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SOME PROPERTIES OF COMPACT NATURAL SETS IN SEVERAL COMPLEX VARIABLES

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1. Definitions and preliminaries.

In this paper we present some results concerning compact natural sets (also called holomorphically convex sets or holomorphic sets). In $\mathbf{C}^n$ these are defined as follows: If K is a compact set in $\mathbf{C}^n$ we let $O(K)$ be the ring of germs of holomorphic functions on K and we denote by $H(K)$ the uniform closure of $O(K)$ in $C(K)$, where $C(K)$ is the algebra of continuous functions on K with the supremum norm. Then $H(K)$ is a uniform algebra and has a maximal ideal space $M_{H(K)}$. K is said to be natural if and only if $K=M_{H(K)}$.

This notion can be generalized in the following way (cf. [3], [4], [14], [15] and [16]): Let A be an algebra of continuous complex-valued functions on a Hausdorff space S. We suppose that A contains the constants, that the topology in A is given by uniform convergence on compact subsets and that the topology in S is the weakest under which all functions in A are continuous. Furthermore it is assumed that every continuous homomorphism of A into $\mathbf{C}$ is given by evaluation at a point of S.

For an open set U in S we define $O_A(U)$ as the algebra of all A-holomorphic functions on U, that is, all continuous functions on U which locally are uniformly approximable by functions from A. We give to $O_A(U)$ the topology of uniform convergence on compact subsets and then we can define the spectrum $\tilde{U}_A$ of $O_A(U)$ as the space of all non-zero continuous complex-valued homomorphisms of $O_A(U)$.

If K is a compact subset of S, we let

$$O_A(K) = \operatorname{ind\,lim}_{U \supset K} O_A(U) ,$$

where $\{U\}$ runs over a fundamental system of open neighborhoods of K. $O_A(K)$ is given the inductive limit topology and then we define its spectrum $\tilde{K}_A$ in the usual way. Note that each germ in $O_A(K)$ by restriction gives rise to a continuous function on K. Finally we let $H_A(K)$ be the algebra of all continuous functions on K which can be uniformly approximated

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on K by functions from $O_A(K)$. Then $H_A(K)$ is a uniform algebra and it has a maximal ideal space which we denote by $M_{H_A(K)}$. We say that K is natural precisely when $K = M_{H_A(K)}$.

In particular, A could be the algebra of all polynomials in n complex variables and then $S = \mathbf{C}^n$, which means that we have the case considered at first. Another interesting example is when S is a Stein space with A as its algebra of holomorphic functions; we refer to [1] for the details.

Since each function in A by restriction may be considered as an element in the various spaces $O_A(U)$, $O_A(K)$ and $H_A(K)$ we get a mapping π from $\tilde{U}_A$, $\tilde{K}_A$ and $M_{H_A(K)}$ to S. π is defined by taking the restriction of the homomorphisms in $\tilde{U}_A$, $\tilde{K}_A$ and $M_{H_A(K)}$ respectively to A; since each continuous homomorphism of A is given by a point in S, π is a mapping into S.

For each function f in $H_A(K)$ we define its Gelfand transform $\hat{f}$ on $M_{H_A(K)}$ by

$$\hat{f}(\varphi) = \varphi(f) \quad \text{for all } \varphi \in M_{H_A(K)} .$$

We endow $M_{H_A(K)}$ with the weakest topology making the functions $\hat{f}$ continuous and then obviously $\pi\colon M_{H_A(K)} \to S$ is continuous.

The following theorem is presumably well-known, although we have not been able to find it in the literature in this form. For the case that $S = \mathbf{C}^n$ it is treated in [2] and for another variant we refer to [9].

THEOREM 1. *Suppose that S is a metric space and that K is a compact subset of S. Then K is natural if and only if $K = \bigcap_n \pi(\tilde{U}_n)$, where $\{U_n\}$ runs over a fundamental system of open neighborhoods of K and $U_n \subset U_m$ for $n > m$.*

PROOF. Let K be a natural set and suppose that $\bigcap_n \pi(\tilde{U}_n) \supsetneqq K$. Then there exists for all n a homomorphism

$$\varphi_n\colon O_A(U_n) \to \mathbf{C}$$

such that for some compact set $K_n \subset U_n$,

$$|\varphi_n(f)| \leqq \|f\|_{K_n} \quad \text{for all } f \in O_A(U_n)$$

and such that $\pi(\varphi_n) = \alpha_n \notin K$ with $\alpha_n \to \alpha_0 \notin K$. We may assume that $K_n \subset K_m$ for $n > m$.

Since S is metric, K can be given a metric and then $C(K)$ is separable. It follows that there is a countable subset $\{f_i\}_1^\infty$ of $O_A(K)$ which is dense in the supremum norm. By the definition of $O_A(K)$ there exists for every f_i a number m_i such that the germ f_i is represented by a function in $O_A(U_n)$ for all $n \geqq m_i$.

