f$.

LEMMA 4.3. *Let K be a compact absolutely convex set with $\partial_e K \cup \{0\}$ closed. If μ is a boundary measure on K, then* oddμ *is supported by $\partial_e K$.*

PROOF. If μ is a boundary measure on K, then μ is supported by $\overline{\partial_e K}$. But $\overline{\partial_e K} \subseteqq \partial_e K \cup \{0\}$ and

$$\text{odd}\mu(\{0\}) = \tfrac{1}{2}(\mu\{0\}-\mu\{0\}) = 0 .$$

This shows that μ is supported by $\partial_e K$.

THEOREM 4.4. *If K is a compact absolutely convex subset of a locally convex space, then the following are equivalent:*

(i) *K is an L-ball with $\partial_e K \cup \{0\}$ closed*

(ii) *There exists an affine selection φ from K to the weak* compact set $M_1(K)$ such that φ is continuous.*

(iii) *The function* odd$\hat{f}$ *is in $A_0(K)$ for each continuous convex function f on K.*

(iv) *Every continuous function f on $\partial_e K \cup \{0\}$ to* R *such that $f(x) = -f(-x)$ can be extended to a function in $A_0(K)$. Furthermore, the extension is norm preserving.*

PROOF. We will prove the theorem in the following two cycles. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (ii). From the proof of Theorem 3.4 we know that $\varphi(x) = \mathrm{odd}\,\mu_x$ is an affine map from K to $M_1(K)$, where μ_x is a boundary measure representing x. We claim that $\varphi(K)$ is a closed subset in $M_1(K)$. Let $\{\varphi(x_\alpha)\}$ be a net in $\varphi(K)$ converging to a measure ω in $M_1(K)$. We first observe that ω is an odd measure. Indeed, for any continuous function f on K, we have

$$\omega(f) = \lim_\alpha \varphi(x_\alpha)(f) = -\lim_\alpha \varphi(x_\alpha)(\sigma f) = -\omega(\sigma f) .$$

If ω is not supported by $\partial_e K \cup \{0\}$, then there exists a compact subset C in K such that C is disjoint from $\partial_e K \cup \{0\}$ and $\omega(C) > 0$ (or $\omega(C) < 0$). Hence we can find a continuous function f on K such that f is 0 on $\partial_e K \cup \{0\}$ and f is 1 on C; since by Lemma 4.3, $\varphi(x_\alpha)$ is supported by $\partial_e K$, we have

$$0 = \lim_\alpha \varphi(x_\alpha)(f) \neq \omega(f) > 0 .$$

This contradiction shows that ω is supported by $\partial_e K \cup \{0\}$. Since ω is an odd measure, $\omega(\{0\}) = \mathrm{odd}\,\omega(\{0\}) = 0$ (as in Lemma 4.3), so it is in fact supported by $\partial_e K$. Thus, ω is an odd boundary measure. Let x be the resultant of ω. By the uniqueness of the odd boundary measure representation, we have $\omega = \varphi(x)$, which shows that $\varphi(K)$ is closed and thus compact.

We already know that the resultant map $\mu \to r(\mu)$ from $M_1(K)$ onto K is continuous [18, p. 4], hence the map from $\varphi(K)$ onto K such that $\mu \to r(\mu)$ is continuous. Since the map is a continuous bijection from a compact set onto a compact set, the inverse map is continuous.

(ii) $\Rightarrow$ (iii). Without loss of generality, we assume that $\varphi(x) = \mathrm{odd}\,\mu_x$ where μ_x is a boundary measure representing x. By Theorem 2.2 (iii), we have $\mathrm{odd}\hat{f}(x) = \varphi(x)(f)$ for each convex continuous function f. Since the map φ is continuous affine and $\varphi(0)=0$, it is easily seen that odd$\hat{f}$ is a continuous affine symmetric function on K.

