MATH. SCAND. 33 (1973), 343–358

AN APPROXIMATE TAYLOR'S THEOREM FOR $R(X)$

JAMES LI-MING WANG

1. Introduction.

Let X be a compact subset of the complex plane C. We denote by $C(X)$ the algebra of all complex-valued continuous functions on X, by $R_0(X)$ the subalgebra of $C(X)$ consisting of the (restrictions to X of) rational functions having no poles on X, and by $R(X)$ the uniform closure of $R_0(X)$. In this paper, we give some answers to the question: What can be said about the behaviour of the functions in $R(X)$ near a point $x \in X$, beyond the obvious property of continuity? Of course, if x is an interior point of X, each $f \in R(X)$ is analytic in a neighbourhood of x, so the functions in the unit ball of $R(X)$ are equicontinuous at x, and even satisfy a uniform Lipschitz condition at x. On the other hand, if x is a boundary point of X, no such equicontinuity or Lipschitz condition is possible, because there are peak points in any neighbourhood of x. However, it turns out that we can obtain some satisfying results by using the notions of approximate continuity, approximate Hölder condition, etc. For instance, at almost all points of X which are not peak points for $R(X)$, every function in $R(X)$ satisfies an approximate Hölder condition of order α at x for every $\alpha < 1$, and the condition is uniform on the unit ball of $R(X)$.

We investigate also the consequences of the existence of bounded point derivations. We say that there exists a bounded point derivation of order p on $R(X)$ at $x \in X$ if there exists a bounded linear functional $D_x{}^p$ on $R(X)$ such that $D_x{}^p f = f^{(p)}(x)/p!$ for all $f \in R_0(X)$. Wermer [7] constructed an X with $R(X) \neq C(X)$ such that there existed no first order bounded point derivations at any point of X. In the opposite direction, Hallstrom [3] constructed an X with empty interior such that bounded point derivations of all orders exist at almost all points of X. If there exists a sequence $\{x_n\}$ in X, converging to x, such that

$$\lim (f(x_n) - f(x))/(x_n - x) = Lf$$

exists for all $f \in R(X)$, then L is a bounded linear functional (by the uniform boundedness principle), and so $L = D_x{}^1$. No example of a bounded

Received July 18, 1973.

point derivation at a boundary point x of X has ever been constructed by exhibiting such a sequence, to our knowledge, but we show that in fact every bounded first order point derivation does arise in this way, and higher order bounded point derivations are the corresponding limits of higher order difference quotients.

Surprisingly little attention has been paid in the past to our basic question. We are aware only of Browder's result of 1967 asserting the equi-approximate continuity of the functions in the unit ball of $R(X)$ at a non peak point, and the recent work of O'Farrell [4, 5, 6], involving "non-tangential approach" to a boundary point.

The rest of the paper is in two parts. Section 2 contains, besides notation and terminology, the measure theoretic machinery; Section 3 applies this machinery to obtain the results on $R(X)$ mentioned above, and some others.

2. Measures and potentials.

Throughout this paper, z will denote the identity function on $\mathbf{C}$, and m will denote two-dimensional Lebesgue measure. By a *measure*, we understand a complex Borel measure. For $f \in C(X)$, we write $\|f\|$ for $\max_X |f|$. If μ is a measure on X, we denote by $|\mu|$ the associated total variation measure; then $|\mu|(X) = \|\mu\| =$ norm of μ as a continuous linear functional on $C(X)$.

Fix $x \in \mathbf{C}$. We say that a set $E \subset \mathbf{C}$ has *full area density at* x if

$$\lim_{n\to\infty} m(E \cap \Delta_n)/m(\Delta_n) = 1 ,$$

where $\Delta_n = \{y \in \mathbf{C} : |y-x| \leqq n^{-1}\}$. Let F be a function defined on X, $x \in X$. We say that a is the *approximate limit* of F at x, and write

$$\operatorname{applim}_{y\to x} F(y) = a ,$$

if there exists a subset E of X having full area density at x, such that

$$\lim_{\substack{y\to x\\ y\in E}} F(y) = a .$$

If $\operatorname{applim}_{y\to x}(F(y)-F(x))/(y-x)$ exists, we call it the *approximate derivative* of F at x. We say that F is *approximate continuous* at x if

$$\operatorname{applim}_{y\to x} F(y) = F(x) .$$

We say that F admits φ as *modulus of approximate continuity* at x if

$$|F(y)-F(x)| \leqq \varphi(|y-x|)$$

for all y in a set having full area density at x; here φ is a positive function on $(0,\infty)$ with $\lim_{r\to 0}\varphi(r)=0$. We say that F satisfies an *approximate Hölder condition* of order α at x if F admits Cr^α as modulus of approximate continuity at x, for some constant C.

DEFINITION. Let ψ be a positive non-decreasing function on $(0,\infty)$. For each compactly supported measure μ, we define the ψ-potential of μ, $U_\mu{}^\psi$, by

$$U_\mu{}^\psi(y) = \int \frac{d|\mu|}{\psi(|z-y|)}.$$

If $\psi(|z|)^{-1}$ is locally summable with respect to m, an application of Fubini's theorem shows that $U_\mu{}^\psi$ is locally summable; in particular, $U_\mu{}^\psi<\infty$ a.e. (m).

The next important case is $\psi(r)\equiv r$; in this case, we denote $U_\mu{}^\psi$ by $\tilde{\mu}$. We define $\hat{\mu}(y)=\int(z-y)^{-1}d\mu$ for all $y\in \mathbf{C}$ where $\tilde{\mu}(y)<\infty$.

