

ISOMORPHISMS OF SPACES OF AFFINE CONTINUOUS COMPLEX FUNCTIONS

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Abstract

Let X and Y be compact convex sets such that their each extreme point is a weak peak point. We show that $\text{ext } X$ is homeomorphic to $\text{ext } Y$ provided there exists a small-bound isomorphism of the space $\mathfrak{A}(X, \mathbb{C})$ of continuous affine complex functions on X onto $\mathfrak{A}(Y, \mathbb{C})$. Further, we generalize a result of Cengiz to the context of compact convex sets.

1. Introduction

The notation and notions used in the introduction are explained in the next section. The starting point of our paper is the classical Banach-Stone theorem asserting that, given a pair of compact spaces K and L , they are homeomorphic provided $\mathcal{C}(K, \mathbb{F})$ is isometric to $\mathcal{C}(L, \mathbb{F})$ (see [13, Theorem 3.117]).

This can be reformulated in the framework of compact convex sets as follows: if X, Y are Bauer simplices and $\mathfrak{A}(X, \mathbb{F})$ is isometric to $\mathfrak{A}(Y, \mathbb{F})$, then $\text{ext } X$ is homeomorphic to $\text{ext } Y$.

A generalization of this classical result was given by Lazar in [23], who proved that for general simplices X, Y , the spaces $\mathfrak{A}(X, \mathbb{R})$ and $\mathfrak{A}(Y, \mathbb{R})$ are isometric only if X is affinely homeomorphic to Y .

A result of Rao (see [27]) precisely describes isometries of $\mathfrak{A}(X, \mathbb{C})$ for a simplex X .

A remarkable generalization of the Banach-Stone theorem was given by Amir [3] and Cambern [6]. They showed that compact spaces K, L are homeomorphic if there exists an isomorphism $T: \mathcal{C}(K, \mathbb{F}) \rightarrow \mathcal{C}(L, \mathbb{F})$ with $\|T\| \cdot \|T^{-1}\| < 2$. Alternative proofs were given by Cohen [10] and Drewnowski [12].

A reformulation of this result for simplices reads as follows: given Bauer simplices X and Y , the sets $\text{ext } X$ and $\text{ext } Y$ are homeomorphic, provided there exists an isomorphism $T: \mathfrak{A}(X, \mathbb{F}) \rightarrow \mathfrak{A}(Y, \mathbb{F})$ with $\|T\| \cdot \|T^{-1}\| < 2$.

This theorem was improved by Chu and Cohen in [8], who proved that for compact convex sets X and Y , the sets $\text{ext } X$ and $\text{ext } Y$ are homeomorphic

provided there exists an isomorphism $T: \mathfrak{A}(X, \mathbb{R}) \rightarrow \mathfrak{A}(Y, \mathbb{R})$ with $\|T\| \cdot \|T^{-1}\| < 2$ and one of the following conditions hold:

- (i) X and Y are simplices such that their extreme points are weak peak points;
- (ii) X and Y are metrizable and their extreme points are weak peak points;
- (iii) $\text{ext } X$ and $\text{ext } Y$ are closed and extreme points of X and Y are split faces.

In [24], it was showed that extreme points of X and Y are homeomorphic, provided there exists an isomorphism $T: \mathfrak{A}(X, \mathbb{R}) \rightarrow \mathfrak{A}(Y, \mathbb{R})$ with $\|T\| \cdot \|T^{-1}\| < 2$, extreme points are weak peak points and both $\text{ext } X$ and $\text{ext } Y$ are Lindelöf sets.

In [11] the same result is proved without the assumption of the Lindelöf property.

It turns out that this result is in a sense optimal since the bound 2 cannot be improved (see [9]) and the assumption on weak peak points cannot be omitted (see [17]).

The main result of our paper is a variant of [11, Theorem 2.1] for complex spaces. It reads as follows.

THEOREM 1.1. *Let X, Y be compact convex sets and let $T: \mathfrak{A}(X, \mathbb{C}) \rightarrow \mathfrak{A}(Y, \mathbb{C})$ be an isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$.*

If each point of $\text{ext } X$ and $\text{ext } Y$ is a weak peak point, $\text{ext } X$ is homeomorphic to $\text{ext } Y$.

We use the obtained results for affine complex functions on compact convex sets to obtain corollaries on selfadjoint function spaces, see Theorem 5.3.

2. Notation and basic concepts

First we recall several notions. We work within the framework of real or complex vector spaces and write \mathbb{F} for the respective field \mathbb{R} or \mathbb{C} . Further we write \mathbb{T} for the set $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$. If X is a compact convex set in a locally convex space, we write $\mathfrak{A}(X, \mathbb{F})$ for the space of all affine continuous \mathbb{F} -valued functions on X endowed with the sup-norm. Let $\text{ext } X$ stand for the set of all extreme points of X . If K is compact (Hausdorff) topological space, let $\mathcal{C}(K, \mathbb{F})$ stand for the space of all continuous \mathbb{F} -valued functions on K endowed with the sup-norm.

We identify the dual space $(\mathcal{C}(K, \mathbb{F}))^*$ with the space $\mathcal{M}(K, \mathbb{F})$ of all Radon measures on K . We write $\mathcal{M}^+(K)$ for positive Radon measures and $\mathcal{M}^1(K)$ for probability Radon measures on K .

A point x in a compact convex set X is called a *weak peak point* if

$$\begin{aligned} &\text{given } \varepsilon \in (0, 1) \text{ and an open set } U \subset X \text{ containing } x, \\ &\text{there exists } a \text{ in the unit ball } B_{\mathfrak{A}(X, \mathbb{C})} \text{ of } \mathfrak{A}(X, \mathbb{C}) \text{ such that} \\ &|a| < \varepsilon \text{ on } \text{ext } X \setminus U \text{ and } a(x) > 1 - \varepsilon. \end{aligned} \quad (2.1)$$

For any $\mu \in \mathcal{M}^1(X)$ there exists a unique point $r(\mu) \in X$ such that $\mu(a) = a(r(\mu))$, $a \in \mathfrak{A}(X, \mathbb{C})$, see [2, Proposition I.2.1]. We call $r(\mu)$ the *barycenter* of μ . A function $f: X \rightarrow \mathbb{F}$ satisfies the *barycentric formula* (or is called *strongly affine*) if $\mu(f) = f(r(\mu))$, $\mu \in \mathcal{M}^1(X)$.

If $\mu, \nu \in \mathcal{M}^+(X)$, then $\mu \prec \nu$ if $\mu(k) \leq \nu(k)$ for each convex continuous function k on X . A measure $\mu \in \mathcal{M}^+(X)$ is *maximal* if μ is \prec -maximal. A measure $\mu \in \mathcal{M}(X, \mathbb{F})$ is called *boundary* if its total variation $|\mu|$ is maximal.

By the Choquet-Bishop-de-Leeuw representation theorem (see e.g. [2, Theorem I.4.8]), for each $x \in X$ there exists a maximal measure $\mu \in \mathcal{M}^1(X)$ with $r(\mu) = x$. If this measure is uniquely determined for each $x \in X$, the set X is called a *simplex*. It is called a *Bauer simplex* if, moreover, the set $\text{ext } X$ is closed. In this case, the space $\mathfrak{A}(X, \mathbb{F})$ is isometric to the space $\mathcal{C}(\text{ext } X, \mathbb{F})$ (see [2, Theorem II.4.3]). On the other hand, given a space $\mathcal{C}(K, \mathbb{F})$, it is isometric to $\mathfrak{A}(\mathcal{M}^1(K), \mathbb{F})$ ([2, Corollary II.4.2]).

A face $F \subset X$ is called a *split face* if the complementary set F' (i.e., the union of all faces disjoint from F) is convex and X is a direct convex sum of F and F' , i.e., every point in X can be uniquely represented as a convex combination of a point in F and a point in F' (see [2, p. 133]). If F is a closed split face, then the *upper envelope* of the characteristic function χ_F defined as

$$\chi_F^*(x) = \inf\{a(x) : a \in \mathfrak{A}(X, \mathbb{R}), a > \chi_F\}, \quad x \in X,$$

is upper semicontinuous and affine, $F = (\chi_F^*)^{-1}(1)$ and $F' = (\chi_F^*)^{-1}(0)$, see [2, Propositions II.6.5 and II.6.9]. Moreover, the family $\{a \in \mathfrak{A}(X, \mathbb{R}) : \chi_F < a\}$ is downward directed.