For every i we thus obtain a sequence $\{\varphi_n(f_i)\}$ for $n \geqq m_i$ and

$$|\varphi_n(f_i)| \leqq \|f_i\|_{K m_i} .$$

By the usual diagonal procedure we may then pick out a subsequence $\{n_j\}$ of all n such that

$$\Phi(f_i) = \lim_{n_j \to \infty} \varphi_{n_j}(f_i)$$

exists for all i. Since $K_n \downarrow K$ it follows that Φ defines an element in the spectrum $\tilde{K}_A$ of $O_A(K)$. But $\tilde{K}_A$ is equal to $M_{H_A(K)}$ as the following argument from [9] shows. Suppose there exists a germ $f \in O_A(K)$ with $\|f\|_K < 1$ and $\Phi(f) = 1$. Then $(1-f)^{-1} \in O_A(K)$, which implies that

$$1 = \Phi(1) = \Phi[(1-f)^{-1} \cdot (1-f)] = \Phi[(1-f)^{-1}] \cdot 0 = 0 ,$$

a contradiction. Hence every element in $\tilde{K}_A$ can be lifted to a homomorphism of $H_A(K)$ and trivially we can go in the other direction too. So in fact $\tilde{K}_A = M_{H_A(K)}$ and especially $\Phi \in M_{H_A(K)}$.

As noted earlier, each function in A can be considered as an element in $H_A(K)$, $O_A(K)$ and in $O_A(U_n)$. From the construction of Φ we se that

$$\lim_{n_j \to \infty} |\Phi(f) - \varphi_{n_j}(f)| = 0$$

for all f in A. This means that $\alpha_{n_j} = \pi(\varphi_{n_j}) \to \pi(\Phi)$, since the topology in S is determined by the functions in A. But K is natural and therefore $\Phi = \pi(\Phi)$ is a point in K, that is, $\alpha_0 = \pi(\Phi) \in K$, a contradiction. Hence $K = \bigcap_n \pi(\tilde{U}_n)$.

Conversely, assume that $K = \bigcap_n \pi(\tilde{U}_n)$. Every $\varphi \in M_{H_A(K)}$ defines by restriction an element in each $\tilde{U}_n$, and hence $\pi(\varphi) \in K$, that is, $\pi(M_{H_A(K)}) = K$. So to finish the proof we need the following theorem, which is proved in [3] and [4].

Theorem 2. *If $\pi(M_{H_A(K)}) = K$, then K is natural.*

Proof. Since the argument in [4] is very short, we reproduce it here. We have to prove that the fibers in $M_{H_A(K)}$ over points in K each contains only one point. To see this we consider an arbitrary germ $g \in O_A(K)$ and define $\tilde{g}(x) = g(\pi(x))$ for all $x \in M_{H_A(K)}$ (this is possible since $\pi(M_{H_A(K)}) = K$). As π is continuous it follows that $\tilde{g}$ is $H_A(K)$-holomorphic on $M_{H_A(K)}$. Hence the uniform algebra generated by $H_A(K)$ and $\tilde{g}$ has the same Shilov boundary as $H_A(K)$ (see [6]) and this is situated in K. But $g = \tilde{g}$ on K, so $g = \tilde{g}$ on all of $M_{H_A(K)}$. Since $\tilde{K}_A = M_{H_A(K)}$, $M_{H_A(K)}$ must then be equal to K. This proves theorem 2 and hence also theorem 1.

If U is an open set in $\mathbf{C}^n$, $U = \tilde{U}$ (that is U is "natural") precisely when U is a domain of holomorphy as is shown in [8]. Now theorem 1 can be reformulated to make sense also for open sets. Namely, a compact set K is natural if and only if for every open neighborhood W of K there exists a smaller open neighborhood V such that $\pi(\tilde{V}) \subset W$. But this statement is in general false for open sets. Consider for instance the open Reinhardt set

$$U = \{z \in \mathbf{C}^2 : \ |z_2| < |z_1| < 1\} .$$

U is a domain of holomorphy and has a fundamental system of open neighborhoods $\{U_n\}$ which are Reinhardt sets too. Since every U_n contains the origin, each $\tilde{U}_n$ contains the unit polydisc (we assume that $U_n \supset\supset U$). Because $\pi(\tilde{U}_n) = \tilde{U}_n$, we see that $\pi(\tilde{U}_n)$ is not close to U for any U_n. — In passing we have showed that $\overline{U}$ is not natural, that is, the closure of a relatively compact domain of holomorphy need not be a natural set.

On the other hand, if a compact set K in $\mathbf{C}^n$ is an intersection of open domains of holomorphy, then it follows immediately from theorem 1 that K is natural (this was originally proved by Rossi in [17]). That not all natural sets in $\mathbf{C}^n$ arise in this manner was shown by means of an example in [4]. So we see that although the theories of open domains of holomorphy and of compact natural sets clearly are related to each other, the relation is not too close.

With this general set-up we can now discuss some problems which have been posed by Jan-Erik Björk and have been solved with his very generous assistance. For more facts about compact natural sets we refer to his papers [3] and [4] and also to [19].

2. A Levi problem.

We say that a C^2 real-valued function $p(z)$ in $\mathbf{C}^n$ (here $z = (z_1, \ldots, z_n)$) is plurisubharmonic at a point z_0 if there is a neighborhood W of z_0 such that the Hessian of p,

$$\left\{\frac{\partial^2 p}{\partial z_i \partial \bar{z}_j}(z)\right\}_{i,j=1}^{n},$$

is positive semi-definite for all z in W. If the Hessian is positive definite, p is said to be strictly plurisubharmonic at z_0.