For each $\delta>0$, we set

$$E_\mu{}^\psi(\delta) = \{y\in \mathbf{C}:\ \psi(|y-x|)U_\mu{}^\psi(y)<\delta\}$$

and

$$E_\mu(\delta) = \{y\in \mathbf{C}:\ |y-x|\tilde{\mu}(y)<\delta\}\ .$$

DEFINITION. We say that φ is an admissible function if

a) φ is a positive, non-decreasing function defined on $(0,\infty)$, and
b) the associated function ψ, defined by $\psi(r)=r/\varphi(r)$, is non-decreasing, with $\psi(0+)=0$.

Examples of admissible functions are

i) $$\varphi(r) = r^\alpha,\quad \text{with } 0\leqq\alpha<1;$$
ii) $$\varphi(r) = r\log(r^{-1})^\beta \quad \text{for } 0<r\leqq r_0$$
$$= r_0\log(r_0{}^{-1})^\beta \quad \text{for } r\geqq r_0\,,$$

where $\beta>1$ and $r_0>0$ is sufficiently small.

REMARK. If φ is an admissible function, then $\varphi(r)\leqq\varphi(r_1)+\varphi(r_2)$ whenever $r\leqq r_1+r_2$, since

$$\varphi(r) \leqq \varphi(r_1+r_2) = (r_1+r_2)/\psi(r_1+r_2)$$
$$\leqq r_1/\psi(r_1)+r_2/\psi(r_2) = \varphi(r_1)+\varphi(r_2)\ .$$

LEMMA 2.1. *Let ψ be a positive non-decreasing function on $(0,\infty)$, and for each positive integer n let m_n be a positive measure on Δ_n such that*

i) $\psi(n^{-1})U^{\psi}_{m_n}(y) \leqq C$ *for all* $y \in \mathbf{C}$, *all* n;
ii) $\psi(n^{-1})\|m_n\| \to 0$ *for* $n \to \infty$.

Then for every compactly supported measure σ *with* $\sigma(\{x\})=0$,

$$\int \psi(|y-x|)U_\sigma^{\psi}(y)\,dm_n(y) \to 0\ .$$

PROOF. Let $F_n(\omega)=\int [\psi(|y-x|)/\psi(|y-\omega|)]\,dm_n(y)$. Then

$$F_n(\omega) \leqq \psi(n^{-1})U^{\psi}_{m_n}(\omega) \leqq C \quad \text{for all } \omega\ ,$$

and for $\omega \neq x$ we have for large n

$$F_n(\omega) \leqq \psi(n^{-1})\psi(|\omega-x|-n^{-1})^{-1}\|m_n\| \to 0 \quad \text{for } n \to \infty\ .$$

Thus $\{F_n\}$ converges boundedly to 0 a.e. (σ), so $\int F_n\,d|\sigma| \to 0$, and applying Fubini's theorem we have the desired conclusion.

LEMMA 2.2. *Let* E *be a measurable subset of* $\mathbf{C}$, *and define* ϱ_n *by* $\pi\varrho_n{}^2 = m(\Delta_n \setminus E)$. *Set* $m_n = n\varrho_n{}^{-1}m\,|\,(\Delta_n \setminus E)$. *Then*

$$\int \psi(|y-x|)U_\sigma^{\psi}(y)\,dm_n(y) \to 0$$

for any ψ *associated to an admissible function, and any compactly supported measure* σ *with* $\sigma(\{x\})=0$.

PROOF. We show that ψ and m_n satisfy the hypotheses of Lemma 2.1. Let D_n be the disk with center x and raduis ϱ_n. Then

$$\begin{aligned}
\psi(n^{-1})U^{\psi}_{m_n}(y) &= \psi(n^{-1})n\varrho_n{}^{-1}\int_{\Delta_n\setminus E}\frac{dm}{\psi(|z-y|)}\\
&\leqq \psi(n^{-1})n\varrho_n{}^{-1}\int_{D_n}\frac{dm}{\psi(|z-x|)}\\
&= \psi(n^{-1})n\varrho_n{}^{-1}\,2\pi\int_0^{\varrho_n}\psi(r)^{-1}r\,dr\\
&= \psi(n^{-1})n\varrho_n{}^{-1}\,2\pi\int_0^{\varrho_n}\varphi(r)\,dr\\
&\leqq \psi(n^{-1})n\varrho_n{}^{-1}\,2\pi\varrho_n\varphi(\varrho_n)\\
&\leqq 2\pi n\psi(n^{-1})\varphi(n^{-1}) = 2\pi\ .
\end{aligned}$$

(The first inequality above used only that ψ was positive and nondecreasing; compare [2, p. 151].) Thus hypothesis a) of Lemma 2.1 is satisfied. Next,

$$\psi(n^{-1})\|m_n\| = \psi(n^{-1})n\varrho_n^{-1}\pi\varrho_n^2 \leqq \pi\psi(n^{-1}) \to 0 ,$$

so hypothesis b) is satisfied as well. The lemma follows.

LEMMA 2.3. *Let ψ be the function associated to some admissible function, and let σ be a compactly supported measure with $\sigma(\{x\})=0$. Then $E_\sigma^\psi(\delta)$ has full area density at x for every $\delta>0$.*

PROOF. Taking E to be the empty set in Lemma 2.2, we obtain

$$m(\Delta_n)^{-1}\int_{\Delta_n}\psi(|y-x|)U_\sigma^\psi(y)dm(y) \to 0 \quad \text{for } n\to\infty .$$

Now

$$m(\Delta_n\setminus E_\sigma^\psi(\delta)) \leqq \delta^{-1}\int_{\Delta_n}\psi(|y-x|)U_\sigma^\psi(y)dm(y) ,$$

so the conclusion follows.

