MATH. SCAND. 33 (1973), 375–381

LIMIT MEASURES ON COMPACT SEMITOPOLOGICAL SEMIGROUPS

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Let S be a compact semitopological semigroup, i.e. the multiplication is separately continuous. A probability measure on S is a positive regular Borel measure with norm one, and the set $P(S)$ of probability measures is also a compact semitopological semigroup under convolution and the weak* topology [6]. In this paper we are concerned with limit theorems about $P(S)$ by extending some results known for compact jointly continuous topological semigroups to the separately continuous situation.

For a measure $\mu \in P(S)$ we use $\operatorname{supp}\mu$ to denote its support which is the smallest closed subset of S with μ-mass one. A very useful property of probability measures is formulated in the next theorem, where the bar denotes closure.

THEOREM 1. (Glicksberg [6].) *For* $\mu,\nu = P(S)$,

$$\operatorname{supp}\mu\nu = (\operatorname{supp}\mu \operatorname{supp}\nu)^- .$$

It follows by induction and [1, II.3.1] that for any finite family of measures $\mu_i \in P(S)$, $i=1,\ldots,n$,

$$\operatorname{supp}(\mu_1 \ldots \mu_n) = (\operatorname{supp}\mu_1 \ldots \operatorname{supp}\mu_n)^- .$$

A semigroup S is said to be topologically simple if every two-sided ideal is dense.

THEOREM 2. (Pym [8].) *The support of an idempotent measure in* $P(S)$ *is a topologically simple semigroup in* S.

The result in Theorem 2 is best possible in the sense that the word "topologically" cannot be omitted, for it may occur that the support of an idempotent measure is not its own minimal ideal. This can be seen from the idempotent η given in Example 2 of [3]. Incidentally, we can also see from that example — however, one has no difficulty in obtain-

Received March 12, 1973.

ing simpler examples — that the result in Theorem 1 requires the closure sign since $\operatorname{supp}\eta \operatorname{supp}\eta \neq \operatorname{supp}\eta$.

For a compact jointly continuous semigroup the following theorem was stated in [4, § 3.5], and interested readers can find references listed there.

THEOREM 3. *Suppose $\mu \in P(S)$ and let $\nu_N = N^{-1}(\mu + \ldots + \mu^N)$. Then the sequence (ν_N) converges, as $N \to \infty$, to an idempotent measure $L(\mu) \in P(S)$. Moreover, $L(\mu)\mu = \mu L(\mu) = L(\mu)$ and*

$$\operatorname{supp} L(\mu) = \big(K(S(\mu))\big)^- ,$$

where $K(S(\mu))$ is the minimal ideal of the compact semigroup $S(\mu)$ generated by $\operatorname{supp}\mu$.

PROOF. The existence of the limit $L(\mu)$ with the property that $L(\mu)\mu = \mu L(\mu) = L(\mu)$ and that $\operatorname{supp} L(\mu)$ is an ideal in

$$S(\mu) = \left(\bigcup_{i=1}^{\infty} (\operatorname{supp}\mu)^i\right)^-$$

can be proved by an argument similar to that given in Proposition 3.4 of [7]. Thus

$$\operatorname{supp} L(\mu) \supseteq K(S(\mu)) .$$

Since $K(S(\mu))$ is an ideal of $\operatorname{supp} L(\mu)$, the result follows from Theorem 2.

Throughout the remainder of this paper the notations in Theorem 3 will be used.

We say a measure $\mu \in P(S)$ is central if $\mu\nu = \nu\mu$ for all $\nu \in P(S)$.

THEOREM 4. *Suppose $\mu, \nu \in P(S)$. If $L(\mu\nu)$ and $L(\nu\mu)$ are central, then $L(\mu\nu) = L(\nu\mu)$.*

PROOF. Since $L(\mu\nu)\mu\nu = L(\mu\nu)$ which is central, we see that

$$\begin{aligned} L(\mu\nu)(\nu\mu)^i &= \\ &= L(\mu\nu)\nu\mu\nu\mu \ldots \nu\mu = \nu\big(L(\mu\nu)\mu\nu\big)\mu \ldots \nu\mu = \\ &= \nu L(\mu\nu)\mu \ldots \nu\mu = \ldots = \nu L(\mu\nu)\mu = L(\mu\nu)\nu\mu \end{aligned}$$

for all i, and so

$$L(\mu\nu)N^{-1}\textstyle\sum_{i=1}^{N} (\nu\mu)^i = L(\mu\nu)\nu\mu .$$

It follows that $L(\mu\nu)L(\nu\mu) = L(\mu\nu)\nu\mu$. Thus

$$\mu L(\mu\nu)L(\nu\mu)\nu = \mu L(\mu\nu)\nu\mu\nu = L(\mu\nu)\mu\nu\mu\nu = L(\mu\nu)\mu\nu = L(\mu\nu) .$$