In what follows, we consider the weak*-topology on $(\mathfrak{A}(X, \mathbb{C}))^*$, and we understand X as a subset of $B_{(\mathfrak{A}(X, \mathbb{C}))^*}$ via the evaluation mapping, i.e., $x(f) = f(x)$, $f \in \mathfrak{A}(X, \mathbb{C})$, $x \in X$.

3. Isomorphisms with a small bound

The proof of Theorem 1.1 follows the strategy of the proof of [8], but we use [11, Corollary 1.3(b)] as the main tool. Thus we need the key Lemma 3.4 which allows us to represent upper semicontinuous affine functions on X as elements of $(\mathfrak{A}(X, \mathbb{C}))^{**}$.

We start with a lemma describing extreme points of $B_{(\mathfrak{A}(X, \mathbb{C}))^*}$, see also Lemma 1 in [19].

LEMMA 3.1. *Let X be a compact convex set. Then*

$$\text{ext } B_{(\mathfrak{A}(X, \mathbb{C}))^*} = \mathbb{T} \cdot \text{ext } X.$$

PROOF. We denote $K = B_{(\mathfrak{A}(X, \mathbb{C}))^*}$. First we prove that $\text{ext } K \subset \mathbb{T} \cdot X$. Since $\mathbb{T} \cdot X$ is compact, by the Milman theorem it is enough to show that $\overline{\text{co}}(\mathbb{T} \cdot X) = K$.

Assuming the contrary, there exist $s \in K \setminus \overline{\text{co}}(\mathbb{T} \cdot X)$, $\alpha \in \mathbb{R}$, and $f \in \mathfrak{A}(X, \mathbb{C})$ such that

$$\text{Re } s(f) > \alpha > \sup\{\text{Re } r(f) : r \in \overline{\text{co}}(\mathbb{T} \cdot X)\}.$$

Let $\mu \in B_{\mathcal{M}(X, \mathbb{C})}$ be a Hahn-Banach extension of s . Since

$$\text{Re } t f(x) < \alpha, \quad t \in \mathbb{T}, x \in X,$$

we obtain $|f(x)| < \alpha, x \in X$. Then

$$\alpha < \text{Re } s(f) = \text{Re } \mu(f) \leq \left| \int_X f \, d\mu \right| \leq \int_X |f| \, d|\mu| < \int_X \alpha \, d|\mu| = \alpha$$

gives a contradiction. Thus $\text{ext } K \subset \mathbb{T} \cdot X$.

Now we show that $\text{ext } K = \mathbb{T} \cdot \text{ext } X$. Let $s \in \text{ext } K$ be given. Then $s = tx$ for some $t \in \mathbb{T}$ and $x \in X$. If $x = \frac{1}{2}(x_1 + x_2)$ for some distinct points $x_1, x_2 \in X$, then

$$s = \frac{1}{2}(tx_1 + tx_2),$$

where $tx_1 \neq tx_2$. Thus $s \notin \text{ext } K$, which is impossible. This proves “ \subset ”.

On the other hand, let $tx = \frac{1}{2}(s_1 + s_2)$ for some $t \in \mathbb{T}, x \in \text{ext } X$ and $s_1, s_2 \in K$. Then $x = \frac{1}{2}(t^{-1}s_1 + t^{-1}s_2)$. Let $\mu_1, \mu_2 \in B_{\mathcal{M}(X, \mathbb{C})}$ be Hahn-Banach extensions of $t^{-1}s_1$ and $t^{-1}s_2$, respectively. Then

$$f(x) = \frac{1}{2}(\mu_1(f) + \mu_2(f)), \quad f \in \mathfrak{A}(X, \mathbb{C}).$$

If $\mu = \frac{1}{2}(\mu_1 + \mu_2)$, then $\mu(1) = 1 = \|\mu\|$, and thus $\mu \in \mathcal{M}^1(X)$. Further, x is the barycenter of μ . Since $x \in \text{ext } X$, we obtain that $\mu = \varepsilon_x$, the Dirac measure centered at the point x . Since

$$\mu_1(1) = \mu_2(1) = \|\mu_1\| = \|\mu_2\|,$$

it follows that $\mu_1 = \mu_2 = \varepsilon_x$, and thus $t^{-1}s_1 = t^{-1}s_2 = x$. Thus $s_1 = s_2 = tx$ and $tx \in \text{ext } K$.

This proves “ \supset ” and finishes the proof.

LEMMA 3.2. Let $\mu \in \mathcal{M}^1(K)$ be a probability measure on a compact space K and let $\{f_j\}_{j \in J}$ be a bounded downward directed net of functions in $\mathcal{C}(K, \mathbb{R})$ converging to a function f . Then for any $g \in \mathcal{C}(K, \mathbb{C})$ it holds

$$\lim_{j \in J} \int_K g f_j \, d\mu = \int_K g f \, d\mu.$$

PROOF. It is known (see [15, Corollary 414B]), that for a bounded downward directed net $\{h_j\}_{j \in J}$ in $\mathcal{C}(K, \mathbb{R})$ pointwise converging to h we have $\mu(h) = \lim_{j \in J} \mu(h_j)$. We decompose $g = \sum_{k=0}^3 i^k g_k$, where $g_0, \dots, g_3 \in \mathcal{C}(K, \mathbb{R})$ positive, and apply this fact to the bounded downward directed nets $\{g_k f_j\}_{j \in J}$, $k = 0, \dots, 3$, and obtain

$$\mu(gf) = \sum_{k=0}^3 i^k \mu(g_k f) = \sum_{k=0}^3 i^k \lim_{j \in J} \mu(g_k f_j) = \lim_{j \in J} \mu(g f_j),$$

which finishes the proof.

If K is a compact topological space and T is a topological space, a function $f: K \rightarrow T$ is said to be *of the first Borel class* if, for each $U \subset T$ open, the set $f^{-1}(U)$ can be written as a countable union of differences of closed sets in K , see [29, Definition 3.2]. In case of $T = \mathbb{R}$, any semicontinuous function is of the first Borel class.

LEMMA 3.3. Let K be a compact topological space. If $f_1, f_2: K \rightarrow \mathbb{C}$ are two functions of the first Borel class, then their product

$$h: x \in K \mapsto f_1(x) f_2(x) \in \mathbb{C}$$

is of the first Borel class as well.

PROOF. Consider the mappings $f: x \in K \mapsto (f_1(x), f_2(x)) \in \mathbb{C}^2$ and $\phi: (y, z) \in \mathbb{C}^2 \mapsto yz \in \mathbb{C}$. Since ϕ is continuous and $h = \phi \circ f$, it is enough to show that the mapping f is of the first Borel class. To this end, let \mathcal{U} be a countable basis of \mathbb{C} and $U_1, U_2 \in \mathcal{U}$. Write

$$f_i^{-1}(U_i) = \bigcup_{n \in \mathbb{N}} (H_i^n \setminus F_i^n), \quad i = 1, 2,$$

where $F_i^n, H_i^n \subseteq K$ are closed sets. Then

$$\begin{aligned} f^{-1}(U_1 \times U_2) &= f_1^{-1}(U_1) \cap f_2^{-1}(U_2) \\ &= \left(\bigcup_{n \in \mathbb{N}} (H_1^n \setminus F_1^n) \right) \cap \left(\bigcup_{m \in \mathbb{N}} (H_2^m \setminus F_2^m) \right) \\ &= \bigcup_{n, m \in \mathbb{N}} (H_1^n \setminus F_1^n) \cap (H_2^m \setminus F_2^m) \\ &= \bigcup_{n, m \in \mathbb{N}} (H_1^n \cup H_2^m) \setminus (F_1^n \cup F_2^m). \end{aligned}$$

Further, any open subset V of \mathbb{C}^2 may be written as a countable union of sets R_i of the form $R_i = U_i^1 \times U_i^2$, where $U_i^1, U_i^2 \in \mathcal{U}$. Thus we have that $f^{-1}(V)$ can be written as a countable union of differences of closed sets in K , i.e., f is of the first Borel class.