REMARK. Suppose that the support of σ lies in a compact set X which satisfies a "cone condition" at x, that is, there exist $r_0>0$ and an open interval I such that the sector

$$\{y:\ 0<|y-x|<r_0,\ \arg(y-x)\in I\}$$

is disjoint from X. Let J be a closed interval contained in I, and put

$$C_\varepsilon = \{y:\ 0<|y-x|\leqq\varepsilon,\ \arg(y-x)\in J\} .$$

We assert that, under the hypotheses of Lemma 2.3, $C_\varepsilon\subset E_\sigma^\psi(\delta)$ for $\varepsilon>0$ sufficiently small. To see this, we observe that if $0<r_1<r_0$, and k is a sufficiently large positive integer, we have $|y-x|\leqq k|y-\omega|$ for all $\omega\in X$, $y\in C_{r_1}$. Now

$$\psi(|y-x|)U_\sigma^\psi(y) = \left(\int_{D_r}+\int_{X\setminus D_r}\right)\psi(|y-x|)\psi(|y-z|)^{-1}d|\sigma|$$

where D_r is the disk of center x and radius r. Using the remark following the definition of admissible functions, we have

$$\psi(|y-x|) \leqq \psi(k|y-\omega|) \leqq k\psi(|y-\omega|)$$

for all $y\in C_{r_1}$, $\omega\in X$. Choosing $r<r_1$ so that $|\sigma|(D_r)<\delta/2k$, and then $\varepsilon>0$ so that $\psi(\varepsilon)<\frac{1}{2}\delta\psi(r)/\|\sigma\|$, we obtain

$$\psi(|y-x|)U_\sigma^\psi(y) \leqq k|\sigma|(D_r)+\psi(\varepsilon)\psi(r)^{-1}\|\sigma\| < \delta$$

for all $y\in C_\varepsilon$, thus verifying the assertion that $E_\sigma^\psi(\delta)\supset C_\varepsilon$.

We remark that Lemma 2.3 also holds when $\psi(r)=r^\beta$ $(1<\beta<2)$, though ψ is not admissible. We shall not make use of this fact.

LEMMA 2.4. *Let φ be an admissible function, p a non-negative integer, and μ a measure on X satisfying $\int [|z-x|^p\varphi(|z-x|)]^{-1}d|\mu| < \infty$. Set*

$$\lambda_j = \mu(z-x)^{-j} \quad (0 \leqq j \leqq p),$$

$$\sigma = \frac{|\mu|}{|z-x|^p\,\varphi(|z-x|)}.$$

Then

(i) *for any $F \in C(X)$, $y \in \mathbf{C}$, we have*

$$\int F(z-y)^{-1}d\mu = \sum_{j=1}^{p} (y-x)^{j-1} \int F\,d\lambda_j + (y-x)^p \int F(z-y)^{-1}d\lambda_p,$$

and

(ii) $$|y-x|\tilde{\lambda}_p(y) \leqq \varphi(|y-x|)[\psi(|y-x|)U_\sigma^{\psi}(y) + |y-x|\tilde{\sigma}(y)]$$

for all $y \in \mathbf{C}$, where ψ is the function associated to φ.

PROOF. Since $\lambda_j = (z-x)\lambda_{j+1}$ for $0 \leqq j \leqq p$, we have

$$\begin{aligned}\int F(z-y)^{-1}d\lambda_j &= \int F(z-x)(z-y)^{-1}d\lambda_{j+1}\\ &= \int F\,d\lambda_{j+1} + (y-x)\int F(z-y)^{-1}d\lambda_{j+1},\end{aligned}$$

from which (i) follows easily. We also have

$$\begin{aligned}\tilde{\lambda}_p(y) &= \int \varphi(|z-x|)|z-y|^{-1}d\sigma\\ &\leqq \int [\varphi(|z-y|) + \varphi(|y-x|)]|z-y|^{-1}\,d\sigma\\ &= U_\sigma^{\psi}(y) + \varphi(|y-x|)\tilde{\sigma}(y),\end{aligned}$$

from which (ii) is obtained on multiplying by $|y-x|$.

PROPOSITION 2.5. *Let φ be an admissible function, p a non-negative integer, and μ a compactly supported measure such that $\mu(\{x\}) = 0$ and $\int [|z-x|^p\varphi(|z-x|)]^{-1}d|\mu| < \infty$. Let $\delta > 0$, and $E = E_\mu(\delta)$. Then*

$$m(\Delta_n \setminus E) = o\big(\varphi(n^{-1})^2 n^{-2p-2}\big).$$

PROOF. Let ψ be the function associated with φ. Let σ and λ_j $(0 \leqq j \leqq p)$ be defined as in Lemma 2.4. Applying conclusion (i) of that lemma, we have

$$\begin{aligned}\tilde{\mu}(y) &= \sup\{|\textstyle\int F(z-y)^{-1}d\mu| : F \in C(X), \|F\| = 1\}\\ &\leqq \textstyle\sum_{j=1}^{p} |y-x|^{j-1}\|\lambda_j\| + |y-x|^p\tilde{\lambda}_p(y)\\ &\leqq C + |y-x|^p\tilde{\lambda}_p(y)\end{aligned}$$

for $y \in \Delta_1$. Now applying (ii) of Lemma 2.4, we get

$$|y-x|\tilde{\mu}(y) \leqq C|y-x| + |y-x|^p\varphi(|y-x|)[\psi(|y-x|)U_\sigma^{\,\nu}(y) + |y-x|\tilde{\sigma}(y)] .$$