A domain D in $\mathbf{C}^n$ is called (strictly) pseudoconvex if there is a C^2-function $p(z)$ defined and (strictly) plurisubharmonic in a neighborhood W of D's topological boundary bd$D$ such that

$$D \cap W = \{z \in W : \ p(z) < 0\} .$$

These notions can then be carried over to complex manifolds; we refer to [8] for the details.

It has become known as Levi's problem to show that a domain which is pseudoconvex (in some sense) is a domain of holomorphy or at least is holomorphically convex. Grauerts solution of this problem goes as follows (see [8]):

THEOREM 3. *If D is a relatively compact and strictly pseudoconvex domain in a complex manifold, then D is holomorphically convex.*

We now give a theorem for compact natural sets which bears a slight resemblence to theorem 3.

THEOREM 4. *Let K be a compact natural set in $\mathbf{C}^n$ and let L be a subset of K with the following property: there exists a C^2 real-valued function $p(z)$ defined in a neighborhood of K such that*

$$L = \{z \in K : p(z) \leqq 0\}$$

and such that $p(z)$ is plurisubharmonic on L. Then L is natural.

PROOF. According to theorem 1 it is enough to prove the following: For a given open neighborhood W of L in $\mathbf{C}^n$ there always exists a smaller open neighborhood V of L such $\pi_V(\tilde{V}) \subset W$, where $\pi_V : \tilde{V} \to \mathbf{C}^n$ is given by restricting each element $\tilde{V}$ to the polynomials in $\mathbf{C}^n$. So let us fix an open neighborhood W of L in $\mathbf{C}^n$.

That $p(z)$ is plurisubharmonic on L really means that $p(z)$ is plurisubharmonic in some neighborhood of L since plurisubharmonicity is a local property. Shrinking W if necessary we may hence suppose that $p(z)$ is defined and plurisubharmonic on W.

Being a continuous function, $p(z)$ assumes a minimum on the compact set $K \setminus W$. Suppose that the minimum value is 2δ; then δ is a positive number according to the definition of L. This means that the set $\{z \in \mathbf{C}^n : p(z) > \delta\}$ is a neighborhood of $K \setminus W$ in $\mathbf{C}^n$. It is then clear that we can choose an open neighborhood U_1 of K and a positive number ε such that

$$L \subset \{z \in U_1 : p(z) < \varepsilon\} \subset W .$$

Due to the fact that K is natural there exists an open neighborhood U of K in $\mathbf{C}^n$ such that $\pi_U(\tilde{U}) \subset U_1$; here $\pi_U : \tilde{U} \to \mathbf{C}^n$ is defined by restriction to the polynomials. As is shown in [8], π_U is a local homeomorphism which gives to $\tilde{U}$ an analytic structure so that $\tilde{U}$ becomes a Stein manifold. By means of π_U we can explain how to differentiate functions on $\tilde{U}$. Namely, we set

$$D^i f(a) = \frac{\partial(f\circ\pi_U{}^{-1})}{\partial z_i}(\pi_U(a))$$

if the latter expression exists. Here a is a point in $\tilde{U}$ and f is a function defined near a.

We now define a function p^* on $\tilde{U}$ by $p^*(x)=p(\pi_U(x))$ for all $x\in\tilde{U}$. Since $\pi_U(\tilde{U})$ is close to K we can assume (shrinking the neighborhood U_1 of K a little if necessary) that $p(z)$ is defined on this set, and then p^* is well-defined on $\tilde{U}$. With derivation as above we see that p^* is plurisubharmonic at $x\in\tilde{U}$ if and only if p is plurisubharmonic at $\pi_U(x)\in\mathbf{C}^n$.

$\tilde{U}$ being a Stein manifold, there exists a real-valued function $\psi\in C^\infty(\tilde{U})$ which is strictly plurisubharmonic on $\tilde{U}$ and has the following property:

$$\{x\in\tilde{U}:\ \psi(x)<r\}\subset\subset\tilde{U}\quad\text{for every } r\in\mathbf{R}^+$$

(see [10]). Let $M=\|\psi\|_L$ (under the identification $U\subsetneq\tilde{U}$) and define

$$\varphi(x) = \sup\{p^*(x),\psi(x)-M-1\}\quad\text{for all } x \text{ in } \tilde{U}\ .$$

Let $V_\varepsilon=\{x\in\tilde{U}:\varphi(x)-\varepsilon<0\}$ with ε as before. Due to the function ψ, V_ε is then a relatively compact subset of $\tilde{U}$. Since $\varphi\leqq 0$ on L, we see that $L\subset V_\varepsilon$. For every x in V_ε, $p^*(x)=p(\pi_U(x))<\varepsilon$ and furthermore $\pi_U(x)\in U_1$ since $V_\varepsilon\subset\tilde{U}$. This means that $\pi_U(V_\varepsilon)\subset W$ and hence p^* is plurisubharmonic on V_ε. Being the supremum of two plurisubharmonic functions φ is also plurisubharmonic on V_ε.