LEMMA 3.4. *Let X be a compact convex set and $f: X \rightarrow \mathbb{R}$ be an upper semicontinuous affine function. Then the following assertions hold.*

- (a) *There exists an element $a^{**} \in (\mathfrak{A}(X, \mathbb{C}))^{**}$ such that $a^{**}(x) = f(x)$, $x \in X$.*
- (b) *If $\{a_j\}_{j \in J}$ is a bounded downward directed net in $\mathfrak{A}(X, \mathbb{R})$ satisfying*

$$f(x) = \lim_{j \in J} a_j(x) = \inf_{j \in J} a_j(x), \quad x \in X,$$

*then $a_j \rightarrow a^{**}$ weak*.*

- (c) *The function a^{**} is of the first Borel class on $\mathbb{T} \cdot X$.*
- (d) *For each $\mu \in \mathcal{M}^1(\mathbb{T} \cdot X)$, it holds $\mu(a^{**}) = a^{**}(r(\mu))$.*
- (e) *The function a^{**} is of the first Borel class on any $rB_{(\mathfrak{A}(X, \mathbb{C}))^*}$, $r > 0$.*
- (f) *There is only one element $a^{**} \in (\mathfrak{A}(X, \mathbb{C}))^{**}$ extending f .*

PROOF. (a) Since f is upper semicontinuous and affine, it is bounded (see e.g. [26, Lemma 4.20 and Theorem 4.21]). Hence we may assume that $-M < f < M$ for some constant $M > 0$. By [26, Proposition 4.12],

$$f(x) = \inf\{a(x) : a \in \mathfrak{A}(X, \mathbb{R}), f < a \leq M\}, \quad x \in X,$$

in other words, the downward directed net $\{a \in \mathfrak{A}(X, \mathbb{R}), f < a \leq M\}$ converges pointwise to f . We consider the family

$$\{a \in \mathfrak{A}(X, \mathbb{R}) : f < a \leq M\}$$

as a net of elements in $(\mathfrak{A}(X, \mathbb{C}))^{**}$. We claim that this net converges weak* to an element $a^{**} \in (\mathfrak{A}(X, \mathbb{C}))^{**}$.

Indeed, let $s \in \mathfrak{A}(X, \mathbb{C})^*$ be given. We extend s by the Hahn-Banach theorem to an element $\mu \in \mathcal{M}(X, \mathbb{C})$ with $\|\mu\| = \|s\|$ and write $\mu = \sum_{k=0}^3 i^k c_k \mu_k$, where $c_k \geq 0$ and $\mu_k \in \mathcal{M}^1(X)$, $k = 0, \dots, 3$. Let x_k be the barycenter of μ_k , $k = 0, \dots, 3$. Then

$$s(a) = \mu(a) = \sum_{k=0}^3 i^k c_k \mu_k(a) = \sum_{k=0}^3 i^k c_k a(x_k), \quad a \in \mathfrak{A}(X, \mathbb{C}).$$

Since the net $\{a(x_k) : a \in \mathfrak{A}(X, \mathbb{R}), f < a \leq M\}$ converges to $f(x_k)$ for each $k = 0, \dots, 3$, the net $\{s(a) : a \in \mathfrak{A}(X, \mathbb{R}), f < a \leq M\}$ converges.

By setting

$$a^{**}(s) = \lim\{s(a) : a \in \mathfrak{A}(X, \mathbb{R}), f < a \leq M\}, \quad s \in (\mathfrak{A}(X, \mathbb{C}))^*,$$

we obtain a linear functional on $(\mathfrak{A}(X, \mathbb{C}))^*$.

To conclude the proof it is enough to show that it is bounded. Considering $s \in (\mathfrak{A}(X, \mathbb{C}))^*$ as above, let $\mu \in \mathcal{M}(X, \mathbb{C})$ be again a Hahn-Banach extension of s , i.e., $\|\mu\| = \|s\|$. Then we can decompose μ as $\mu = \sum_{k=0}^3 i^k c_k \mu_k$, where $c_k \geq 0$ and $\mu_k \in \mathcal{M}^1(X)$, $k = 0, \dots, 3$, and, moreover, $c_0 + c_1 + c_2 + c_3 \leq 2\|\mu\|$.

Then the inequalities

$$\begin{aligned} |a^{**}(s)| &= \left| \lim\{s(a) : a \in \mathfrak{A}(X, \mathbb{R}), f < a \leq M\} \right| \\ &= \left| \lim\left\{ \sum_{k=0}^3 i^k c_k a(r(\mu_k)) : a \in \mathfrak{A}(X, \mathbb{R}), f < a \leq M \right\} \right| \\ &\leq \sum_{k=0}^3 c_k M \leq 2M\|\mu\| = 2M\|s\| \end{aligned}$$

imply the boundedness of a^{**} , i.e., $a^{**} \in (\mathfrak{A}(X, \mathbb{C}))^{**}$.

(b) Let $\{a_j\}_{j \in J}$ be a bounded downward directed net in $\mathfrak{A}(X, \mathbb{R})$ pointwise converging to f on X . Given $s \in (\mathfrak{A}(X, \mathbb{C}))^*$, we extend s to $\mu \in \mathcal{M}(X, \mathbb{C})$ as above and write $\mu = \sum_{k=0}^3 i^k c_k \mu_k$, where $c_k \geq 0$, $\mu_k \in \mathcal{M}^1(X)$. Then

$$\begin{aligned} a^{**}(s) &= \lim\{s(a) : a \in \mathfrak{A}(X, \mathbb{R}), f < a \leq M\} = \sum_{k=0}^3 i^k c_k f(r(\mu_k)) \\ &= \sum_{k=0}^3 i^k c_k \lim_{j \in J} a_j(r(\mu_k)) = \lim_{j \in J} \left(\sum_{k=0}^3 i^k c_k r(\mu_k) \right) (a_j) = \lim_{j \in J} s(a_j). \end{aligned}$$

(c) We remind that we understand X as a subset of $(\mathfrak{A}(X, \mathbb{C}))^*$, and we consider a homeomorphic mapping $\varphi: \mathbb{T} \times X \rightarrow \mathbb{T} \cdot X$ defined by $\varphi(t, x) = tx$, $(t, x) \in \mathbb{T} \times X$.

Then the function $h: \mathbb{T} \times X \rightarrow \mathbb{C}$ defined as $h(t, x) = tf(x)$, $(t, x) \in \mathbb{T} \times X$, as a product of a continuous function and a function of the first Borel class, is of the first Borel class as well by Lemma 3.3.

Thus $a^{**} = h \circ \varphi^{-1}$ is of the first Borel class on $\mathbb{T} \cdot X$ also.

(d) Let $\mu \in \mathcal{M}^1(\mathbb{T} \cdot X)$ be given. Then its barycenter $r(\mu)$ belongs to the set $B_{(\mathfrak{A}(X, \mathbb{C}))^*}$. We denote $\nu = \varphi^{-1}\mu \in \mathcal{M}^1(\mathbb{T} \times X)$ and pick a bounded downward directed net $\{a_j\}_{j \in J}$ in $\mathfrak{A}(X, \mathbb{R})$ converging to f .

Then we have using Lemma 3.2 and (b)

$$\begin{aligned} \mu(a^{**}) &= (\varphi\nu)(a^{**}) = \nu(a^{**} \circ \varphi) = \nu(h) = \lim_{j \in J} \int_{\mathbb{T} \times X} t a_j(x) d\nu(t, x) \\ &= \lim_{j \in J} \int_{\mathbb{T} \cdot X} a_j d\mu = \lim_{j \in J} a_j(r(\mu)) = a^{**}(r(\mu)). \end{aligned}$$

(e) We first show that a^{**} is of the first Borel class on $B_{(\mathfrak{A}(X, \mathbb{C}))^*}$. To this end we recall that

$$\overline{\text{ext}}B_{(\mathfrak{A}(X, \mathbb{C}))^*} \subset \mathbb{T} \cdot X.$$

We know from (d) that the function a^{**} satisfies the barycentric formula for each $\mu \in \mathcal{M}^1(\overline{\text{ext}}B_{(\mathfrak{A}(X, \mathbb{C}))^*})$. By [28, Theorem 3.3], a^{**} is strongly affine on $B_{(\mathfrak{A}(X, \mathbb{C}))^*}$. Since a^{**} is of the first Borel class on $\overline{\text{ext}}B_{(\mathfrak{A}(X, \mathbb{C}))^*}$, [25, Theorem 3.5] implies that a^{**} is of the first Borel class on $B_{(\mathfrak{A}(X, \mathbb{C}))^*}$.