Define ϱ_n by $\pi\varrho_n^{\,2} = m(\Delta_n \setminus E)$, and put $m_n = n\varrho_n^{\,-1}m \,|\, \Delta_n \setminus E$. We have

$$\begin{aligned} m(\Delta_n \setminus E) &\leqq \delta^{-1} \int_{\Delta_n \setminus E} |y-x|\tilde{\mu}(y)\,dm \\ &= \varrho_n(\delta n)^{-1} \int |y-x|\tilde{\mu}(y)\,dm_n \end{aligned}$$

or

$$\pi\varrho_n^{\,2} \leqq C(\delta n)^{-1}m(\Delta_n \setminus E) + \frac{\varphi(n^{-1})}{\delta n^p}\,\frac{\varrho_n}{n}\,[\int \psi(|y-x|)U_\sigma^{\,\nu}(y)\,dm_n(y) + \int |y-x|\tilde{\sigma}(y)\,dm_n(y)]$$

so for large n

$$\pi\varrho_n \leqq \frac{2\varphi(n^{-1})}{\delta n^{p+1}}\,[\int \psi(|y-x|)U_\sigma^{\,\nu}(y)\,dm_n(y) + \int |y-x|\tilde{\sigma}(y)\,dm_n(y)] .$$

Applying Lemma 2.2, we obtain

$$\varrho_n = o(\varphi(n^{-1})n^{-p-1}) ,$$

and the proposition follows on squaring both sides.

We remark that the hypothesis $\mu(\{x\}) = 0$ is needed only for the case $p=0$, $\varphi(0+) > 0$, when the proposition reduces to Lemma 2.3.

DEFINITION. We say that the admissible function φ is *nice* if $\int_0^1 \varphi(r)^{-1}dr < \infty$.

The examples previously given of admissible functions are nice. Nice admissible functions will be our favoured candidates for moduli of approximate continuity.

LEMMA 2.6. *Let μ be a compactly supported measure with $\mu(\{x\}) = 0$. Then there exists a nice admissible function φ, with $\varphi(0+) = 0$, such that $\int \varphi(|z-x|)^{-1}d|\mu| < \infty$.*

PROOF. Let $D_r = \{y \in \mathbf{C}\colon |y-x| < r\}$ and let $M(r) = |\mu|(D_r)$. Since $M(0+) = 0$ by regularity, we can choose r_1 so that $M(r_1) < 1$, and then select inductively $r_2, r_3, \ldots$, so that $M(r_j) < j^{-3}$ and $r_j < \min\{\frac{1}{2}r_{j-1}, j^{-3}\}$ for $j > 1$. We define φ in $[r_{j+1}, r_j]$ to be the linear function having values $1/(j+1)$ and $1/j$ at r_{j+1} and r_j respectively, and set $\varphi(r) = 1$ for $r \geqq r_1$. It is clear that φ is a continuous non-decreasing function on $(0, \infty)$, and that

$\varphi(0+)=0$. To see that φ is admissible, we must check that the associated function ψ is increasing. Since ψ has the form $r/(A+Br)$ on each interval $[r_{j+1}, r_j]$, it suffices to check that $\psi(r_{j+1})<\psi(r_j)$, that is that

$$\varphi(r_j)/r_j < \varphi(r_{j+1})/r_{j+1} \cdot$$

But $\varphi(r_j)/\varphi(r_{j+1})=(j+1)/j \leqq 2 < r_j/r_{j+1}$. Thus ψ is increasing, and since $\psi(r_j)=jr_j<j^{-2}$, we have $\psi(0^+)=0$. Thus φ is admissible. Furthermore,

$$\begin{aligned}\int_0^{r_1} \frac{dr}{\varphi(r)} &\leqq \sum_1^\infty \frac{r_j - r_{j+1}}{\varphi(r_{j+1})} \\ &< \sum_1^\infty r_j/\varphi(r_{j+1}) < \sum_1^\infty (j+1)j^{-3} < \infty ,\end{aligned}$$

so φ is nice as well.

Finally,

$$\begin{aligned}\int \frac{d|\mu|}{\varphi(|z-x|)} &= |\mu|(\mathbf{C}\setminus D_{r_1}) + \sum_1^\infty \int_{D_{r_j}\setminus D_{r_{j+1}}} \frac{d|\mu|}{\varphi(|z-x|)} \\ &\leqq \|\mu\| + \sum_1^\infty M(r_j)/\varphi(r_{j+1}) \\ &< \|\mu\| + \sum_1^\infty (j+1)j^{-3} < \infty ,\end{aligned}$$

and the proof is concluded.

3. The approximate Taylor's formula.

We call the measure μ on X a *representing measure* for $x \in X$ if $\int f d\mu = f(x)$ for all $f \in R(X)$, an *annihilating measure* if $\int f d\mu = 0$ for all $f \in R(X)$. We observe that if μ is a representing measure for x, then $\hat{\mu}(y) = 1/(x-y)$ for all $y \notin X$, since $1/(z-y) \in R(X)$, and if μ is an annihilating measure, then $\hat{\mu}(y)=0$ for $y \notin X$, for the same reason. Our main tool is the following simple lemma of Bishop:

LEMMA 3.1. *Let μ be an annihilating measure. If $\hat{\mu}(y)$ is defined and $\neq 0$, then $\hat{\mu}(y)^{-1}(z-y)^{-1}\mu$ is a representing measure for y.*

PROOF. If $f \in R_0(X)$, then $f = f(y) + (z-y)g$ for some $g \in R_0(X)$, whence

$$\int f(z-y)^{-1} d\mu = f(y)\hat{\mu}(y) + \int g\, d\mu = f(y)\hat{\mu}(y) .$$

COROLLARY 3.2. *Let μ be a representing measure for x. Let*

$$c(y) = \int (z-x)(z-y)^{-1} d\mu = 1 + (y-x)\hat{\mu}(y) .$$

Then $c(y)^{-1}(z-x)(z-y)^{-1}\mu$ is a representing measure for y, whenever $c(y)$ is defined and $\neq 0$, in particular for $y \in E_\mu(1)$.