With $N=\|\psi\|_{V_\varepsilon}$ we put $\alpha(x)=(\varepsilon/4N)\psi(x)$ and define

$$V_0 = \{x\in V_\varepsilon:\ \varphi(x)-\tfrac{1}{2}\varepsilon+\alpha(x)<0\}\ .$$

From the fact that $\|\alpha\|_{V_\varepsilon}\leqq\frac{1}{4}\varepsilon$ we see that $L\subset V_0\subset\subset V_\varepsilon$. Furthermore the function $\varphi(x)+\alpha(x)-\frac{1}{2}\varepsilon$ is strictly plurisubharmonic on V_ε. Hence V_0 is a strictly pseudoconvex and relatively compact domain in the Stein manifold $\tilde{U}$. So now we can use theorem 3 to infer that V_0 is holomorphically convex. As a subset of a Stein manifold V_0 is then a Stein manifold itself.

Next we define V as $V_0\cap U$. Then V is an open neighborhood on L in $\mathbf{C}^n$ and we want to prove that $\pi_V(\tilde{V})\subset W$.

The mapping $\pi_V\colon\tilde{V}\to\mathbf{C}^n$ can be factored as follows. If x is a homomorphism in V we first take its restriction to the algebra $O(\tilde{U})\subset O(V)$. Since $\tilde{U}$ is a Stein manifold this new homomorphism is a point $\pi_1(x)$ in $\tilde{U}$. Then restricting this homomorphism to the algebra of polynomials in the coordinate functions for $\mathbf{C}^n$ we get a point $\pi_U(\pi_1(x))\in\mathbf{C}^n$. Clearly

$\pi_U \circ \pi_1(x) = \pi_V(x)$ as the latter mapping is obtained by taking the restriction of x to the polynomials at once. It might be helpful to visualize this result in a commutative diagram:

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\pi_1} & \tilde{U} \\ \cup \big\uparrow & \searrow^{\pi_V} & \big\downarrow \pi_U \\ V & \xrightarrow{\subset} & \mathbf{C}^n \end{array}$$

Now V is a subset of V_0 and as V_0 is a Stein manifold, $\pi_1(\tilde{V}) \subset V_0$. But $V_0 \subset V_\varepsilon$ and, as shown before, $\pi_U(V_\varepsilon) \subset W$. Hence $\pi_V(\tilde{V}) = \pi_U \circ \pi_1(\tilde{V}) \subset W$, so V is the sought for neighborhood of L. This proves our theorem.

3. The union of two compact natural sets.

In [3] it was proved that an intersection of natural sets always is natural. But of course it is generally not true that a union of natural sets is natural. We will now discuss a case where this actually is true.

Let A and S be as before. We say that a subset V of S is an A-analytic variety in S if for every x in S there exists a neighborhood U_x of x in S such that the set $V \cap U_x$ consists of the common zeros for some family (which may be infinite) of functions that are defined and are A-holomorphic in U_x. V is then obviously a closed set in S.

THEOREM 5. *Let K_1 be a compact subset of an A-analytic variety V in S, let K_2 be a compact natural set in S and set $K = K_1 \cup K_2$. Suppose furthermore that*

(i) *$K \cap V$ is natural;*

(ii) *$H_A(K) | K \cap V$ is dense in $H_A(K \cap V)$;*

(iii) *$H_A(K) | K_2$ is dense in $H_A(K_2)$.*

Then K is a natural set.

PROOF. Put $D = M_{H_A(K)} \setminus K$ and suppose that $D \neq \emptyset$. The mapping

$$\pi : M_{H_A(K)} \to S$$

is continuous so $\Omega = D \setminus \pi^{-1}(V)$ is an open subset of $M_{H_A(K)}$. We assume that $\Omega \neq \emptyset$.

If $\operatorname{bd}\Omega$ is the topological boundary of Ω in $M_{H_A(K)}$ we let

$$\begin{aligned} L &= \operatorname{Hull}_{H_A(K)}(\operatorname{bd}\Omega) \\ &= \{x \in M_{H_A(K)} : \ |f(x)| \leqq \|f\|_{\operatorname{bd}\Omega}, \ \forall f \in H_A(K)\} \end{aligned}$$

(here and in the following we in general do not distinguish between a function and its Gelfand transform). We define the algebra B as the uniform closure of $H_A(K)|\Omega$ in $C(\bar{\Omega})$. Then $\Omega \subseteqq L = M_B \subseteqq M_{H_A(K)}$ and from Rossi's local maximum modulus principle we infer that

$$\partial_B \subseteqq \mathrm{bd}\Omega \subseteqq \pi^{-1}(V) \cup (K_2 \setminus V) ,$$

where ∂_B is the Shilov boundary of B.

Now ∂_B is the closure of the set of strong boundary points for B. Suppose that all these points are situated in $(K_2 \setminus V)^-$ (the bar denotes closure). Then $\partial_B \subseteqq (K_2 \setminus V)^-$ and consequently every point in Ω has a minimal support in K_2. If $x \in \Omega$ has the support $F_x \subset K_2$ it follows that $|x(f)| \leqq \|f\|_{F_x} \leqq \|f\|_{K_2}$ for all $f \in H_A(K)$. Because $H_A(K)|K_2$ is dense in $H_A(K_2)$, x can then be lifted to a homomorphism of $H_A(K_2)$. But K_2 is natural, so x must be a point in K_2, which contradicts the fact that $x \in \Omega$.

