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A NOTE ON LIFTING OF MATRIX UNITS IN C^*-ALGEBRAS

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1. Introduction.

A C^*-algebra A is said to be sequentially monotone closed if every norm-bounded increasing sequence of self-adjoint elements has a least upper bound. In this note we show that if I is a closed, two-sided cofinite ideal in a sequentially monotone closed C^*-algebra A then a set of matrix units of A/I lifts to a set of matrix units of A. As an application of this result we improve on a result concerning continuity of separable linear maps from [2] (for definitions see below).

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2. Lifting projections.

We begin by proving a result on lifting one projection from a quotient algebra. A similar result may be found in [3, Theorem 3.2].

Lemma 1. *Let A be a sequentially monotone closed C^*-algebra and I a closed two-sided ideal in A. Then any projection in A/I is the image of a projection in A.*

Proof. Suppose $x \in A$ is an element for which $x^2 - x \in I$; we may assume $0 \leq x \leq 1$. Consider the following real-valued continuous functions on $[0,1]$.

$$g(t) = \begin{cases} 2t & 0 \leqq t \leqq \frac{1}{2}, \\ 1 & \frac{1}{2} \leqq t \leqq 1, \end{cases}$$

$$f_n(t) = \begin{cases} 0 & 0 \leqq t \leqq \frac{1}{2}, \\ n(t-\frac{1}{2}) & \frac{1}{2} \leqq t \leqq \frac{1}{2}+n^{-1}, \\ 1 & \frac{1}{2}+n^{-1} \leqq t \leqq 1, \end{cases} \qquad n = 2,3,\ldots,$$

$$h(t) = \begin{cases} 0 & 0 \leqq t \leqq \frac{1}{2}, \\ 2(t-\frac{1}{2}) & \frac{1}{2} \leqq t \leqq 1. \end{cases}$$

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We note that $g \geq f_n \geq h$ for all n so that $g(x) \geq f_n(x) \geq h(x)$ in A. Let $p = \sup f_n(x)$; it is easy to see that p is a projection and that $g(x) \geq p \geq h(x)$. If π denotes the canonical mapping $\pi: A \to A/I$ then

$$\pi(g(x)) \geq \pi(p) \geq \pi(h(x)) .$$

But since $g(0) = h(0) = 0$ and $g(1) = h(1) = 1$ we get that $\pi(g(x)) = \pi(h(x)) = \pi(x)$ and consequently $\pi(x) = \pi(p)$.

Once we know that individual projections can be lifted it is easy to extend this fact to countable families.

LEMMA 2. *If A is a sequentially monotone closed C^*-algebra and I a closed two-sided ideal in A, if $\{E_j\}$ is a countable set of orthogonal projections in A/I then there exists a set $\{e_j\}$ of orthogonal projections in A such that $\pi(e_j) = E_j$, $j = 1, 2, \ldots$.*

PROOF. Once we have Lemma 1 we can quote the proof of Proposition 5.4 [2] to which we refer.

3. Lifting the off-diagonal elements.

To handle the lifting of the matrix units sitting off the diagonal we impose the further restriction on I that A/I be finite dimensional. To simplify notation we shall assume that A/I is a full matrix algebra. The general result will follow by a routine direct sum argument. Specifically we prove the following

THEOREM 3. *If A is a sequentially monotone closed C^*-algebra and I a cofinite closed two-sided ideal then the matrix units of A/I lift to a set of matrix units of A.*

PROOF. To be precise the claim of the theorem is the following: With π denoting as usual the canonical mapping, if $\{E_{ij}\}$ is a set of elements in A/I such that

$$\textstyle\sum_i E_{ii} = 1, \quad E_{ij} = E_{ji}^* ,$$
$$E_{ij} E_{kl} = \delta_{jk} E_{il}, \quad i,j,k,l = 1, \ldots, n ,$$

then we can find $\{e_{ij}\}$ in A such that

$$\pi(e_{ij}) = E_{ij} ,$$
$$1 - \textstyle\sum e_{ii} \in I, \quad e_{ij} = e_{ji}^*$$
$$e_{ij} e_{kl} = \delta_{jk} e_{il} \quad i,j,k,l = 1, \ldots, n .$$

Suppose first (Lemma 2) we have lifted $\{E_{ii}\}$ to $\{e_{ii}\}$; to begin the construction of $\{e_{ij}\}$ when $i \neq j$ consider first E_{21} and let $p_{21} \in E_{21}$. Using polar decomposition (see the proof of Proposition 2.3 in [1]) on p_{21} and $|p_{21}| = (p_{21}^* p_{21})^{\frac{1}{2}}$ we get $c_{21} \in A$ such that

$$p_{21} = c_{21}|p_{21}|, \quad \text{and} \quad |p_{21}| = c_{21}^* p_{21} .$$

Replacing p_{21} by $e_{22}p_{21}e_{11}$ we see that we may assume $p_{21} = e_{22}p_{21}e_{11} \in e_{22}Ae_{11}$. Consequently, $p_{21}^*p_{21} \in e_{11}Ae_{11}$, in fact $p_{21}^*p_{21} \in E_{11}$, and also $|p_{21}| \in e_{11}Ae_{11}$. From this it follows that

$$\begin{aligned} p_{21} &= e_{22}p_{21}e_{11} = e_{22}c_{21}|p_{21}|e_{11} \\ &= e_{22}c_{21}e_{11}|p_{21}|e_{11} = (e_{22}c_{21}e_{11})|p_{21}|e_{11} \\ &= (e_{22}c_{21}e_{11})|p_{21}| . \end{aligned}$$