PROOF. $(z-x)\mu$ is an annihilating measure.

The next lemma, generalizing Lemma 3.1, was first published by Wilken [8].

LEMMA 3.3. *Suppose there exists a representing measure μ for x and a positive integer p such that $\int |z-x|^{-p} d|\mu| < \infty$. Let $c_j = \int (z-x)^{-j} d\mu$ $(0 \leqq j \leqq p)$ and define $\mu_0, \mu_1, \ldots, \mu_p$ by:*

$$\mu_0 = \mu, \quad \mu_j = \mu(z-x)^{-j} - \textstyle\sum_{k<j} c_{j-k}\mu_k .$$

Then D_x^j exists, and $D_x^j f = \int f d\mu_j$ for all $f \in R(X)$, $0 \leqq j \leqq p$.

PROOF. We proceed by induction on j. For $j=0$, there is nothing to prove. Suppose $D_x^k f = \int f d\mu_k$ for $k<j$, where $j>0$. We observe that, for all $g \in R_0(X)$

$$\int g(z-x)^{j+1} d\mu_j = \int g(z-x)\, d\mu - \textstyle\sum_{k<j} c_{j-k} D_x^k[(z-x)^{j+1} g] = 0 ,$$

$$\int (z-x)^j d\mu_j = \int d\mu - \textstyle\sum_{k<j} c_{j-k} D_x^k[(z-x)^j] = 1 ,$$

and for $k<j$,

$$\begin{aligned}\int (z-x)^k d\mu_j &= c_{j-k} - \textstyle\sum_{l<j} c_{j-l} D_x^l (z-x)^k \\ &= c_{j-k} - c_{j-k} = 0 .\end{aligned}$$

Hence for any $f \in R_0(X)$ we can write

$$f = \textstyle\sum_{k=0}^{j} (D_x^k f)(z-x)^k + (z-x)^{j+1} g$$

with $g \in R_0(X)$, and conclude

$$\begin{aligned}\int f d\mu_j &= \textstyle\sum_{k=0}^{j} (D_x^k f) \int (z-x)^k d\mu_j + \int (z-x)^{j+1} g\, d\mu_j \\ &= D_x^j f ,\end{aligned}$$

thus completing the induction.

REMARK. It is even easier to see that conversely, if D_x^p exists, then there exists a representing measure μ for x such that $\int |z-x|^{-p} d|\mu| < \infty$. For let ν be a measure representing D_x^p, that is $D_x^p f = \int f d\nu$ for all $f \in R(X)$ (such ν exists by the Hahn–Banach and Riesz representation theorems). Put $\mu = (z-x)^p \nu$; then $\int f d\mu = D_x^p[(z-x)^p f] = f(x)$ by Leibniz' rule, and

$$\int |z-x|^{-p} d|\mu| = |\nu|(X \setminus \{x\}) < \infty .$$

We are led to the following definition:

DEFINITION. Let t be a positive real number. We say that there exists a t-th order bounded point derivation on $R(X)$ at x if there is a representing measure η for x satisfying

$$\int |z-x|^{-t} d|\mu| < \infty .$$

In view of Lemma 3.3 and the following remark, this definition agrees with the previous one for t a positive integer.

We now come to the principal theorem of this paper.

THEOREM 3.4. *Let φ be an admissible function and p a non-negative integer. Suppose there exists a representing measure μ for x such that $\mu(\{x\})=0$ and*

$$\int \frac{d|\mu|}{|z-x|^p \varphi(|z-x|)} < \infty .$$

Then for every $\varepsilon > 0$ there exists a subset E of X having full area density at x, such that for every $f \in R(X)$,

(i) $$f = \sum_{j=0}^{p} (D_x^j f)(z-x)^j + R$$

where $R \in R(X)$ satisfies

(ii) $$|R(y)| \leqq \varepsilon |y-x|^p \varphi(|y-x|) \|f\| \quad \textit{for all } y \in E ,$$

and

(iii) $$\text{app} \lim_{y \to x} \frac{R(y)}{|y-x|^p \varphi(|y-x|)} = 0 .$$

PROOF. Since $\int |z-x|^{-p} d|\mu| < \infty$, Lemma 3.3 applies and tells us that D_x^j exist for $0 \leqq j \leqq p$. Let R be defined by equation (i). Then $R \in R(X)$, $D_x^j R = 0$ for $0 \leqq j \leqq p$, and $\|R\| \leqq C\|f\|$ for some constant C.

Let $\lambda_j = (z-x)^{-j}\mu$, $0 \leqq j \leqq p$, and $\sigma = \varphi(|z-x|)^{-1}|\lambda_p|$. By Lemma 3.3, λ_j is a linear combination of measures representing D_x^k $(0 \leqq k \leqq j)$, so $\int R\, d\lambda_j = 0$ for $0 \leqq j \leqq p$.