Hence there exists a strong boundary point δ for B such that $\delta \in \pi^{-1}(V) \setminus (K_2 \setminus V)^-$ and $\delta \in \mathrm{bd}\Omega$. Then $\pi(\delta)$ belongs to V. We let U be an open neighborhood of $\pi(\delta)$ in S such that

$$V \cap U$$
$$= \{x \in S : f_\alpha(x) = 0, \text{ where the } f_\alpha\text{'s are defined and } A\text{-holomorphic in } U\} .$$

For $x \in U$ there exists a neighborhood N_x of x in U and functions $\{f_{\alpha,n}\} \subset A$ such that

$$\lim_{n\to\infty} \|f_\alpha - f_{\alpha,n}\|_{N_x} = 0 .$$

The Gelfand transform of $f_{\alpha,n}$ is $f_{\alpha,n} \circ \pi$. Since

$$\lim_{n\to\infty} \|f_\alpha \circ \pi - f_{\alpha,n} \circ \pi\|_{\pi^{-1}(N_x)} = \lim_{n\to\infty} \|f_\alpha - f_{\alpha,n}\|_{N_x} = 0 ,$$

we see that $f_\alpha \circ \pi$ is $H_A(K)$-holomorphic on $\pi^{-1}(U)$ (note that this argument was used already in the proof of theorem 2, although it was not written out in detail there).

Let U_0 be an open neighborhood of δ in $M_{H_A(K)}$ such that $U_0 \cap (K_2 \setminus V)^- = \emptyset$ and $U_0 \subset \pi^{-1}(U)$. From the fact that δ is a strong boundary point for B it follows from a lemma of Rickart (see the proof of lemma 3.1 in [14]) that there is a neighborhood U_1 of δ with $U_1 \subset U_0$ and such that all functions which are $H_A(K)$-holomorphic on $U_0 \cap M_B$ and vanish on $U_0 \cap \pi^{-1}(V) = U_0 \cap \mathrm{bd}\Omega$ also vanish on $U_1 \cap M_B$. In particular, this is true for the functions $f_\alpha \circ \pi$. Now $\Omega \subseteqq M_B$ and since $\delta \in \mathrm{bd}\Omega$ there are points x_ν in Ω converging towards δ. For ν large enough, $x_\nu \in U_1$, that is, $f_\alpha \circ \pi(x_\nu) = 0$ for all α. But this means that $x_\nu \in \pi^{-1}(V)$

when ν is sufficiently large which is a contradiction. Hence $\Omega = \emptyset$ and $D = \pi^{-1}(V) \setminus V$.

The topological boundary $\mathrm{bd}D$ of D in $M_{H_A(K)}$ is situated in $K \cap V$. For otherwise there exists $\{y_n\} \subset D$ with

$$y_n \to y \in \mathrm{bd}D \cap (K \setminus V) .$$

π being continuous it follows that $\pi(y_n) \to \pi(y) = y$. But $\pi(y_n) \in V$ and V is closed, so $y \in V$ too, which is a contradiction.

From the local maximum modulus principle it now follows that $|x(f)| \leqq \|f\|_{K \cap V}$ for every $x \in D$ and for all $f \in H_A(K)$. Since $H_A(K) \mid K \cap V$ is dense in $H_A(K \cap V)$ every $x \in D$ therefore defines an element in $H_A(K \cap V)$. Due to the fact that $K \cap V$ is natural, x must then be a point in $K \cap V$. Hence $D = \emptyset$, which proves the theorem.

Conversely, if K is natural it follows from a theorem of Rickart (theorem 3.2 and lemma 1.1 in [14]) that $K \cap V$ also is natural, so condition (i) is necessary.

Suppose now that S is a Stein manifold and that A is its algebra of holomorphic functions. For this case it was shown in [9] that theorems A and B of Cartan are valid for coherent analytic sheaves on compact natural sets. As a direct consequence of this every element in $O_A(K \cap V)$ is the restriction to V of an element in $O_A(K)$, if V is an analytic variety in S. So then also condition (ii) in theorem 5 is necessary.

As Rickarts theorem is very easy when S is a Stein manifold, we give a proof here for that special case.

Proposition. *Let K be a compact natural set in a Stein manifold S and let V be an analytic variety in S. Then $K \cap V$ is a natural set.*

Proof. Theorems A and B imply that

$$K \cap V = \{x \in K : f_i(x) = 0 \text{ for } f_i \in O_A(K)\} ,$$

that is, $K \cap V$ is globally defined. Since $A = O(S) \subsetneq H(K) \subsetneq H(K \cap V)$ and K is natural, we can in the usual way define mappings making the diagram

$$\begin{array}{ccc} M_{H_A(K \cap V)} & \longrightarrow & \\ \downarrow \pi & & \downarrow \pi_1 \\ S \supsetneq K & \xleftarrow{\ \cong\ } & M_{H_A(K)} \end{array}$$

commutative. According to theorem 2 it suffices to show that $\pi(M_{H_A(K\cap V)})=K\cap V$, and from the diagram we see that it is even sufficient to show that $\pi_1(M_{H_A(K\cap V)})=K\cap V$.