If $r > 0$ is arbitrary, we realize that $rB_{(\mathfrak{A}(X, \mathbb{C}))^*}$ is affinely homeomorphic to $B_{(\mathfrak{A}(X, \mathbb{C}))^*}$ and a^{**} is linear. Hence a^{**} is of the first Borel class on $rB_{(\mathfrak{A}(X, \mathbb{C}))^*}$.

(f) It is enough to show that, given $a^{**} \in (\mathfrak{A}(X, \mathbb{C}))^{**}$, $a^{**} = 0$ provided $a^{**} = 0$ on X . Let $s \in (\mathfrak{A}(X, \mathbb{C}))^*$ be arbitrary. We extend s to an element $\mu \in \mathcal{M}(X, \mathbb{C})$ and write $\mu = \sum_{k=0}^3 i^k c_k \mu_k$, where $c_k \geq 0$, $\mu_k \in \mathcal{M}^1(X)$, $k = 0, \dots, 3$. Then $s = \sum_{k=0}^3 i^k c_k r(\mu_k)$, and thus

$$a^{**}(s) = \sum_{k=0}^3 i^k c_k a^{**}(r(\mu_k)) = 0.$$

Hence $a^{**} = 0$ as needed. The proof is finished.

Next we need a decomposition lemma which is well known for real spaces $\mathfrak{A}(X, \mathbb{R})$.

LEMMA 3.5. *Let X be a compact convex set and $F \subset X$ be a closed split face. Let F' be the complementary face of F . Then $(\mathfrak{A}(X, \mathbb{C}))^* = \text{span } F \oplus_{\ell_1} \text{span } F'$.*

PROOF. Let $s \in \mathfrak{A}(X, \mathbb{C})^*$ be given. We extend s to $\mu \in \mathcal{M}(X, \mathbb{C})$ which is boundary (see [19, Theorem], [18] and [16, Theorem 1.2]). Then $|\mu|(\chi_F) = |\mu|(\chi_F^*)$ (see [2, Proposition I.4.5 and the subsequent Remark]), and thus μ is carried by the set $\{\chi_F = \chi_F^*\} = F \cup F'$. We write $\mu|_F = \sum_{k=0}^3 i^k c_k \mu_k$ and $\mu|_{F'} = \sum_{k=0}^3 i^k d_k \nu_k$, where $c_k, d_k \geq 0$, $\mu_k \in \mathcal{M}^1(F)$ and $\nu_k \in \mathcal{M}^1(F')$, $k = 0, \dots, 3$. Let $x_k = r(\mu_k)$, $y_k = r(\nu_k)$, $k = 0, \dots, 3$. By [2, Corollary II.6.11], $x_k \in F$ and $y_k \in F'$. Thus

$$s_F = \sum_{k=0}^3 i^k c_k x_k \in \text{span } F, \quad s_{F'} = \sum_{k=0}^3 i^k d_k y_k \in \text{span } F'$$

and for $a \in \mathfrak{A}(X, \mathbb{C})$ we have

$$\begin{aligned} s(a) &= \mu(a) = \mu|_F(a) + \mu|_{F'}(a) = \sum_{k=0}^3 i^k c_k a(x_k) + \sum_{k=0}^3 i^k d_k a(y_k) \\ &= s_F(a) + s_{F'}(a). \end{aligned}$$

Thus $s = s_F + s_{F'} \in \text{span } F + \text{span } F'$.

Further, $\|s_F\| \leq \|\mu|_F\|$ and $\|s_{F'}\| \leq \|\mu|_{F'}\|$, and thus

$$\|s\| = \|\mu\| = \|\mu|_F\| + \|\mu|_{F'}\| \geq \|s_F\| + \|s_{F'}\|.$$

Hence $\|s\| = \|s_F\| + \|s_{F'}\|$.

Let $s \in \text{span } F \cap \text{span } F'$. Then there exists $c_k \geq 0$, $d_k \geq 0$, $x_k \in F$, $y_k \in F'$, $k = 0, \dots, 3$, such that

$$s = (c_0 x_0 - c_1 x_1) + i(c_2 x_2 - c_3 x_3) = (d_0 y_0 - d_1 y_1) + i(d_2 y_2 - d_3 y_3).$$

If we apply s to an arbitrary $a \in \mathfrak{A}(X, \mathbb{R})$, we obtain

$$(c_0 x_0 - c_1 x_1)(a) = (d_0 y_0 - d_1 y_1)(a), \quad (c_2 x_2 - c_3 x_3)(a) = (d_2 y_2 - d_3 y_3)(a).$$

Thus

$$c_0 x_0 - c_1 x_1 = d_0 y_0 - d_1 y_1, \quad c_2 x_2 - c_3 x_3 = d_2 y_2 - d_3 y_3.$$

An application of s to a constant function 1 yields

$$c = c_0 + d_1 = d_0 + c_1, \quad d = c_2 + d_3 = d_2 + c_3.$$

If $c = 0$, then $c_0 = c_1 = d_0 = d_1 = 0$, and thus $c_0 x_0 - c_1 x_1 = 0$. Otherwise we have the following equality

$$\frac{c_0}{c} x_0 + \frac{d_1}{c} y_1 = \frac{d_0}{c} y_0 + \frac{c_1}{c} x_1.$$

Since X is a direct convex sum of F and F' , these convex combinations must be equal, i.e.,

$$c_0x_0 = c_1x_1, \quad d_1y_1 = d_0y_0.$$

Hence $c_0x_0 - c_1x_1 = 0$.

Similarly we handle the second term $c_2x_2 - c_3x_3 = d_2y_2 - d_3y_3$ and obtain $s = 0$.

Thus $\mathfrak{A}(X, \mathbb{C})^* = \text{span } F \oplus_{\ell_1} \text{span } F'$ as claimed.

LEMMA 3.6. *Let X be a compact convex set and $f: X \rightarrow \mathbb{C}$ be an affine function of the first Borel class. Then*

$$\sup_{x \in X} |f(x)| = \sup_{x \in \text{ext } X} |f(x)|.$$

PROOF. It is proved in [22, Theorem 2.3] that every complex function of the first Borel class on a compact space has the point of continuity property. For the rest of the proof see [11, Corollary 1.5(b)].

LEMMA 3.7. *Let x be a weak peak point of a compact convex set X . Then $\{x\}$ is a split face of X .*

PROOF. Suppose that x is a weak peak point. First we prove that x is an extreme point. To this end, let $\mu \in \mathcal{M}^1(X)$ be a maximal measure representing x . For the proof that x is extreme it is enough to show that $\mu = \varepsilon_x$, the Dirac measure centered at the point x . We fix an arbitrary closed neighborhood U of x and $\varepsilon > 0$. Then there is a function $a \in B_{\mathfrak{A}(X, \mathbb{C})}$ satisfying

$$a(x) > 1 - \varepsilon \quad \text{and} \quad |a| < \varepsilon \quad \text{on } \text{ext } X \setminus U.$$

Since a is continuous and U is closed, it even holds that $|a| \leq \varepsilon$ on the set $\overline{\text{ext } X} \setminus U \subseteq \overline{\text{ext } X} \setminus U$. So, since μ is maximal measure, we have by [2, Proposition I.4.6]

$$\begin{aligned} 1 - \varepsilon < a(x) &\leq \int_X |a| d\mu = \int_{\overline{\text{ext } X}} |a| d\mu \\ &= \int_U |a| d\mu + \int_{\overline{\text{ext } X} \setminus U} |a| d\mu \leq \mu(U) + \varepsilon. \end{aligned}$$

In other words, $\mu(U) > 1 - 2\varepsilon$. Since $\varepsilon > 0$ is chosen arbitrarily, we have that $\mu(U) = 1$. Hence $\mu(V) = 1$ for each closed neighborhood V of x . From this it easily follows that $\mu = \varepsilon_x$.

For the fact that x is actually a split face it is enough to follow the proof of [8, Proposition 1].

Now we can prove Theorem 1.1.