As $\mu L(\mu\nu)L(\nu\mu)\nu = L(\mu\nu)\mu\nu L(\nu\mu) = L(\mu\nu)L(\nu\mu)$, we get

$$L(\mu\nu) = L(\mu\nu)L(\nu\mu) .$$

Similarly $L(\nu\mu) = L(\mu\nu)L(\nu\mu)$, and the result follows.

The theorem of § 3.6 of [4] for compact jointly continuous semigroups is improved and extended to compact semitopological semigroups in the next theorem.

THEOREM 5. *Suppose $\mu, \nu \in P(S)$ such that $L(\mu\nu)$ and $L(\nu)$ are central. If the family*

$$\{\operatorname{supp} \mu^i : i = 1, 2, \ldots\}$$

has the finite intersectiom property, then $L(\mu\nu) = L(\mu)L(\nu)$.

PROOF. Since $L(\mu\nu) = L(\mu\nu)\mu\nu$ and $L(\mu\nu)$ is central, it follows that

$$L(\mu\nu) = \mu^i L(\mu\nu)\nu^i, \quad i = 1, 2, \ldots .$$

Because S is compact we see that $\bigcap_{i=1}^{\infty} \operatorname{supp} \mu^i \neq \emptyset$, that is, there exists a point $a \in \operatorname{supp} \mu^i$ for all i. Hence for each i,

$$a \operatorname{supp} L(\mu\nu) \operatorname{supp} \nu^i \subseteq \operatorname{supp} \mu^i \operatorname{supp} L(\mu\nu) \operatorname{supp} \nu^i \subseteq \operatorname{supp} \mu^i L(\mu\nu)\nu^i = \operatorname{supp} L(\mu\nu) .$$

Consequently,

$$a \operatorname{supp} L(\mu\nu) \operatorname{supp} L(\nu) \subseteq \operatorname{supp} L(\mu\nu) ,$$

whence we have, by Lemma 3 of [8], that

$$L(\mu\nu) = L(\mu\nu)\delta(a)L(\mu\nu)L(\nu)L(\mu\nu) = \delta(a)L(\mu\nu)L(\nu) ,$$

where $\delta(a)$ is the unit point mass at a. Since $L(\nu)$ is central, we obtain

$$\begin{aligned} N^{-1}\textstyle\sum_{i=1}^{N} (\mu\nu)^i L(\nu) &= N^{-1}\textstyle\sum_{i=1}^{N} (\mu\nu L(\nu))^i \\ &= N^{-1}\textstyle\sum_{i=1}^{N} (\mu L(\nu))^i = N^{-1}\textstyle\sum_{i=1}^{N} \mu^i L(\nu) \end{aligned}$$

and so $L(\mu\nu)L(\nu) = L(\mu)L(\nu)$. It follows that

$$\begin{aligned} \delta(a)L(\mu\nu)L(\nu) &= L(\mu\nu)L(\nu)\delta(a)L(\mu\nu)L(\nu) = L(\mu)L(\nu)\delta(a)L(\mu)L(\nu) \\ &= L(\mu)\delta(a)L(\mu)L(\nu) = L(\mu)L(\mu)\delta(a)L(\mu)L(\nu) \\ &= L(\mu)L(\nu) , \end{aligned}$$

by applying again Lemma 3 of [8] since $\operatorname{supp} L(\mu)\delta(a) \subseteq \operatorname{supp} L(\mu)$. Hence $L(\mu\nu) = L(\mu)L(\nu)$.

COROLLARY 6. *Suppose* $\mu,\nu \in P(S)$ *such that* $L(\mu\nu)$ *and* $L(\nu)$ *are central. If* $\operatorname{supp}\mu \subseteqq \operatorname{supp}\mu^2$, *or there exists an idempotent* $e \in \operatorname{supp}\mu$, *then* $L(\mu\nu) = L(\mu)L(\nu)$.