The functions $f_i \in H_A(K)$ defining $K\cap V$ can be considered as elements in $H_A(K\cap V)$, and their Gelfand transforms on $M_{H_A(K\cap V)}$ are $f_i\circ\pi_1$. Let x be an arbitrary point in $M_{H(K\cap V)}$. Then $|x(f_i)| \leqq \|f_i\|_{K\cap V}=0$, that is, $f_i(\pi_1(x))=x(f_i)=0$ for all i. Hence $\pi_1(x) \in K\cap V$, which proves the proposition.

In order to see that theorem 5 may fail if not all of the conditions (i), (ii) and (iii) are satisfied, we consider the following example in $\mathbf{C}^2$. Let

$$\begin{aligned} K_1 &= \{z \in \mathbf{C}^2 :\ z_1=0,\ b \leqq |z_2| \leqq d\}\ , \\ K_2 &= \{z \in \mathbf{C}^2 :\ 0 \leqq |z_1| \leqq a,\ b < c \leqq |z_2| \leqq d\}\ , \\ V &= \{z \in \mathbf{C}^2 :\ z_1=0\} \end{aligned}$$

and set $K=K_1\cup K_2$.

For $b>0$ all conditions in theorem 5 are fulfilled and K is natural. If $b=0$ condition (i) and (ii) are still valid, but condition (iii) is no longer satisfied. For instance, the function z_2^{-1} belongs to $H(K_2)$ but cannot be approximated with functions from $H(K)$. In this case

$$M_{H(K)} = \{z \in \mathbf{C}^2 :\ 0 \leqq |z_1| \leqq a,\ 0 \leqq |z_2| \leqq d\} \supsetneqq K\ .$$

We also give an example which indicates the difficulty in generalizing theorem 5 to give a positive answer to the following question: If K_1, K_2 and $(K_1\cup K_2)\cap V$ are natural, is then also $K_1\cup K_2$ natural? Let

$$\begin{aligned} K_1 &= \{z \in \mathbf{C}^2 :\ 0 \leqq |z_1| \leqq a,\ |z_2| = b\} \cup \\ &\quad \cup \{z \in \mathbf{C}^2 :\ z_1=0,\ b \leqq |z_2| \leqq d\}\ , \\ K_2 &= \{z \in \mathbf{C}^2 :\ 0 \leqq |z_1| \leqq a,\ |z_2| = e\} \cup \\ &\quad \cup \{z \in \mathbf{C}^2 :\ z_1=0,\ c \leqq |z_2| \leqq e\}\ , \end{aligned}$$

where $0<b<c<d<e$ and let $V=\{z \in \mathbf{C}^2: z_1=0\}$. Then

$$M_{H(K_1\cup K_2)} = \{z \in \mathbf{C}^2 :\ 0 \leqq |z_1| \leqq a,\ b \leqq |z_2| \leqq e\} \supsetneqq K_1 \cup K_2\ ,$$

so $K_1\cup K_2$ is not natural.

4. Holomorphic mappings of compact natural sets.

As was hinted at in section 1, it makes sense to talk about compact natural sets in Stein spaces. Actually, of course, a compact set in an analytic space is nothing but a compact set in some analytic variety in some $\mathbf{C}^n$.

If $F: X \to Y$ is a proper, surjective, holomorphic and finite mapping from an analytic space X to a Stein space Y, it is well-known that also X is Stein (see [7] for instance). Now an analytic space X is Stein precisely when X is equal to the spectrum of the algebra $O(X)$ of holomorphic functions on X (see [1]). Therefore it is natural to ask whether the inverse image under F of a natural set is natural. As an easy consequence of theorem 1 we obtain the following result.

THEOREM 6. *Let X and Y be Stein spaces and let $F: X \to Y$ be a proper, holomorphic and surjective mapping. If K is a compact natural set in Y, then the compact set $F^{-1}(K)$ in X is natural too.*

PROOF. The holomorphy of F means that we have an injection $O(Y) \hookrightarrow O(X)$ given by $O(Y) \ni f \mapsto f \circ F$. So we can consider $O(Y)$ as a subalgebra of $O(X)$.

Let $\{U_\alpha\}$ be a fundamental system of open neighborhoods of K in Y. If V is an open neighborhood of $F^{-1}(K)$ in X, the topological boundary $\operatorname{bd} V$ of V is closed, and hence also $F(\operatorname{bd} V)$ is closed since F is proper. Because of the fact that $K \cap F(\operatorname{bd} V) = \emptyset$ (here it is important that $F^{-1}(K)$ is an inverse set), there is an open neighborhood U of K in Y such that $U \cap F(\operatorname{bd} V) = \emptyset$ and then $F^{-1}(K) \subset F^{-1}(U) \subset V$. We set $V_\alpha = F^{-1}(U_\alpha)$ and infer that $\{V_\alpha\}$ is a fundamental system of open neighborhoods of $F^{-1}(K)$ in X (cf. lemma 4D on p. 292 in Whitney [18]).