(iii) $\Rightarrow$ (i). By Theorem 2.2 (iii), we need only show that $\partial_e K \cup \{0\}$ is closed. For this we will show

$$\partial_e K \cup \{0\} = \bigcap_{f\in Q(K)} \{x: \operatorname{odd}\hat{f}(x) = \operatorname{odd} f(x)\}$$

where $Q(K)$ is the set of continuous convex functions on K. It is known that $\hat{f}=f$ on $\partial_e K$ for any continuous function f. Thus

$$\partial_e K \cup \{0\} \subseteq \bigcap_{f\in Q(K)} \{x: \operatorname{odd}\hat{f}(x) = \operatorname{odd} f(x)\}.$$

To show the reverse inclusion, we let $x \notin \partial_e K \cup \{0\}$ and consider the following two cases.

Case I. $\|x\| < 1$. Let $y = \|x\|^{-1}x$ and $z = -\|x\|^{-1}x$. Then

$$x = \lambda y + (1-\lambda)z \quad \text{where } \lambda = \tfrac{1}{2}(1+\|x\|).$$

By Lemma 4.1 (i), there exists an a in $A(K)$ such that $\|a\|=1$, $a(z) > \frac{1}{2}(1+\|x\|)$ and $a(x)=0$. Let $f = a \vee 0$; we then have

$$\operatorname{odd} f(x) = \tfrac{1}{2}(-a(-x)) = -\|x\|a(z)/(1+\|x\|) < -\tfrac{1}{2}\|x\|.$$

On the other hand, since $\|f\| \leqq 1$, we have $\hat{f}(z) \leqq 1$, and $\hat{f}(y) \geqq 0$. By linearity of odd$\hat{f}$, we have

$$\begin{aligned}
\operatorname{odd}\hat{f}(x) &= \lambda \operatorname{odd}\hat{f}(y) + (1-\lambda)\operatorname{odd}\hat{f}(z) \\
&= \tfrac{1}{2}(1+\|x\|)\tfrac{1}{2}(\hat{f}(y)-\hat{f}(-y)) + \tfrac{1}{2}(1-\|x\|)\tfrac{1}{2}(\hat{f}(z)-\hat{f}(-z)) \\
&= \tfrac{1}{4}(1+\|x\|)(\hat{f}(y)-\hat{f}(z)) + \tfrac{1}{4}(1-\|x\|)(\hat{f}(z)-\hat{f}(y)) \\
&= \tfrac{1}{2}(\|x\|\hat{f}(y) - \|x\|\hat{f}(z)) \geqq -\tfrac{1}{2}\|x\| > \operatorname{odd} f(x).
\end{aligned}$$

Hence $x \notin \bigcap_{f\in Q(K)}\{x: \operatorname{odd}\hat{f}(x) = \operatorname{odd} f(x)\}$.

Case II. $\|x\|=1$. Since x is not an extreme point, there exist y,z in K such that $x=\frac{1}{2}(y+z)$, where $\|y\|=\|z\|=1$, and $y,z \neq x$. We can find a_1, a_2 in $A(K)$ such that

$$a_1(-y) > 0, \quad a_1(-x) = 0, \quad a_1(-z) < 0, \quad \|a_1\| < \tfrac{1}{2}$$

and

$$a_2(-y) < 0, \quad a_2(-x) = 0, \quad a_2(-z) > 0, \quad \|a_2\| < \tfrac{1}{2}.$$

Choose ε such that $0 < \varepsilon < \min\{a_1(-y), a_2(-z)\} < \frac{1}{2}$. By Lemma 4.1 (ii), there exists $a_3 \in A_0(K)$ such that

$$a_3(x), a_3(y), a_3(z) > 1-\varepsilon > \tfrac{1}{2}.$$

Let $f = a_1 \vee a_2 \vee a_3$, then f is continuous convex function and each of $f(x)$, $f(y)$, $f(z)$ is greater than $1-\varepsilon$ and

$$f(-x) = 0, \quad f(-z) = a_2(-z) > \varepsilon, \quad f(-y) = a_1(-y) > \varepsilon .$$