Let $\delta = \varepsilon/(2C+\varepsilon)$, and set $E = E_\mu(\delta) \cap E_\sigma(\delta) \cap E_\sigma^\psi(\delta)$, where ψ is the function associated with φ. Then E has full area density at x by Lemma 2.3, and $E \subset X$ since $E_\mu(\delta) \subset X$. We observe that $c(y) = 1 + (y-x)\hat{\mu}(y)$ is well-defined and $\neq 0$ for $y \in E$, in fact $|c(y)| \geqq 1 - \delta$. By Corollary 3.2, we have, for $y \in E$,

$$\begin{aligned} R(y) &= c(y)^{-1} \int [R(z-x)/(z-y)]\, d\mu \\ &= c(y)^{-1} \int R[1 + (y-x)/(z-y)]\, d\mu \\ &= c(y)^{-1}(y-x) \int [R/(z-y)]\, d\mu \end{aligned}$$

since $R(x)=0$. Applying Lemma 2.4 i), we obtain

$$R(y) = c(y)^{-1}(y-x)^{p+1} \int R(z-y)^{-1}\, d\lambda_p .$$

so

$$\begin{aligned}|R(y)| &\leqq |c(y)|^{-1}|y-x|^{p+1}\|R\|\tilde{\lambda}_p(y)\\ &\leqq C\|f\|(1-\delta)^{-1}|y-x|^p|y-x|\tilde{\lambda}_p(y)\\ &\leqq C\|f\|(1-\delta)^{-1}|y-x|^p\varphi(|y-x|)[\psi(|y-x|)U_\sigma^{\psi}(y)+|y-x|\tilde{\sigma}(y)]\end{aligned}$$

by Lemma 2.4 (ii). Thus for $y \in E$,

$$\begin{aligned}|R(y)| &\leqq 2\delta C(1-\delta)^{-1}\|f\|\,|y-x|^p\varphi(|y-x|)\\ &= \varepsilon\|f\|\,|y-x|^p\varphi(|y-x|)\ ,\end{aligned}$$

and (ii) is proved.

Let $L_y f = R(y)/[|y-x|^p\varphi(|y-x|)]$ for $f \in R(X)$. Then $\{L_y : y \in E\}$ is a uniformly bounded set of linear functionals on $R(X)$, for (ii) states that $\|L_y\| \leqq \varepsilon$ for $y \in E$. Since $L_y f \to 0$ as $y \to x$ for $f \in R_0(X)$, it follows that $L_y f \to 0$ as $y \to x$ in E for all $f \in R(X)$, and thus (iii) is proved.

REMARK. The hypothesis $\mu(\{x\})=0$ was included only to cover the case $p=0$, $\lim_{r\to 0}\varphi(r)>0$. The theorem in this case specializes to a theorem of Browder [2, p. 177].

COROLLARY 3.5. *If there exists a t-th order bounded point derivation on $R(X)$ at x, then for every $f \in R(X)$ we have*

$$f = \textstyle\sum_{j=0}^{[t]} (D_x{}^j f)(z-x)^j + R$$

where $\operatorname{applim}_{y\to x} R(y)|y-x|^{-t}=0$.

PROOF. Take $p=[t]$ and $\varphi(r)=r^\alpha$ with $\alpha=t-[t]$ in Theorem 3.4.

Taking $t=1$ in this corollary, we deduce immediately: If there exists a first order bounded point derivation on $R(X)$ at x, then

$$D_x{}^1 f = \operatorname{applim}_{y\to x}\,(f(y)-f(x))/(y-x) \quad \text{for all } f \in R(X)\ .$$

This observation extends easily to higher order bounded point derivations.

For f a function defined on a subset of X, $h \in \mathbf{C}$, we set $\Delta_h f = f(z+h)-f$, so $\Delta_h f$ is a function defined on a (possibly empty) subset of X. We define, inductively, $\Delta_h{}^0=\mathrm{id}$, $\Delta_h{}^j=\Delta_h\circ\Delta_h{}^{j-1}$ for $j \geqq 1$.

COROLLARY 3.6. *If there exists a p-th order bounded point derivation on $R(X)$ at x, then for all $f \in R(X)$*

$$D_x{}^p f = \operatorname{applim}_{h\to 0} \Delta_h{}^p f(x)/(p!\, h^p) .$$

PROOF. By Corollary 3.5, there exists a subset E of X having full area density at x such that

$$f = \sum_{j=0}^{p} (D_x{}^j f)(z-x)^j + R ,$$

where $R(y)|y-x|^{-p} \to 0$ as $y \to x$ in E. Let

$$F_j = \{h \in C : x+jh \in E\}$$

for $j=1,\ldots,p$. Then each F_j has full area density at 0, hence $F=\bigcap_1^p F_j$ has full area density at 0. We note that $\Delta_h f(x)$ is a linear combination of $f(x+jh)$, $0 \leqq j \leqq p$, hence is well-defined for $h \in F$. Now it is classical that

$$\lim_{h\to 0} \Delta_h{}^p P(x)/(p!\, h^p) = D_x{}^p P$$

for a polynomial P. Since

$$h^{-p} R(x+jh) \to 0 \quad \text{as } h \to 0 \text{ in } F,\ 1 \leqq j \leqq p,$$

we have

$$\operatorname{applim}_{h\to 0} \Delta_h{}^p R(x)/(p!\, h^p) = 0 ,$$

so taking $P=\sum_{j=0}^{p}(D_x{}^j f)(z-x)^j$, and noting $D_x{}^p P = D_x{}^p f$, the desired conclusion follows.

REMARK. Taking into account the remark made after Lemma 2.3, we see that the set E of Theorem 3.4 contains any sector lying in the interior of X, ,,non-tangentially" with respect to X, having x as vertex and sufficiently small radius. Hence we can deduce from Theorem 3.4 some results on non-tangential limits found by O'Farrell [5, 6].