PROOF OF THEOREM 1.1. We write $\langle \cdot, \cdot \rangle$ for the duality mapping. We write $A = \mathfrak{A}(X, \mathbb{C})$ and $B = \mathfrak{A}(Y, \mathbb{C})$.

We assume that there exist $c' \in \mathbb{R}$ such that $1 < c' < 2$ and $\|T\| < 2$ and $\|Ta\| > c'\|a\|$ for all $a \in A \setminus \{0\}$ (otherwise we would find $1 < c' < 2$ such that $\|T\| \cdot \|T^{-1}\| < \frac{2}{c'} < 2$ and consider the mapping $c'\|T^{-1}\|T$; see [8, p. 76]). We fix $c \in \mathbb{R}$ satisfying $1 < c < c'$.

CLAIM 1. *For any $a^{**} \in A^{**} \setminus \{0\}$ and $b^{**} \in B^{**} \setminus \{0\}$ we have $\|T^{**}a^{**}\| > c\|a^{**}\|$ and $\|(T^{-1})^{**}b^{**}\| > \frac{1}{2}\|b^{**}\|$.*

Indeed, for $a^{**} \in A^{**} \setminus \{0\}$ we have

$$\|a^{**}\| = \|(T^{-1})^{**}T^{**}a^{**}\| \leq (c')^{-1}\|T^{**}a^{**}\| < c^{-1}\|T^{**}a^{**}\|.$$

The second inequality is analogous.

For each $x \in \text{ext } X$ we consider the function $f_x = \chi_{\{x\}}^*$. Since $\{x\}$ is a split face, f_x is an upper semicontinuous affine function on X . We extend f_x using Lemma 3.4 to an element $a_x^{**} \in A^{**}$. By Lemma 3.4(e), a_x^{**} is of the first Borel class on any ball in A^* .

Analogously we define for $y \in \text{ext } Y$ the function g_y and the element $b_y^{**} \in B^{**}$.

We define mappings ρ_X and ρ_Y as follows:

$$\rho_X(x) = \{y \in \text{ext } Y : |\langle x, (T^{-1})^{**}b_y^{**} \rangle| > \frac{1}{2}\}, \quad x \in \text{ext } X, \quad \text{and}$$

$$\rho_Y(y) = \{x \in \text{ext } X : |\langle y, T^{**}a_x^{**} \rangle| > c\}, \quad y \in \text{ext } Y.$$

CLAIM 2. ρ_X and ρ_Y are mappings.

Let $x \in \text{ext } X$ be such that there exist distinct points $y_1, y_2 \in \text{ext } Y$ with

$$|\langle (T^{-1})^*x, b_{y_i}^{**} \rangle| = |\langle x, (T^{-1})^{**}b_{y_i}^{**} \rangle| > \frac{1}{2}, \quad i = 1, 2.$$

Using Lemma 3.5 we write

$$(T^{-1})^*x = \lambda_1 y_1 + \mu_1 = \lambda_2 y_2 + \mu_2,$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$, $\mu_1 \in \text{span}\{y_1\}'$ and $\mu_2 \in \text{span}\{y_2\}'$. Then

$$\frac{1}{2} < |\langle (T^{-1})^*x, b_{y_i}^{**} \rangle| = |\langle \lambda_i y_i, b_{y_i}^{**} \rangle + \langle \mu_i, b_{y_i}^{**} \rangle| = |\lambda_i|, \quad i = 1, 2.$$

Since

$$1 \geq \|(T^{-1})^*x\| = |\lambda_i| + \|\mu_i\| > \frac{1}{2} + \|\mu_i\|, \quad i = 1, 2,$$

we obtain

$$1 > \|\mu_1\| + \|\mu_2\| \geq \|\mu_1 - \mu_2\| = \|\lambda_1 y_1 - \lambda_2 y_2\| = |\lambda_1| + |\lambda_2| > 1,$$

i.e., a contradiction.

Analogously we show that $\rho_Y(y)$ is at most single-valued.

Let \widehat{X} and \widehat{Y} denote the domain of ρ_X and ρ_Y , respectively.

CLAIM 3. *The mappings $\rho_X: \widehat{X} \rightarrow \text{ext } Y$ and $\rho_Y: \widehat{Y} \rightarrow \text{ext } X$ are surjective.*

Let $y \in \text{ext } Y$ be given. We assume that $|\langle x, (T^{-1})^{**}b_y^{**} \rangle| \leq \frac{1}{2}$ for each $x \in \text{ext } X$ and seek a contradiction.

We show that the element $(T^{-1})^{**}b_y^{**} \in A^{**}$ is of the first Borel class on B_{A^*} .

Indeed, we know that b_y^{**} is of the first Borel class on any ball in B^* , in particular on $2B_{B^*}$. Since $(T^{-1})^*$ is a weak*-weak* homeomorphism, $(T^{-1})^*(B_{A^*}) \subset 2B_{B^*}$ and $(T^{-1})^{**}b_y^{**} = b_y^{**} \circ (T^{-1})^*$, it follows that $(T^{-1})^{**}b_y^{**}$ is of the first Borel class on B_{A^*} as well.

By Lemma 3.6,

$$\begin{aligned} \frac{1}{2} &\leq \frac{1}{2} \|b_y^{**}\| < \|(T^{-1})^{**}b_y^{**}\| = \sup_{a^* \in \text{ext } B_{A^*}} |\langle a^*, (T^{-1})^{**}b_y^{**} \rangle| \\ &= \sup_{a^* \in \mathbb{T} \cdot \text{ext } X} |\langle a^*, (T^{-1})^{**}b_y^{**} \rangle| = \sup_{x \in \text{ext } X} |\langle x, (T^{-1})^{**}b_y^{**} \rangle| \leq \frac{1}{2}. \end{aligned}$$

This contradiction implies that ρ_X is surjective.

Analogously we check that ρ_Y is surjective.

CLAIM 4. *We have $\widehat{X} = \text{ext } X$ and $\widehat{Y} = \text{ext } Y$ and $\rho_Y(\rho_X(x)) = x$, $x \in \text{ext } X$, and $\rho_X(\rho_Y(y)) = y$, $y \in \text{ext } Y$.*

Let $y \in \widehat{Y}$ be given. We want to show that $\rho_X(\rho_Y(y)) = y$, i.e., that

$$|\langle \rho_Y(y), (T^{-1})^{**}b_y^{**} \rangle| > \frac{1}{2}. \tag{3.1}$$

We have

$$\begin{aligned} d &= \sup_{x \in \text{ext } X} |\langle x, (T^{-1})^{**}b_y^{**} \rangle| = \sup_{s \in \mathbb{T} \cdot \text{ext } X} |\langle s, (T^{-1})^{**}b_y^{**} \rangle| \\ &= \|(T^{-1})^{**}b_y^{**}\| > \frac{1}{2} \|b_y^{**}\| \geq \frac{1}{2}. \end{aligned}$$

Since $c > 1$, we have $d > \max\{\frac{d}{c}, \frac{1}{2}\}$. Hence there exists $x \in \text{ext } X$ such that

$$|\langle x, (T^{-1})^{**}b_y^{**} \rangle| > \max\{\frac{d}{c}, \frac{1}{2}\} \geq \frac{1}{2}.$$

Thus $y = \rho_X(x)$.

Assume that (3.1) does not hold. Then $\rho_Y(y) \neq x$. By Claim 3 there exists $\widehat{y} \in \widehat{Y}$ such that $\rho_Y(\widehat{y}) = x$. Then $\widehat{y} \in \{y\}'$, and thus $\langle \widehat{y}, b_y^{**} \rangle = 0$. We write

$$T^*\widehat{y} = \lambda x + \mu, \quad \lambda \in \mathbb{C}, \mu \in \text{span}\{x\}'.$$

Then

$$\begin{aligned} 0 &= \langle \widehat{y}, b_y^{**} \rangle = \langle \widehat{y}, T^{**}(T^{-1})^{**}b_y^{**} \rangle = \langle T^*\widehat{y}, (T^{-1})^{**}b_y^{**} \rangle \\ &= \langle \lambda x, (T^{-1})^{**}b_y^{**} \rangle + \langle \mu, (T^{-1})^{**}b_y^{**} \rangle. \end{aligned}$$

Since $x = \rho_Y(\widehat{y})$, we have

$$d < |\langle \widehat{y}, T^{**}a_x^{**} \rangle| = |\langle T^*\widehat{y}, a_x^{**} \rangle| = |\langle \lambda x + \mu, a_x^{**} \rangle| = |\lambda|.$$

Since

$$\|\mu\| + |\lambda| = \|T^*\widehat{y}\| < 2\|\widehat{y}\| = 2,$$

we obtain $\|\mu\| < 2 - c$. By putting everything together we get

$$\begin{aligned} d &< |\lambda| \frac{d}{c} < |\lambda| |\langle x, (T^{-1})^{**}b_y^{**} \rangle| = |\langle \lambda x, (T^{-1})^{**}b_y^{**} \rangle| \\ &= |\langle \mu, (T^{-1})^{**}b_y^{**} \rangle| \leq d\|\mu\| \leq d(2 - c) < d, \end{aligned}$$

a contradiction. Thus (3.1) holds, which means that $\rho_X(\rho_Y(y)) = y$, $y \in \widehat{Y}$.