We thus have

$$\operatorname{odd} f(x) = \tfrac{1}{2}(f(x)-f(-x)) = \tfrac{1}{2}f(x) > \tfrac{1}{2}(1-\varepsilon) ,$$

and since $\operatorname{odd} f$ is affine

$$\begin{aligned}\operatorname{odd}\hat{f}(x) &= \tfrac{1}{2}[\operatorname{odd}\hat{f}(y)+\operatorname{odd}\hat{f}(z)] \\ &= \tfrac{1}{4}[\hat{f}(y)-\hat{f}(-y)+\hat{f}(z)-\hat{f}(-z)] \\ &\leqq \tfrac{1}{4}[2-\hat{f}(-y)-\hat{f}(-z)] \\ &< \tfrac{1}{4}(2-2\varepsilon) = \tfrac{1}{2}(1-\varepsilon) < \operatorname{odd} f(x) .\end{aligned}$$

Hence $x \notin \bigcap_{f\in Q(K)}\{x : \operatorname{odd}\hat{f}(x) = \operatorname{odd} f(x)\}$.

We conclude that the two sets are equal and since $\operatorname{odd}\hat{f}$ and $\operatorname{odd} f$ are both continuous, the set

$$\bigcap\nolimits_{f\in Q(K)} \{x : \operatorname{odd}\hat{f}(x) = \operatorname{odd} f\}$$

is closed. That is $\partial_e K \cup \{0\}$ is closed.

(ii) $\Rightarrow$ (iv). We may assume that $\varphi(x) = \operatorname{odd}\mu_x$, where μ_x is a boundary measure representing x. Suppose f is a continuous function on $\partial_e K \cup \{0\}$ with $f(x) = -f(-x)$, define $\bar{f}$ on K by $\bar{f}(x) = \varphi(x)(f)$. Since $\varphi(x)$ is an odd boundary measure, it is supported by $\partial_e K$ (Lemma 4.3) and the map $\bar{f}$ is well defined. If $x \in \partial_e K \cup \{0\}$, then $\varphi(x)(f) = f(x)$ and we see that $\bar{f}$ is an affine extension of f. To show that $\bar{f}$ is continuous, we need only observe that $x \to \varphi(x)$ and $\varphi(x) \to \varphi(x)(f)$ are continuous for $x \in K$ (the last map is continuous since f is defined on the closed set $\partial_e K \cup \{0\}$ and can be extended to a continuous function on K).

(iv) $\Rightarrow$ (iii). Let f be a continuous convex function on K and consider $\operatorname{odd} f$ restricted to $\partial_e K \cup \{0\}$. By (iv), there exists a continuous affine function $\bar{f}$ such that $\bar{f} = \operatorname{odd} f$ on $\partial_e K \cup \{0\}$. We want to show that $\bar{f}$ equals $\operatorname{odd}\hat{f}$. Let

$$g(x) = 2\bar{f}(x)+f(-x), \quad x \in K .$$

Then $f(x) = g(x)$, $x \in \partial_e K$ and by Lemma 4.2, we have $\hat{f} = \hat{g}$ on K. Since $\bar{f}$ is affine and continuous,

$$\hat{g}(x) = 2\bar{f}(x)+\hat{f}(-x), \quad x \in K$$

[18, p. 19], and thus we have $\bar{f}(x) = \operatorname{odd}\hat{f}$.