So far, we have emphasized the consequences of Theorem 3.4 in taking $\varphi \equiv 1$. We now turn our attention to the case $p=0$. In this case, equation (ii) of Theorem 3.4 reads

$$|f(y)-f(x)| \leqq \varepsilon\varphi(|y-x|)\|f\|$$

for all $y \in E$, $f \in R(X)$ and is thus the assertion that $\varepsilon\varphi$ is a modulus of approximate continuity at x for every function in the unit ball of $R(X)$.

We say that $x \in X$ is a *peak point* for $R(X)$ if there exists $f \in R(X)$ with $f(x) > |f(y)|$ for all $y \neq x$. It was shown by Bishop (see, e.g., [2, p. 99]) that x is *not* a peak point for $R(X)$ if and only if there exists a representing measure μ for x with $\mu(\{x\})=0$. Taking into account this fact, and Lemma 2.6, we deduce at once from Theorem 3.4:

COROLLARY 3.7. *If x is not a peak point for $R(X)$, there exists a nice admissible function φ with $\varphi(0^+)=0$ such that every function in the unit ball of $R(X)$ admits $\varepsilon\varphi$ as modulus of approximate continuity at x, for every $\varepsilon>0$.*

In fact, any nice admissible φ will work for almost all non-peak points. (By a theorem of Bishop (see, e.g., [2, p. 172]), the set of non-peak points has positive measure whenever $R(X)\neq C(X)$.)

LEMMA 3.8. *Let φ be a nice admissible function. If x is not a peak point for $R(X)$, then*

$$\{y\in X : \text{there exists a representing measure } \mu_y \text{ for } y \text{ such that } \int \varphi(|z-y|)^{-1}d|\mu_y| < \infty\}$$

has full area density at x.

PROOF. Let μ be a representing measure for x with $\mu(\{x\})=0$. Let

$$F = \{y\in \mathbf{C} : \int [|z-y|\varphi(|z-y|)]^{-1}d|\mu|<\infty\}\,.$$

Since $1/|z|\varphi(|z|)$ is locally summable with respect to m, $m(\mathbf{C}\setminus F)=0$ Fix δ, $0<\delta<1$, and put $E=F\cap E_\mu(\delta)$. Then E has full area density at x by Lemma 2.3, and for each $y\in E$, the measure

$$c(y)^{-1}(z-x)(z-y)^{-1}\mu = \mu_y$$

is a representing measure for y (Lemma 3.2). Also

$$\int \frac{d|\mu_y|}{\varphi(|z-y|)} = c(y)^{-1}\int \frac{|z-x|\,d|\mu|}{|z-y|\varphi(|z-y|)}$$
$$\leqq K\int [|z-y|\varphi(|z-y|)]^{-1}d|\mu| < \infty$$

for all $y\in R$. The lemma is proved.

The following corollary is immediate.

COROLLARY 3.9. *Let φ be a nice admissible function. Then for almost all non peak points x, every function in the unit ball of $R(X)$ admits φ as modulus of approximate continuity at x. In particular, at almost all non peak points x, the functions in $R(X)$ satisfy approximate Hölder conditions of order α for every $\alpha<1$.*

On the other hand, no positive function φ with $\varphi(0^+)=0$ can serve as modulus of approximate continuity for every function in the unit ball of every $R(X)$ at every non-peak point. For given such a φ, we construct X as follows:

Let $\bar{D}_0$ be the closed unit disk, and let D_n be the open disk with center a_n and radius ϱ_n, where $a_n \neq 0$ and $\sum_1^\infty \varrho_n/|a_n| < 1$. Let $X = \bar{D}_0 \setminus \bigcup_1^\infty D_n$. Then $0 \in X$, and 0 is not a peak point for $R(X)$. For let $X_N = \bar{D}_0 \setminus \bigcup_1^N D_n$; then 0 lies in the interior of X_N for each N, and

$$f(0) = (2\pi i)^{-1} \int_{\partial X_N} f z^{-1} dz = \int f d\mu_N$$

for $f \in R_0(X_N)$, by Cauchy's integral formula. Since $\bigcap X_N = X$, each $f \in R_0(X)$ belongs to $R_0(X_N)$ for N sufficiently large. Now

$$\|\mu_N - \mu_M\| \leqq (2\pi)^{-1} \sum_{n=N+1}^{M} \int_{\partial D_n} |z|^{-1} |dz| \quad \text{for } M > N ,$$

and

$$\int_{\partial D_n} |z|^{-1} |dz| \leqq 2\pi \varrho_n/(|a_n| - \varrho_n) \leqq 2\pi C \varrho_n/|a_n| ,$$

where $C = \max(|a_n| - \varrho_n)^{-1}|a_n| < \infty$. Hence $\{\mu_N\}$ converges in norm to a measure μ, which represents 0 for $R(X)$ and has no mass at 0. Thus 0 is not a peak point for $R(X)$.

Now choose a_n so that $0 < a_{n+1} < a_n$, $\lim a_n = 0$, and $\varphi(r) < \varepsilon_n$ for $0 < r \leqq a_n$, where $\{\varepsilon_n\}$ is any sequence of positive numbers with $\sum_1^\infty \varepsilon_n < 1$. Put $\varrho_n = \varepsilon_n a_n$, and form X as above. We assert that

$$Y = \{y \in X : \ |f(y) - f(0)| \leqq \varphi(|y|) \text{ for all } f \in R(X), \|f\| \leqq 1\}$$

does not have full area density at 0. To see this, let $f_n = \varrho_n (z - a_n)^{-1}$, so $f_n \in R(X)$ and $\|f_n\| \leqq 1$. Thus

$$Y \subset \{y \in \mathbf{C} : \ |f_n(y) - f_n(0)| \leqq \varphi(|y|), \ n = 1, 2, \ldots\} .$$