Now, let $x \in \text{ext } X$ be given. Then there exists $y \in \widehat{Y}$ such that $\rho_Y(y) = x$. Then $y = \rho_X(\rho_Y(y)) = \rho_X(x)$, which means that $x \in \widehat{X}$.

Let $y \in \text{ext } Y$ be given. Then we can find $x \in \widehat{X} = \text{ext } X$ with $\rho_X(x) = y$ and further we can select $\widehat{y} \in \widehat{Y}$ such that $\rho_Y(\widehat{y}) = x$. Then

$$y = \rho_X(x) = \rho_X(\rho_Y(\widehat{y})) = \widehat{y} \in \widehat{Y}.$$

Hence $\widehat{Y} = \text{ext } Y$.

Finally, if $x \in \text{ext } X$, we find $y \in \text{ext } Y$ with $\rho_Y(y) = x$ and obtain

$$\rho_Y(\rho_X(x)) = \rho_Y(\rho_X(\rho_Y(y))) = \rho_Y(y) = x.$$

Till now we have proved that $\rho_X: \text{ext } X \rightarrow \text{ext } Y$ is a bijection with ρ_Y being its inverse. Now we use the assumption on weak peak points to check that ρ_X is a homeomorphism. To this end it is enough to follow the proof of [8, Theorem 7].

4. Cardinality of extreme points

The second result of our paper generalizes a theorem of Cengiz [7] who proved that a pair of locally compact spaces K, L has the same cardinality provided $\mathcal{C}_0(K, \mathbb{F})$ is isomorphic to $\mathcal{C}_0(L, \mathbb{F})$.

We show in Theorem 4.2 the same result in the framework of compact convex sets. Before its proof we need the following lemma on finite-dimensional compact convex sets.

LEMMA 4.1. *Let X be a compact convex set in a finite-dimensional space and let each point of $\text{ext } X$ be a split face. Then the set $\text{ext } X$ is finite and X is a simplex.*

PROOF. We identify X with a subset of \mathbb{R}^m for a suitable $m \in \mathbb{N}$.

First we show that the set $\text{ext } X$ is finite. Assuming the contrary, there is a sequence $\{x_n\}_{n=1}^\infty$ of distinct points in $\text{ext } X$ converging to a point $x \in X$. By the Minkowski theorem (see e.g. [2, Corollary I.6.13] or [26, Theorem 2.11]), x belongs to the convex hull of $\text{ext } X$, thus there exist finite sequences $\{\lambda_i\}_{i=1}^k$ in $(0, 1]$ and $\{z_i\}_{i=1}^k$ in $\text{ext } X$ such that

$$\sum_{i=1}^k \lambda_i = 1 \quad \text{and} \quad x = \sum_{i=1}^k \lambda_i z_i.$$

Now, the function $\chi_{z_1}^*$ is affine, and hence continuous. Since $\{z_1\}$ is a closed split face and $\chi_{z_1}^* = 0$ on $\text{ext } X \setminus \{z_1\}$, the sequence of real numbers $\{\chi_{z_1}^*(x_n)\}_{n=n_0}^\infty$ is identically zero for some suitable $n_0 \in \mathbb{N}$. So, by the continuity of $\chi_{z_1}^*$ we have $\chi_{z_1}^*(x) = 0$. On the other hand, it holds by the affinity of $\chi_{z_1}^*$ that

$$\chi_{z_1}^*(x) = \chi_{z_1}^*\left(\sum_{i=1}^k \lambda_i z_i\right) \geq \lambda_1 \chi_{z_1}^*(z_1) = \lambda_1 > 0,$$

which gives a contradiction. Thus $\text{ext } X$ is a finite set.

Now we show that X is a simplex. We write $\text{ext } X = \{x_i\}_{i=1}^k$. We fix an element $x \in X \setminus \text{ext } X$ and assume that there are two convex combinations

$$x = \sum_{i=1}^n \lambda_i x_i = \sum_{i=1}^n \mu_i x_i,$$

where $\lambda_i, \mu_i \in [0, 1)$, $i = 1, \dots, n$. Fix arbitrary $j \in \{1, \dots, n\}$. By the assumption, $\{x_j\}$ is a split face. Since $\{x_i : i \neq j\}$ is contained in the complementary face $\{x_j\}'$ and

$$x = \lambda_j x_j + (1 - \lambda_j) \sum_{i \neq j} \frac{\lambda_i}{1 - \lambda_j} x_i = \mu_j x_j + (1 - \mu_j) \sum_{i \neq j} \frac{\mu_i}{1 - \mu_j} x_i,$$

from the uniqueness of the decomposition of X to $\{x_j\}$ and $\{x_j\}'$ we obtain $\lambda_j = \mu_j$.

Thus x is a unique convex combination of extreme points of X , from which it follows that X is a simplex. This finishes the proof.

THEOREM 4.2. *Let X, Y be compact convex sets such that $\mathfrak{A}(X, \mathbb{C}), \mathfrak{A}(Y, \mathbb{C})$ are isomorphic. If each point of $\text{ext } X$ and $\text{ext } Y$ is a split face, then the cardinality of $\text{ext } X$ is equal to the cardinality of $\text{ext } Y$.*

PROOF. First we suppose that the space $\mathfrak{A}(X, \mathbb{C})$ is finite-dimensional. Then also $\mathfrak{A}(Y, \mathbb{C})$ is finite-dimensional, with $\dim(\mathfrak{A}(Y, \mathbb{C})) = \dim(\mathfrak{A}(X, \mathbb{C}))$, and also the sets X and Y are finite-dimensional as well. By Lemma 4.1 we have that X is a Bauer simplex with finitely many extreme points, and so it holds that

$$\mathfrak{A}(X, \mathbb{C}) = \mathcal{C}(\text{ext } X, \mathbb{C}) = \ell^\infty(\text{ext } X, \mathbb{C}),$$

and the same holds for Y . Thus

$$|\text{ext } X| = \dim(\ell^\infty(\text{ext } X, \mathbb{C})) = \dim(\ell^\infty(\text{ext } Y, \mathbb{C})) = |\text{ext } Y|.$$

Now suppose that the space $\mathfrak{A}(X, \mathbb{C})$ (and hence also the space $\mathfrak{A}(Y, \mathbb{C})$) is infinite-dimensional. Let $T: \mathfrak{A}(X, \mathbb{C}) \rightarrow \mathfrak{A}(Y, \mathbb{C})$ be an isomorphism. We will show that $|\text{ext } X| \leq |\text{ext } Y|$.

To this end, let $y \in \text{ext } Y$ be fixed. For each $x \in \text{ext } X$ we consider the upper semicontinuous affine function $f_x = \chi_{\{x\}}^*$ and its extension $a_x^{**} \in (\mathfrak{A}(X, \mathbb{C}))^{**}$, see Lemma 3.4. Let $\lambda_y(x) = \langle T^*y, a_x^{**} \rangle$. We claim that the set

$$X_y = \{x \in \text{ext } X : \lambda_y(x) \neq 0\}$$

is at most countable. Indeed, let $s = T^*y$ and $\mu \in \mathcal{M}(X, \mathbb{C})$ be a boundary measure extending s . Let $x \in \text{ext } X$ be arbitrary. Let $\{a_j\}_{j \in J}$ be a bounded downward directed net of functions in $\mathfrak{A}(X, \mathbb{R})$ converging to $f_x = \chi_{\{x\}}^*$. Then we have

$$\begin{aligned} \mu(\{x\}) &= \mu(\chi_{\{x\}}) = \mu(\chi_{\{x\}}^*) = \lim_{j \in J} \mu(a_j) = \lim_{j \in J} \langle s, a_j \rangle \\ &= \lim_{j \in J} \langle T^*y, a_j \rangle = \langle T^*y, a_x^{**} \rangle = \lambda_y(x). \end{aligned}$$

Since $\|\mu\| < \infty$, $\mu(\{x\}) \neq 0$ for at most countably many $x \in \text{ext } X$.