Fix an α for a moment. The restriction F_α of F to V_α is a holomorphic mapping F_α: $V_\alpha \to U_\alpha$ and hence we have an injection $O(U_\alpha) \hookrightarrow O(V_\alpha)$. If φ is a homomorphism in $\tilde{V}_\alpha$ we can restrict it first to $O(X)$ and then to $O(Y)$; obviously we obtain the same result by first taking the restriction to $O(U_\alpha)$ and then to $O(Y)$, that is, we have the commutative diagram

$$\begin{array}{ccc} \tilde{V}_\alpha & \xrightarrow{R_\alpha} & \tilde{U}_\alpha \\ {\scriptstyle \pi_{V_\alpha}}\downarrow & & \downarrow{\scriptstyle \pi_{U_\alpha}} \\ X & \xrightarrow{F} & Y \end{array}$$

where $R_\alpha(\varphi) = \varphi \mid O(U_\alpha)$. To prove that $F^{-1}(K)$ is natural we need, according to theorem 1, only to show that for a given open neighborhood W of $F^{-1}(K)$ there exists an α such that $\pi_{V_\alpha}(\tilde{V}_\alpha) \subset W$. But since K is natural there exists an α such that

$$\pi_{U_\alpha} \circ R_\alpha(\tilde{V}_\alpha) \subset \pi_{U_\alpha}(\tilde{U}_\alpha) \subset W' ,$$

where W' is a neighborhood of K such that $F^{-1}(K) \subset F^{-1}(W') \subset W$. From the commutativity in the diagram it then follows that $\pi_{V_\alpha}(\tilde{V}_\alpha) \subset W$, which proves the theorem.

A natural question now is the following: Suppose that K is a compact set in X such that the compact set $F(K)$ in Y is natural. Does it follow that K also is natural? We give an example which shows that this is not true in general. Namely, let

$$X = \{z \in \mathbf{C}^3 : (z_3 - z_2)(z_3 + z_2) = 0\} = X_1 \cup X_2 ,$$

where X_1 and X_2 are the components of X, let

$$Y = \{z \in \mathbf{C}^3 : z_3 = 0\}$$

and let $F: X \to Y$ be the restriction to X of the projection $\mathbf{C}^3 \to \mathbf{C}^2$ taking (z_1, z_2, z_3) to (z_1, z_2). Then F has all the properties required in the theorem. Let D be the closed unit polydisc in Y and put $K = K_1 \cup K_2$, where $K_1 = F^{-1}(D) \cap X_1$ and

$$K_2 = F^{-1}(D \setminus \{z \in Y : 0 \leqq |z_1|, |z_2| < \tfrac{1}{2}\}) \cap X_2 .$$

Then $F(K) = D$, D is natural, but clearly K is not.

In [13] Remmert and Stein prove the following theorem: Let X and Y be connected normal analytic spaces and let $F: X \to Y$ be a proper, discrete, surjective, holomorphic mapping. Then X is Stein precisely when Y is and X is holomorphically convex precisely when Y is. For a related result we refer to [11]. Using the methods of Remmert and Stein we can prove the following analogue for natural sets.

Theorem 7. *Let X, Y be Stein spaces and suppose that Y is normal as well. Let the mapping $F: X \to Y$ be proper, holomorphic, surjective and be such that each irreducible component of X is mapped onto an irreducible component of Y. If K is a compact set in Y such that $F^{-1}(K)$ is natural, then K is natural.*

Remark. The condition that F maps each irreducible component of X onto an irreducible component of Y is used only to ascertain that the inverse image of the set of singular points in Y is a not too big set in any component of X as is seen from the proofs of Satz 2 and Satz 3 in [13]. So, for instance, if the singular set in Y is empty, this condition is superfluous.

Proof. We observe firstly that F is automatically finite since X is a Stein space and hence has point separating holomorphic functions. So $F: X \to Y$ is proper, finite, holomorphic, surjective and maps each irreducible component of X onto an irreducible component of Y; to simplify the notation we say that a mapping with these properties is admissible. Let (X^*, ξ) be the normalization of X. Then ξ is admissible and it follows that X^* is Stein. We can now use Satz 2 in [13] to infer that there exists a proper holomorphic mapping $F^*: X \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} X^* & \longrightarrow & \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle F^*} \\ X & \xrightarrow{F} & Y \end{array}$$

commutes. From the properties of the normalization it follows that F^* is admissible and that

$$\xi^{-1}(F^{-1}(K)) = (F^*)^{-1}(K)\,.$$

To prove the theorem it suffices to show the implications:

$$\begin{aligned} & F^{-1}(K) \text{ is natural} \\ \Rightarrow\ & \xi^{-1}(F^{-1}(K)) = (F^*)^{-1}(K) \text{ is natural} \\ \Rightarrow\ & K \text{ is natural}\,. \end{aligned}$$

The first implication follows from theorem 6 and the second from the lemma below.

