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THE AMALGAMATION PROPERTY, THE UNIVERSAL-HOMOGENEOUS MODELS, AND THE GENERIC MODELS

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In the mid 1960's, the possibility of applying the notion of forcing to model theory must have occurred to and been developed by various people. For instance, in [13] G. Reyes established a substantial body of knowledge in connexion with topological spaces on classes of models. Certainly, a series of papers, which appeared a few years later, by A. Robinson is the main stimulus for the writer's as well as many others' interest in this field. Soon after it was observed by several people that $\boldsymbol{M}$-universal-homogeneous structures and their elementary substructures are generic, where $\boldsymbol{M}$ is the class of all models of a given theory T (cf. [21], [2]). The present paper originates in trying to see if the converse is true. It is, if there are arbitrarily large $\boldsymbol{M}$-universal-homogeneous structures. To guarantee the existence of such structures, Jónsson assumed various properties on the class $\boldsymbol{M}$, of which some are automatically satisfied when $\boldsymbol{M}$ is the class of all models of a theory, but the amalgamation property, the joint embeddability property, the inductiveness of the theory, and an assumption on cardinal arithmetic are not fulfilled gratuitously. The inductiveness of the theory is harmless for our purpose, since it is needed for constructing generic structures anyhow. (Robinson's device in [15] of considering all substructures of models instead of models only amounts to considering another theory which is inductive.) So we try to circumvent the other three assumptions and arrive at the classes of $\varkappa$-objective and $\varkappa$-subjective models, where $\varkappa$ is an infinite cardinal. (In view of 2.10.1 below and a comment after it, these are really generalized universal-homogeneous and homogeneous models, respectively. However we wanted short names and have chosen "objective" and "subjective".) The answer to our original question is given by the combination of 2.5 and 3.2.2:

If the theory T is inductive, then a model of T is generic if and only if it is an elementary submodel of an $\aleph_1$-objective model.

In the first section we study the amalgamation property and amal-

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gamative models; in the second section we define $\varkappa$-objective and $\varkappa$-subjective models and investigate them in terms of universality and injectivity; and in the last section the connexion of the generic models with $\varkappa$-objective models, etc., is considered. In the process, the amalgamation property and the amalgamative models came to play a dominant role. The following three results are easy to state and may be worth quoting here. They are that if the theory is inductive then every model can be extended to an amalgamative model (1.7); that no non-amalgamative model has an amalgamative envelope (a kind of the smallest amalgamative extension) (2.12); and that an inductive theory has the amalgamation property if and only if there are enough $\aleph_1$-weak injectives (2.11). The last may appear to be an extension of Pierce's result (cf. [12]) that an equational class has the amalgamation property if there are enough injectives. However, weak injectivity, which is introduced by Simmons ([17], [18]), is not really a weakened version of injectivity, and so our result does not imply Pierce's. Nevertheless, it is an algebraic characterization of the amalgamation property, and may not be without merit as the converse of Pierce's result is false. (The class of lattices has the amalgamation property [8], but there is no non-trivial injective lattice [5].)

Assumptions, conventions, and terminology.