Now

$$|f_n(y) - f_n(0)| = \varrho_n |(y - a_n)^{-1} + a_n^{-1}| = \varepsilon_n |y|/|y - a_n| .$$

Hence for $|y| \leqq a_n$ and $\operatorname{Re} y \geqq \frac{1}{2} a_n$, we have

$$|f_n(y) - f_n(0)| \geqq \varepsilon_n > \varphi(|y|) ,$$

so

$$Y \cap \{y : \ |y| \leqq a_n\} \subset \{y : \ |y| \leqq a_n, \operatorname{Re} y < \tfrac{1}{2} a_n\} ,$$

whence

$$m(Y \cap \tilde{\Delta}_n)/m(\tilde{\Delta}_n) \leqq \tfrac{2}{3} + \sqrt{3}/(4\pi) < 1 \quad \text{for all } n ,$$

where $\tilde{\Delta}_n = \{y : |y| \leqq a_n\}$, which implies that Y does not have full area density at 0.

It is not hard to modify the construction so that Y has zero lower density at 0.

It follows that there is no ,,universal'' φ such that $\int \varphi(|z-x|)^{-1}d|\mu| < \infty$ for some μ representing x whenever x is a non-peak point for $R(X)$, with X arbitrary.

For $x,y \in X$, we define

$$\|x-y\| = \sup\{|f(x)-f(y)| : f \in R(X), \|f\| \leqq 1\} .$$

This ,,Gleason metric'' has proved important in the general theory of function algebras. In the following, we fix x, and set

$$P_\varepsilon = \{y \in X : \|y-x\| < \varepsilon\} .$$

Many of the preceding results have obvious formulations in terms of the Gleason metric. For example, taking $p=0$ and $\varphi \equiv 1$ in Theorem 3.4, we obtain Browder's result: If x is not a peak point for $R(X)$, then P_ε has full area density at x, for every $\varepsilon > 0$. Taking $p=1$ and $\varphi \equiv 1$, we obtain: If there exists a first-order bounded point derivation at x, then $y \to \|y-x\|$ satisfies an approximate Lipschitz condition at x, or more precisely, for any $\varepsilon > 0$,

$$\|y-x\| \leqq (\|D_x^1\|+\varepsilon)|y-x|$$

for all y in a set having full area density at x.

We conclude by observing that P_ε has more than full area density at x, if x is not a peak point, and much more if there exists a bounded point derivation at x.

Proposition 3.10. i) *If x is not a peak point, then there exists a nice admissible function φ with $\varphi(0^+)=0$ such that*

$$m(\Delta_n \setminus P_\varepsilon) = o\big(\varphi(n^{-1})^2 n^{-2}\big) .$$

ii) *If φ is a nice admissible function, then for almost all x which are not peak points, we have*

$$m(\Delta_n \setminus P_\varepsilon) = o\big(\varphi(n^{-1})^2 n^{-2}\big) .$$

In particular, $m(\Delta_n \setminus P_\varepsilon) = o\big((\log n)^3 n^{-4}\big)$ for almost all non peak points x.

iii) *If there exists a t-th order bounded point derivation at x, then*

$$m(\Delta_n \setminus P_\varepsilon) = o(n^{-2t-2}) .$$

Proof. If μ is any representing measure for x, then $E_\mu(\delta) \subset P_\varepsilon$ for δ sufficiently small (this was shown in the course of proving Theorem 3.4 for the case $p=0$, $\varphi \equiv 1$, or see [2, p. 176]). Having Lemma 3.6 for i), Lemma 3.8 for ii), and the definition (or preceding remark) of t-th order bounded point derivation for iii), the proposition follows at once from Proposition 2.5.

The last part of the above proposition was first found by O'Farrell [4] as a consequence of a deeper theorem. The weaker result $m(\Delta_n \setminus X) = o(n^{-2t-2})$ (t an integer) had been previously found by Hallstrom [3].

Finally we remark that all our results are valid for the algebra $A(X)$ of all continuous functions on X which are holomorphic in the interior of X. For a theorem of Arens [1] tells us that for any $x \in X$, the space of all $f \in A(X)$ which are (restrictions to X of functions) holomorphic in a neighbourhood of x is dense in $A(X)$. This space serves as substitute for $R_0(X)$ in our arguments. (Compare [2, p. 205].)

Acknowledgement.

This paper contains part of the author's Ph.D. thesis. I would like to thank Prof. Andrew Browder for his advice and comments. The author received support from the National Science Foundation during the preparation of this paper. I would like also to thank the Matematisk Institut at Aarhus Universitet for their hospitality during the academic year 1972–73.

REFERENCES

1. R. Arens, *The maximal ideals of certain function algebras*, Pacific J. Math. 8 (1958), 641–648.
2. A. Browder, *Introduction to Function Algebras*, Benjamin, New York, 1969.
3. A. Hallstrom, *On bounded point derivations and analytic capacity*, J. Functional Analysis 4 (1969), 153–165.
4. A. O'Farrell, *Capacities in uniform approximation*, Thesis, Brown University, 1973.
5. A. O'Farrell, *Equiconvergence of derivations*, Preprint, Brown University.
6. A. O'Farrell, *Analytic capacity, Hölder conditions and τ-spikes*, Preprint.
7. J. Wermer, *Bounded point derivations on certain Banach algebras*, J. Functional Analysis 1 (1967), 28–36.
8. D. Wilken, *Bounded point derivations and representing measures on R(X)*, Proc. Amer. Math. Soc. 24 (1970), 371–373.

UNIVERSITY OF AARHUS, DENMARK
AND
BROWN UNIVERSITY, PROVIDENCE, RHODE ISLAND, U.S.A.