PROOF. If $\operatorname{supp}\mu \subseteqq \operatorname{supp}\mu^2$, we can deduce that

$$\operatorname{supp}\mu^2 \subseteqq \operatorname{supp}\mu^3 \subseteqq \ldots \subseteqq \operatorname{supp}\mu^n \subseteqq \ldots .$$

Thus $\operatorname{supp}\mu \subseteqq \bigcap_{i=1}^{\infty} \operatorname{supp}\mu^i$. If there exists an idempotent $e \in \operatorname{supp}\mu$, then

$$e \in \bigcap\nolimits_{i=1}^{\infty} \operatorname{supp}\mu^i .$$

The result follows immediately from Theorem 5.

THEOREM 7. *Suppose* $\mu \in P(S)$. *Then* $L(\mu) = L(\mu^n)$ *if and only if*

$$\operatorname{supp}L(\mu) = \operatorname{supp}L(\mu^n), \quad n = 1,2,\ldots .$$

PROOF. Assume that $\operatorname{supp}L(\mu) = \operatorname{supp}L(\mu^n)$. Then Lemma 3 of [8] implies

$$L(\mu^n)L(\mu)L(\mu^n) = L(\mu^n) .$$

On the other hand, by Theorem 3 we have $L(\mu)\mu^i = \mu^i L(\mu) = L(\mu)$ for all i and so

$$L(\mu)L(\mu^n) = L(\mu^n)L(\mu) = L(\mu) .$$

Accordingly, $L(\mu^n)L(\mu)L(\mu^n) = L(\mu)$, and the result follows.

Theorem 1 of [5] shows that if S is a compact group and $\operatorname{supp}\mu$ contains the identity then

$$L(\mu) = L(\mu^n), \quad n = 1,2,\ldots .$$

Clearly it is an immediate consequence of the corollary below.

COROLLARY 8. *For* $\mu \in P(S)$, *if* $\operatorname{supp}\mu \subseteqq (\operatorname{supp}\mu)^2$, *then*

$$L(\mu) = L(\mu^n), \quad n = 1,2,\ldots .$$

PROOF. Clearly $(\operatorname{supp}\mu)^i \subseteqq (\operatorname{supp}\mu)^{i+1}$ for all i. It follows that $S(\mu) = S(\mu^n)$, $n = 1,2,\ldots$. Thus

$$\operatorname{supp}L(\mu) = \big(K(S(\mu))\big)^- = \big(K(S(\mu^n))\big)^- = \operatorname{supp}L(\mu^n) .$$

Then apply Theorem 7 to complete the proof.

Now we come to the monothetic semigroup $\Gamma(\mu)$ generated by a measure $\mu \in P(S)$, that is

$$\Gamma(\mu) = \{\mu^i : 1,2,\dots\}^- .$$

Since $\Gamma(\mu)$ is a compact commutative semigroup [1, p. 67], its minimal ideal $K(\Gamma(\mu))$ is a compact topological group with identity $E(\mu)$ [1, II 3.3]. Note that $\operatorname{supp} E(\mu) \subseteq S(\mu)$; for if not, by Urysohn's lemma we can find a continuous function f on S such that $f(S(\mu)) = 0$ and $E(\mu)(f) > 0$. But there exists a subnet (μ^α) of the sequence (μ^i) such that $\mu^\alpha \to E(\mu)$, that is $\mu^\alpha(f) \to E(\mu)(f)$. Since $\mu^\alpha(f) = 0$, it follows that $E(\mu)(f) = 0$, a contradiction.

THEOREM 9. *The following statements are equivalent:*

(1) $L(\mu) = E(\mu)$,
(2) $K(\Gamma(\mu)) = \{E(\mu)\}$,
(3) $(\operatorname{supp} E(\mu) \operatorname{supp} \mu \operatorname{supp} E(\mu))^- = \operatorname{supp} E(\mu)$,
(4) $\operatorname{supp} L(\mu) \subseteq \operatorname{supp} E(\mu)$.

PROOF. (1) implies (2). As $L(\mu)\mu = \mu L(\mu) = L(\mu)$ and $L(\mu) = E(\mu) \in \Gamma(\mu)$, we see that $L(\mu)$ is the zero element in the semigroup $\Gamma(\mu)$ and so (2) follows.

(2) implies (3). Obviously we have $E(\mu)\mu E(\mu) = E(\mu)$ which gives (3) by Theorem 1.