COROLLARY 4.5. *Let K be an L-ball and let J be a compact subset of $\partial_e K \cup \{0\}$. Suppose f is a continuous function on J such that $f(x) = f(-x)$ whenever x, $-x$, are in J. Then there exists an extension $\bar{f}$ of f in $A_0(K)$ such that $\|\bar{f}\| = \|f\|$.*

PROOF. Let $H=\overline{\text{conv}}(J\cup -J)$; we first show that H is an L-ball. Let μ be an odd boundary measure on H representing $x\in H$; then μ is supported by the extreme points $(J\cup -J)\subseteqq\partial_e K$ and hence μ is an odd boundary measure on K. By the uniqueness of odd boundary measure representing the points in K, and thus on H, we conclude that H is an L-ball by Theorem 2.2. Extend f from J to a function f_1 on $J\cup -J=\partial_e H \cup\{0\}$ by

$$f_1(x) = \begin{cases} f(x) & x\in J\,, \\ -f(-x) & x\in -J\,. \end{cases}$$

Then f_1 is a continuous function on $\partial_e H\cup\{0\}$ and by Theorem 4.4 (iv), f_1 can be extended to a function $\bar{f}_1$ in $A_0(H)$ such that $\|f_1\|=\|\bar{f}_1\|$. By Theorem 2.3 (ii), (letting $g=-\|f\|$), we can extend $\bar{f}_1$ to $\bar{f}$ in $A_0(K)$ such that $\|\bar{f}\|=\|f\|$.

Let K be a compact Hausdorff space. By a $C_\sigma(K)$-space, we mean the set of continuous functions on K such that $f\circ\sigma=-f$, where $\sigma\colon K\to K$ is a homeomorphism satisfying $\sigma^2(x)=x$ for all x in K. We call a space $C_\Sigma(K)$-space if it is a $C_\sigma(K)$ space and σ does not have any fixed point. We conclude this section by giving two propositions concerning such spaces. The first one was suggested by Effros [4] and has been independently proved by Fakhoury [8]. The second proposition is due to Lindenstrauss and Wulbert [15]. Both of them are corollaries of Theorem 4.4. In the proof, we make use of a well-known fact [10], namely, if $X=C_\sigma(K)$, then the set of extreme points of the unit ball of X^* are the evaluation functions $\hat{x}$ at those points of K for which $\sigma(x)\neq x$. We denote the unit ball of a Banach space X by $B(X)$.

PROPOSITION 4.6. *Let X be a Lindenstrauss space. Then X is a C_σ space if and only if $\partial_e(B(X^*))\cup\{0\}$ is weak*-closed.*

PROOF. *Necessity.* We may assume that X is a subspace of $C(K)$ where K is a compact Hausdorff space with a homeomorphism σ as above and $f\circ\sigma=-f$ for each f in X. The set K can be continuously mapped into $B(X^*)$ such that the set of extreme points of $B(X^*)$ is the set

$$\{\hat{k}:\ \sigma(k)\neq k,\ k\in K\}$$

and $\hat{k}=0$ if $\sigma(k)=k$. Hence $\partial_e B(X^*)\cup\{0\}=\hat{K}$ and $\hat{K}$ being weak*-closed implies that $\partial_e B(X^*)\cup\{0\}$ is weak*-closed.

Sufficiency. Let $K=\partial_e B(X^*)\cup\{0\}$ be closed in the weak*-topology and let $\sigma\colon K\to K$ be such that $\sigma(x)=-x$ for x in $\partial_e(B(X^*))$ and let $\sigma(0)=0$. We can consider X as the subspace $A_0(B(X^*))$ of $C(B(X^*))$, and we will show that the restriction map $f\to f/K$ (which maps X into $C(K)$) is an isometry between X and $C_\sigma(K)$. Indeed, we have $\|f\| = \|f/K\|$ for any f in X. On the other hand, if $f \in C_\sigma(K)$, then by Theorem 4.4, there exists an extension $\bar{f}$ in $A_0(B(X^*)) = X$ such that $\bar{f} = f$ on K and $\|\bar{f}\| = \|f\|$, which completes the proof.

The same method of proof can be used to prove the following proposition.

PROPOSITION 4.7. (Lindenstrauss–Wulbert.) *Let X be a Lindenstrauss space; then X is a C_Σ space if and only if the set of extreme points of $B(X^*)$ is weak*-closed.*

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