LEMMA. *Let X, Y be normal Stein spaces and let $F: X \rightarrow Y$ be an admissible mapping. If K is a compact set in Y such that $F^{-1}(K)$ is natural, then K is natural.*

PROOF. Let $\{U_\alpha\}$ be a fundamental system of open neighborhoods of K in Y and set $V_\alpha = F^{-1}(U_\alpha)$. Then $\{V_\alpha\}$ form a fundamental system of open neighborhoods of $F^{-1}(K)$ in X. Define F_α: $V_\alpha \rightarrow U_\alpha$ as the restriction of F to V_α. Then clearly all the F_α's are admissible and V_α, U_α are all normal analytic spaces.

We now fix an α and as in the proof of theorem 6 we obtain a commutative diagram:

$$\begin{array}{ccc} \tilde{V}_\alpha & \xrightarrow{R_\alpha} & \tilde{U}_\alpha \\ {\scriptstyle \pi V_\alpha}\downarrow & & \downarrow{\scriptstyle \pi U_\alpha} \\ X & \xrightarrow{F} & Y \end{array}$$

We claim that R_α is a surjective mapping. Indeed, Satz 3 in [13] shows that the triple $(V_\alpha, F_\alpha, U_\alpha)$ is an analytic covering and then it readily follows that $O(V_\alpha)$ is integral over the subalgebra $O(U_\alpha)$. Let $\varphi: O(U_\alpha) \to \mathbf{C}$ be an arbitrary element in $\tilde{U}_\alpha$. Due to the integral dependence this homomorphism can be lifted to a homomorphism $\Phi: O(V_\alpha) \to \mathbf{C}$ (see [5] for instance), and we only have to prove that Φ is continuous. To see this we recall from [13] how the integral dependence is explained.

If p is the number of sheets of V_α over U_α we set for every $f \in O(V_\alpha)$:

$$\begin{aligned} P_f(z) &= (z-f(w_1)) \ldots (z-f(w_p)) \\ &\equiv z^p + a_1 z^{p-1} + \ldots + a_p , \end{aligned}$$

where $(w_1, \ldots, w_p)$ are the points (not necessarily distinct) over $w \in U_\alpha$. Then the functions a_i can be considered as elements in $O(U_\alpha)$ and obviously $P_f(f)=0$. For every $\varphi \in \tilde{U}_\alpha$ there exists a compact set $L \subset U_\alpha$ such that $|\varphi(g)| \leqq \|g\|_L$ for all $g \in O(U_\alpha)$. Then

$$|\Phi(f)| \leqq \|f\|_{F^{-1}(L)} \quad \text{for all } f \in O(V_\alpha) ,$$

that is, Φ is continuous. For suppose not. Using the multiplicative structure of $O(V_\alpha)$ there then exists a function $f \in O(V_\alpha)$ such that

$$\|f\|_{F^{-1}(L)} = 1 \quad \text{and} \quad |\Phi(f)| = A ,$$

where A can be chosen as large as we want. Then $|\Phi(a_i)| \leqq \|a_i\|_L \leqq p$ for every i, and hence

$$\Phi^p(f) + \Phi(a_1) \cdot \Phi^{p-1}(f) + \ldots + \Phi(a_p) = 0$$

cannot possibly be true if A is large enough. Hence R_α is surjective.

To finish the proof we fix an open neighborhood W of K in Y. Then there exists an α such that $\pi_{V_\alpha}(\tilde{V}_\alpha) \subset F^{-1}(W)$ and since R_α is surjective,

$$\pi_{U_\alpha}(\tilde{U}_\alpha) = \pi_{U_\alpha} \circ R_\alpha(\tilde{V}_\alpha) = F \circ \pi_{V_\alpha}(\tilde{V}_\alpha) \subset W .$$

This proves the lemma and with it the theorem.

Also in this case it is tempting to ask the following: Let $F: X \to Y$ be as in the theorem. Then if K is a compact natural set in X, is it true that $F(K)$ is natural? The following example shows that the answer in general is no.

Let D be the polydisc

$$D = \{z \in \mathbf{C}^3 : \ 0 \leqq |z_1| \leqq 2,\ 0 \leqq |z_2| \leqq 2,\ 0 \leqq |z_3| \leqq 1\} .$$

Let

$$X = \{z \in \mathbf{C}^3 : \ (z_3 - z_1)(z_3 - z_2) = 0\}$$

and set $K = D \cap X$. Clearly D is natural and the theorem of Rickart which we have used already in section 3 shows that K is natural too. Let $Y = \{z \in \mathbf{C}^3 : z_3 = 0\}$ and let $F: X \to Y$ be the restriction to X of the projection $\mathbf{C}^3 \to \mathbf{C}^2$. Then

$$F(K) = \{z \in Y : 0 \leqq |z_1|, |z_2| \leqq 2\} \setminus \{z \in Y : 1 < |z_1|, |z_2| \leqq 2\}$$

and hence is not natural.

Finally we remark that M. van Kuilenberg in [12] independently has obtained some results in connection with theorems 6 and 7.

NOTE ADDED IN PROOF. Theorem 3 is valid if the strictly plurisubharmonic function $p(z)$ is merely upper-semicontinuous as is shown by Narasimhan in *The Levi problem for complex spaces* II, Math. Ann. 146 (1962), 195–216. This implies that theorem 4 is true under the assumption that $p(z)$ is a continuous (and not necessarily C^2) plurisubharmonic function (alternatively this result can be achieved by regularization of the function φ appearing in the proof).

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