(3) implies (4). Since $E(\mu) \in K(\Gamma(\mu))$ we have $E(\mu)\mu \in K(\Gamma(\mu))$. Clearly $E(\mu)\mu E(\mu) = E(\mu)\mu$, since $E(\mu)$ is the identity in $K(\Gamma(\mu))$. Thus

$$\begin{aligned}\operatorname{supp} E(\mu) &= (\operatorname{supp} E(\mu) \operatorname{supp} \mu \operatorname{supp} E(\mu))^- \\ &= (\operatorname{supp} E(\mu) \operatorname{supp} \mu)^- \supseteq \operatorname{supp} E(\mu) \operatorname{supp} \mu .\end{aligned}$$

As a consequence, we get $\operatorname{supp} E(\mu) \supseteq \operatorname{supp} E(\mu)\,(\operatorname{supp} \mu)^i$ for all i and so

$$\operatorname{supp} E(\mu) \supseteq (\operatorname{supp} E(\mu)) S(\mu) .$$

Similarly, by considering $\mu E(\mu) \in K(\Gamma(\mu))$ we can obtain

$$\operatorname{supp} E(\mu) \supseteq S(\mu) \operatorname{supp} E(\mu) .$$

So $\operatorname{supp} E(\mu)$ is an ideal of $S(\mu)$. It follows that $\operatorname{supp} E(\mu) \supseteq K(S(\mu))$, whence

$$\operatorname{supp} E(\mu) \supseteq K(S(\mu))^- = \operatorname{supp} L(\mu)$$

implying (4).

(4) implies (1). In view of Lemma 3 of [8] we get $E(\mu)L(\mu)E(\mu) = E(\mu)$. On the other hand, since $L(\mu)\mu^i = \mu^i L(\mu) = L(\mu)$ for all i and $E(\mu) \in \Gamma(\mu)$ we see that $L(\mu)E(\mu) = E(\mu)L(\mu) = L(\mu)$ which implies that $E(\mu)L(\mu)E(\mu) = L(\mu)$. Thus $L(\mu) = E(\mu)$, and the proof is complete.

THEOREM 10. *If the sequence* (μ^i) *converges, then each of the conditions in Theorem* 9 *holds. Moreover,* $\lim \mu^i = L(\mu)$.

PROOF. Suppose the sequence (μ^i) converges to $\tau \in \Gamma(\mu)$. Then sequences $(\mu\mu^i)$ and $(\mu^i\mu)$ must converge to $\mu\tau$ and $\tau\mu$, respectively. Therefore $\mu\tau = \tau\mu = \tau$. Clearly $K(\Gamma(\mu)) = \{\tau\}$. It follows that $E(\mu) = \tau$ and so $K(\Gamma(\mu)) = \{E(\mu)\}$, that is Theorem 9 (2) holds. That $L(\mu) = \tau$ is clear.

COROLLARY 11. *If the sequence* (μ^i) *converges, then* $L(\mu) = L(\mu^n)$, $n = 1, 2, \ldots$.

PROOF. If (μ^i) converges to τ, then $L(\mu) = \tau$. Since the subsequence $((\mu^n)^i)_{i=1}^{\infty}$ also converges to τ, we must have $L(\mu^n) = \tau$ which gives the result.

The converse of Theorem 10 holds for compact jointly continuous semigroups, that is each of the conditions in Theorem 9 implies that the sequence (μ^i) converges [5, Theorem 2], due to the fact that the cluster points of (μ^i) belong to $K(\Gamma(\mu))$. But the situation is different for a compact separately continuous monothetic semigroup since it may contain more than one idempotent ([9], [2]), each of which is a cluster point of (μ^i).

EXAMPLE 12. Take the semigroup $S = S_w(\mu)$, the monothetic semigroup generated by u (in the notation of [2]), with μ defined in Example 2 of [2], which has zero 0 and identity 1 such that $u^{n!/2} \to 0$ and $u^{n!} \to 1$. Now we let $\nu = \delta(u) \in P(S)$. It follows that $\delta(0) \in \Gamma(\nu)$, $\delta(1) \in \Gamma(\nu)$, and so $K(\Gamma(\nu)) = \{\delta(0)\}$, that is Theorem 9 (2) holds. However, as both $\delta(0)$ and $\delta(1)$ are cluster points of (μ^i), we conclude that the sequence (μ^i) fails to converge.

ACKNOWLEDGEMENT. I am grateful to my supervisor, Dr. G. Brown, for many useful discussions, and to the Association of Commonwealth Universities for financial support.

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