Now we prove that for each $x \in \text{ext } X$ there exists $y \in \text{ext } Y$ such that $x \in X_y$. To this end, we assume the contrary. Let $x \in \text{ext } X$ be such that

$$\langle T^*y, a_x^{**} \rangle = 0, \quad y \in \text{ext } Y.$$

Using the same argument as in the proof of Theorem 1.1, Lemma 3.6 yields

$$\begin{aligned} 0 &= \sup_{y \in \text{ext } Y} |\langle T^*y, a_x^{**} \rangle| = \sup_{y \in \text{ext } Y} |\langle y, T^{**}a_x^{**} \rangle| = \sup_{s \in \mathbb{T} \cdot \text{ext } Y} |\langle s, T^{**}a_x^{**} \rangle| \\ &= \sup_{s \in B_{\mathfrak{A}(Y, \mathbb{C})}^*} |\langle s, T^{**}a_x^{**} \rangle| = \|T^{**}a_x^{**}\| \neq 0, \end{aligned}$$

i.e., a contradiction.

Now both the spaces $\mathfrak{A}(X, \mathbb{C})$ and $\mathfrak{A}(Y, \mathbb{C})$ are infinite-dimensional, and thus the sets $\text{ext } X$ and $\text{ext } Y$ are infinite. Indeed, if $\text{ext } X$ were finite, by the minimum principle we would obtain that the space $\mathfrak{A}(X, \mathbb{C}) \subset \ell^\infty(\text{ext } X, \mathbb{C})$ is finite-dimensional.

Now, since we have $\text{ext } X = \bigcup_{y \in \text{ext } Y} X_y$, we get $|\text{ext } X| \leq |\text{ext } Y|$.

By reversing the role of X and Y we obtain the converse inequality, which concludes the proof.

5. Isomorphisms of complex function spaces

This section uses the results of the previous sections to deduce analogous theorems on selfadjoint function spaces. Throughout this section we consider a compact (Hausdorff) space K and a closed subspace $\mathcal{H} \subset \mathcal{C}(K, \mathbb{C})$ which contains constants and separates points of K . By $\mathbf{S}(\mathcal{H})$ we denote the *state space* of \mathcal{H} , i.e., the set

$$\mathbf{S}(\mathcal{H}) = \{s \in \mathcal{H}^* : \|s\| = s(1) = 1\}$$

endowed with the weak* topology. Let $\phi: K \rightarrow \mathbf{S}(\mathcal{H})$ be the evaluation mapping, then ϕ homeomorphically embeds K into the compact convex set $\mathbf{S}(\mathcal{H})$. The *Choquet boundary* $\text{Ch}_{\mathcal{H}} K$ of \mathcal{H} is defined as

$$\text{Ch}_{\mathcal{H}} K = \{x \in K : \phi(x) \in \text{ext } \mathbf{S}(\mathcal{H})\}.$$

By [5, Theorem 2.2.8], $\text{ext } \mathbf{S}(\mathcal{H}) = \phi(\text{Ch}_{\mathcal{H}} K)$. Let $\Phi: \mathcal{H} \rightarrow \mathfrak{A}(\mathbf{S}(\mathcal{H}), \mathbb{C})$ be defined as $\Phi(h)(s) = s(h)$, $s \in \mathbf{S}(\mathcal{H})$, $h \in \mathcal{H}$. Then we have the following identification.

LEMMA 5.1. *Let \mathcal{H} be a selfadjoint closed subspace of $\mathcal{C}(K, \mathbb{C})$ for some compact space K such that \mathcal{H} contains constants and separates points of K . Then the mapping Φ is a isometric isomorphism of \mathcal{H} onto $\mathfrak{A}(\mathbf{S}(\mathcal{H}), \mathbb{C})$.*

PROOF. Clearly, Φ is linear and of norm 1. Since

$$\|h\| \geq \|\Phi(h)\|_{\mathfrak{A}(\mathbf{S}(\mathcal{H}), \mathbb{C})} = \sup_{s \in \mathbf{S}(\mathcal{H})} |s(h)| \geq \sup_{x \in K} |h(x)| = \|h\|,$$

Φ is an isometry. It remains to show that Φ is onto.

To this end, let $f \in \mathfrak{A}(\mathbf{S}(\mathcal{H}), \mathbb{C})$ be given. Any $s \in \mathcal{H}^*$ can be written as $s = \sum_{k=0}^3 i^k a_k s_k$, where $a_k \geq 0$, $s_k \in \mathbf{S}(\mathcal{H})$, $k = 0, \dots, 3$. We define $\tilde{f}: \mathcal{H}^* \rightarrow \mathbb{C}$ as

$$\tilde{f}(s) = \sum_{k=0}^3 i^k a_k f(s_k), \quad s = \sum_{k=0}^3 i^k a_k s_k, \quad a_k \geq 0, \quad s_k \in \mathbf{S}(\mathcal{H}), \quad k = 0, \dots, 3. \quad (5.1)$$

We have to check that this definition is correct, i.e., that

$$\sum_{k=0}^3 i^k a_k f(s_k) = \sum_{k=0}^3 i^k b_k f(t_k),$$

whenever $\sum_{k=0}^3 i^k a_k s_k = \sum_{k=0}^3 i^k b_k t_k$, $a_k, b_k \geq 0$, $s_k, t_k \in \mathbf{S}(\mathcal{H})$, $k = 0, \dots, 3$.

So let

$$(a_0 s_0 - a_1 s_1) + i(a_2 s_2 - a_3 s_3) = (b_0 t_0 - b_1 t_1) + i(b_2 t_2 - b_3 t_3). \quad (5.2)$$

Since any $s \in \mathbf{S}(\mathcal{H})$ can be extended by the Hahn-Banach theorem to a measure $\mu \in \mathcal{M}^1(K)$, $s(\operatorname{Re} h) \in \mathbb{R}$ for each $h \in \mathcal{H}$. (We remind that $\operatorname{Re} h, \operatorname{Im} h \in \mathcal{H}$ for each $h \in \mathcal{H}$ since \mathcal{H} is selfadjoint.) An application of (5.2) to the constant function 1 yields

$$a = a_0 + b_1 = b_0 + a_1, \quad b = a_2 + b_3 = b_2 + a_3.$$

If $a = 0$, $a_0 = a_1 = b_0 = b_1 = 0$, and thus $a_0 f(s_0) - a_1 f(s_1) = b_0 f(t_0) - b_1 f(t_1)$. Otherwise we have for each $h \in \mathcal{H}$ equality

$$((a_0 s_0 - a_1 s_1) + i(a_2 s_2 - a_3 s_3))(\operatorname{Re} h) = ((b_0 t_0 - b_1 t_1) + i(b_2 t_2 - b_3 t_3))(\operatorname{Re} h),$$

which implies

$$(a_0 s_0 - a_1 s_1)(\operatorname{Re} h) = (b_0 t_0 - b_1 t_1)(\operatorname{Re} h), \quad h \in \mathcal{H}.$$

In other words,

$$a \left(\frac{a_0}{a} s_0 + \frac{b_1}{a} t_1 \right) (\operatorname{Re} h) = a \left(\frac{b_0}{a} t_0 + \frac{a_1}{a} s_1 \right) (\operatorname{Re} h), \quad h \in \mathcal{H}.$$

Since $\operatorname{Im} h \in \mathcal{H}$ and $\operatorname{Re}(\operatorname{Im} h) = \operatorname{Im} h$ for each $h \in \mathcal{H}$,

$$\frac{a_0}{a} s_0 + \frac{b_1}{a} t_1 = \frac{b_0}{a} t_0 + \frac{a_1}{a} s_1.$$

Since f is affine, we obtain

$$\begin{aligned} \frac{a_0}{a}f(s_0) + \frac{b_1}{a}f(t_1) &= f\left(\frac{a_0}{a}s_0 + \frac{b_1}{a}t_1\right) \\ &= f\left(\frac{b_0}{a}t_0 + \frac{a_1}{a}s_1\right) = \frac{b_0}{a}f(t_0) + \frac{a_1}{a}f(s_1), \end{aligned}$$

i.e.,

$$a_0f(s_0) - a_1f(s_1) = b_0f(t_0) - b_1f(t_1).$$

Similarly we get

$$a_2f(s_2) - a_3f(s_3) = b_2f(t_2) - b_3f(t_3),$$

which shows that \tilde{f} is by (5.1) well defined.