This means we can assume that

$$c_{21} = e_{22}c_{21}e_{11};$$

in fact $c_{21} \in \lambda E_{21}$ for some λ so that

$$E_{21} = \lambda E_{21}E_{11} = \lambda E_{21} ,$$

i.e. $\lambda = 1$, and $c_{21} \in E_{21}$.

Next we observe that if we use [ ] to denote range projections then

$$|p_{21}| = c_{21}^* c_{21}|p_{21}|$$

implies

$$[|p_{21}|] = c_{21}^* c_{21}[|p_{21}|] ,$$

and consequently, defining

$$d_{21} = c_{21}[|p_{21}|] ,$$

we get

$$d_{21}^* d_{21} = [|p_{21}|]c_{21}^* c_{21}[|p_{21}|] = [|p_{21}|] .$$

This means that d_{21} is a "partial isometry". We must next check that d_{21} has the right properties,

i) $d_{21} \in E_{21}$ and
ii) $[|p_{21}|] \in E_{11}$

Addressing ourselves first to ii) we see that this follows since $|p_{21}| \in e_{11}Ae_{11}$ which is a sequentially monotone closed C^*-algebra in which the range projection $[|p_{21}|]$ may be considered computed.

Concerning i):

$$d_{21} = c_{21}[|p_{21}|] \in e_{22}Ae_{11}$$

implies

$$\pi(d_{21}) \in E_{22}(A/I)E_{11} = \{\lambda E_{21}\} ,$$

so

$$\pi(d_{21}) = \pi(c_{21})\pi([|p_{21}|])$$

implies

$$\lambda E_{21} = E_{21}E_{11} = E_{21}$$

and hence $\lambda = 1$ which shows that $d_{21} \in E_{21}$.

The reason it suffices to know that $[|p_{21}|] \in E_{11}$ is that since $[|p_{21}|] \leq e_{11}$ we may replace e_{11} by $[|p_{21}|]$.

We now repeat the above procedure for p_{31}, replacing, if necessary, $[|p_{21}|]$ by $[|p_{31}|]$. After n steps we have

$$[|p_{21}|] \geq [|p_{31}|] \geq \ldots \geq [|p_{n1}|] = e_{11}$$

where the last equation is to be taken as a final redefinition of e_{11}. We then define

$$e_{i1} = d_{i1}[|p_{n1}|], \quad i = 2,\ldots,n$$

and note the following:

a) $e_{i1}^* e_{i1} = [|p_{n1}|]d_{i1}^* d_{i1}[|p_{n1}|] = [|p_{n1}|] = e_{11}$.

b) $e_{i1} = d_{i1}[|p_{n1}|] = c_{i1}[|p_{n1}|]$, hence $\pi(e_{i1}) = E_{i1}E_{11} = E_{11}$.

Moreover, for the projections $e_{i1}e_{i1}^*$ we have

$$\pi(e_{i1}e_{i1}^*) = E_{i1}E_{i1}^* = E_{ii},$$

so we can redefine

$$e_{ii} = e_{i1}e_{i1}^*$$

without disturbing the orthogonality properties.

As the last step we define

$$e_{ij} = e_{i1}e_{j1}^*, \quad i,j = 1,\ldots,n .$$

It is then a simple matter to check that e_{ij} thus defined is a set of matrix units with all the desired properties.

4. An application.

A separable linear map T from a Banach algebra A to a Banach space B is a linear map for which there exists a map $f\colon A \to \mathbb{R}_+$ such that

$$\|T(xy)\| \leqq f(x)f(y)$$

for all $x,y \in A$. Such linear maps have been studied in [2]. The next result is an improvement of [2, Theorem 5.6].

THEOREM 4. *Let T be a separable linear map defined on a sequentially monotone closed C^*-algebra A. Then T is continuous on a dense sub-*-algebra of A.*

PROOF. Since the generalized polar decomposition [1, Proposition 2.3] is available in a sequentially monotone closed C^*-algebra the results of sections 3–5 of [2] carry directly over to our present setting. Using the notation from [2] and letting M be the closure of $(J_T)_0$ we suppose $\{E_{ij}\}$ is a set of matrix units in A/M with lifting $\{e_{ij}\} \subset A$ (Theorem 3). Define

$$B = \operatorname{span}(e_{ij}) \oplus (J_T)_0$$

and note that B is a dense sub-*-algebra of A. Since $\operatorname{span}(e_{ij})$ is finite-dimensional the continuity of T on B follows.

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