It follows from (5.1) that $\tilde{f}: \mathcal{H}^* \rightarrow \mathbb{C}$ is linear. Indeed, let $s, t \in \mathcal{H}^*$ be given and let

$$s = \sum_{k=0}^3 i^k a_k s_k, \quad t = \sum_{k=0}^3 i^k b_k t_k,$$

where $a_k, b_k \geq 0$, $s_k, t_k \in \mathbf{S}(\mathcal{H})$, $k = 0, \dots, 3$. We select $u \in \mathbf{S}(\mathcal{H})$ and define

$$u_k = \begin{cases} \frac{a_k}{a_k + b_k} s_k + \frac{b_k}{a_k + b_k} t_k, & a_k + b_k > 0, \\ u, & a_k = b_k = 0, \end{cases}$$

and

$$c_k = a_k + b_k, \quad k = 0, \dots, 3.$$

Then $u_k \in \mathbf{S}(\mathcal{H})$ and

$$s + t = \sum_{k=0}^3 i^k c_k u_k.$$

Since f is affine on $\mathbf{S}(\mathcal{H})$, we obtain

$$\tilde{f}(s + t) = \sum_{k=0}^3 i^k c_k f(u_k) = \sum_{k=0}^3 i^k (a_k f(s_k) + b_k f(t_k)) = \tilde{f}(s) + \tilde{f}(t).$$

It is even more straightforward to verify that $\tilde{f}(\lambda s) = \lambda \tilde{f}(s)$, whenever $s \in \mathcal{H}^*$ and $\lambda \geq 0$, $\lambda = -1$, or $\lambda = i$. Thus \tilde{f} is linear.

To check that \tilde{f} is given by an element from \mathcal{H} it is enough to verify its weak* continuity on \mathcal{H}^* . Since \tilde{f} is linear, it is enough to check its weak* continuity on $B_{\mathcal{H}^*}$ (see [13, Corollary 3.94]). We assume that this is not the

case and seek a contradiction. So let $\{s_j\}_{j \in J}$ be a net in $B_{\mathcal{H}^*}$ weak* converging to $s \in B_{\mathcal{H}^*}$ such that $|\tilde{f}(s_j) - \tilde{f}(s)| \geq \eta$ for some $\eta > 0$. Using the Hahn-Banach theorem and the decomposition of a complex measure we write each s_j as $s_j = \sum_{k=0}^3 i^k a_k^j s_k^j$, where $a_k^j \geq 0$, $s_k^j \in \mathbf{S}(\mathcal{H})$ and $a_0^j + a_1^j + a_2^j + a_3^j \leq 2$. By a compactness argument we may assume that $a_k^j \rightarrow a_k$ and $s_k^j \rightarrow s_k$ in the weak* topology, $k = 0, \dots, 3$. Then $s = \sum_{k=0}^3 i^k a_k s_k$. By the continuity of f on $\mathbf{S}(\mathcal{H})$, $f(s_k^j) \rightarrow f(s_k)$ for each $k = 0, \dots, 3$. But then

$$\eta \leq \lim_{j \in J} |\tilde{f}(s_j) - \tilde{f}(s)| = \lim_{j \in J} \left| \sum_{k=0}^3 i^k a_k^j f(s_k^j) - \sum_{k=0}^3 i^k a_k f(s_k) \right| = 0$$

gives a contradiction. Hence \tilde{f} is weak* continuous on $B_{\mathcal{H}^*}$, and thus on \mathcal{H}^* .

Thus there exists an element $h \in \mathcal{H}$ such that $\tilde{f}(s) = s(h)$, $s \in \mathcal{H}^*$. In particular, $\Phi(h) = f$. The proof is finished.

As in the first section we say that $x \in K$ is a *weak peak point* if

$$\begin{aligned} &\text{given } \varepsilon \in (0, 1) \text{ and an open set } U \subset K \text{ containing } x, \\ &\text{there exists } f \in B_{\mathcal{H}} \text{ such that } |f| < \varepsilon \text{ on } \text{Ch}_{\mathcal{H}} K \setminus U \text{ and} \quad (5.3) \\ &f(x) > 1 - \varepsilon. \end{aligned}$$

LEMMA 5.2. *Let $x \in K$ be a weak peak point in the sense of (5.3). Then $\phi(x)$ is a weak peak point of $\mathbf{S}(\mathcal{H})$ in the sense of (2.1).*

PROOF. Suppose that $x \in K$ is a weak peak point in the sense of (5.3), and that we are given $\varepsilon > 0$ and an open neighborhood V of $\phi(x)$ in $\mathbf{S}(\mathcal{H})$. Then we have that $U = \phi^{-1}(V)$ is an open neighborhood of x . So there exists $f \in B_{\mathcal{H}}$ such that $|f| < \varepsilon$ on the set $\text{Ch}_{\mathcal{H}} K \setminus U$ and $f(x) > 1 - \varepsilon$. We denote $a = \Phi(f) \in B_{\mathfrak{A}(\mathbf{S}(\mathcal{H}), \mathbb{C})}$ and we show that a is witnessing the fact that $\phi(x)$ is a weak peak point of $\mathbf{S}(\mathcal{H})$. Firstly, we have that

$$a(\phi(x)) = \Phi(f)(\phi(x)) = \phi(x)(f) = f(x) > 1 - \varepsilon.$$

Now, suppose that $s \in \text{ext } \mathbf{S}(\mathcal{H}) \setminus V$. There is $y \in \text{Ch}_{\mathcal{H}} K$ such that $s = \phi(y)$. Then $\phi(y) \notin V$, and hence $y \notin U$. Thus

$$|a(s)| = |\Phi(f)(\phi(y))| = |f(y)| < \varepsilon,$$

which concludes the proof.

Now we can extend the results of the previous sections to the context of function spaces.

THEOREM 5.3. *For $i = 1, 2$, let K_i be a compact space and \mathcal{H}_i be a self-adjoint closed subspace of $\mathcal{C}(K_i, \mathbb{C})$ which contains constants and separates points of K_i . Let each point of $\text{Ch}_{\mathcal{H}_i} K_i$ be a weak peak point.*

- (i) *If there exists an isomorphism $T: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ satisfying $\|T\| \cdot \|T^{-1}\| < 2$, then $\text{Ch}_{\mathcal{H}_1} K_1$ is homeomorphic to $\text{Ch}_{\mathcal{H}_2} K_2$.*
- (ii) *If there exists an isomorphism $T: \mathcal{H}_1 \rightarrow \mathcal{H}_2$, then $\text{Ch}_{\mathcal{H}_1} K_1$ has the same cardinality as $\text{Ch}_{\mathcal{H}_2} K_2$.*

PROOF. By the identification given by Lemma 5.1, the space $\mathfrak{A}(\mathbf{S}(\mathcal{H}_1), \mathbb{C})$ is isomorphic to $\mathfrak{A}(\mathbf{S}(\mathcal{H}_2), \mathbb{C})$, and by an isomorphism T satisfying $\|T\| \cdot \|T^{-1}\| < 2$ in case (i). Moreover, Lemma 5.2 allows us to use Theorems 1.1 and 4.2 to conclude that $\text{ext } \mathbf{S}(\mathcal{H}_1)$ is homeomorphic to $\text{ext } \mathbf{S}(\mathcal{H}_2)$ in case (i), and in case (ii) those sets have the same cardinality. Hence the assertion follows.

We refer the reader to [21] and [20] for results on function algebras in the spirit of the above theorems. The case of vector-valued Banach-Stone type theorems is treated e.g. in [4], [14] or [1].

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