We assume that a theory T under consideration is formulated in a countable language L, and has infinite models. From time to time, we consider substructures of models which may not be models. "Structures" is used in this sense, while "systems" is used to talk about relational systems of an appropriate similarity type. We notice here that the class of structures is the class of models of a new theory $T_\forall$, that consists of universal sentences in L provable from the given theory T. Given a system S, $L(S)$ is the language of S, that is, L augmented with new constants denoting members of S; and $Dg(S)$ is the diagram of S. Notations like "$\varphi \in \Sigma_1$" and "$\varphi \in \Sigma_1(S)$" express that φ is (logically equivalent to) an existential formula in L and in $L(S)$, respectively. Similar conventions apply to Σ_n and $\Pi_n(S)$, etc. A bold face lower case letters like $\boldsymbol{a}$ denote a sequence — finite, usually — of constants, and $\boldsymbol{a} \in S$ means that $\boldsymbol{a}$ is a sequence of constants denoting members of the universe of the system S. For a system C, $\mathrm{cd}(C)$ denotes the cardinality of the universe of C, and the cofinality of a cardinal $\varkappa$ is denoted by $\mathrm{cf}(\varkappa)$, and $\varkappa^+$ is the smallest cardinality larger than $\varkappa$. In most cases we use the same capital letter to denote a system and its underlying set (that is universe). When it is advisable to do so, we denote by '$|M|$' the underlying set of the system M. All mappings we consider are into isomorphisms, and hence we use the word "map" in this sense. "Embedding", "injection" and "inclusion map" are used also for emphasis and qualification. A map $f: A \to B$ is considered to be a set of ordered pairs, and hence when $C \supseteq B$ we have no hesitation in considering f as a map of A to C as well. By an isomorphism, we mean an onto map, and use notations like "$f: A \cong B$". "$i: A \to B$" always means the inclusion map of A to B. Given a map $f: A \to B$ and a sentence $\varphi(\boldsymbol{a}) \in L(A)$, by $\varphi(f(\boldsymbol{a}))$ we denote the sentence of $L(B)$ that is obtained from φ by replacing the new constants of $L(A)$ by those that denote the image under f. Given classes $\boldsymbol{K}$ and $\boldsymbol{C}$ of systems, we say $\boldsymbol{C}$ is *cofinal* in $\boldsymbol{K}$ if for each $K \in \boldsymbol{K}$ there is a $C \in \boldsymbol{C}$ such that $C \supseteq K$. When $\boldsymbol{C}$ is included in $\boldsymbol{K}$ in addition, we also say that there are *enough* $\boldsymbol{C}$ in $\boldsymbol{K}$. For instance, the class of models and that of structures (in the above sense) are cofinal in each other, and there are enough models in the structures. Often we denote by '$\boldsymbol{CK}$' the intersection of the classes $\boldsymbol{C}$ and $\boldsymbol{K}$. We use '$\boldsymbol{G}$' and '$\boldsymbol{E}$' to denote the classes of generic, and of existentially closed structures. We use the abbreviations 'AP', 'JE', and 'LST' for 'the amalgamation property', 'the joint embeddability property', and 'the Löwenheim–Skolem theorem in the sense of [22]', respectively.

There are two obvious directions of generalization. The first is to consider languages of higher cardinalities, and the second is to consider those maps that preserve the truth of, say, $\forall\exists$-sentences, in place of injection, which preserves the truth of open sentences. We refrained from these generalizations because no new difficulty or gain is anticipated.

1.

A model A of T is called *amalgamative*, in symbols $A \in \boldsymbol{A}$, if for any two given injections f, g of A to models M, N there exist injections f_1 and g_1 of M and N, respectively, to a model K such that $f_1 f = g_1 g$.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & M \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_1} \\
N & \xrightarrow{\ g_1\ } & K
\end{array}
$$

An extension model A' of A is an *amalgamator* of A, in symbols $A' \in \boldsymbol{A}(A)$, if any two injections f and g of A can be amalgamated, provided f

and g can be *factored through* A', that is, there are maps f' and g' of A' such that $f=f'\circ i$ and $g=g'\circ i$ where i is the inclusion map of A into A'.

The following propositions are trivial consequences of the definitions:

PROPOSITIONS.

1.1.1. *A model is amalgamative if and only if it is its own amalgamator.*

1.1.2. *If A' is an amalgamator of A and f is a map of A', then $f(A')$ is an amalgamator of $f(A)$. Hence, an isomorphic image of an amalgamative model is amalgamative.*

1.1.3. *A model B is an amalgamator of A in the class of models if and only if it is so in the class of structures.*

1.1.4. *A theory has the amalgamation property if and only if all models are amalgamative.*

Because of 1.1.3, when we talk of amalgamatives and amalgamators, we shall not always specify whether we are considering them in terms of models or in terms of structures. The next one can also be proved easily from the definitions.

PROPOSITIONS.

1.2.1. *Each extension of an amalgamator of a model is an amalgamator of the model.*

1.2.2. *An amalgamator of a structure is an amalgamator of any substructure of the structure.*

1.2.3. *Each extension of A is an amalgamator of A, if it is amalgamative.*

Here we prove only 1.2.2, because a proof of 1.2.1 will be given after the next theorem as an illustration, and 1.2.3 follows immediately from 1.2.2 and 1.1.1.

PROOF OF 1.2.2. Let $B' \in \boldsymbol{A}(B)$ and $A \subseteq B$. We are to show $B' \in \boldsymbol{A}(A)$. For given maps $f: A \to M$ and $g: A \to N$, assume that there are $f': B' \to M$ and $g': B' \to N$ such that $f=f'\circ i$ and $g=g'\circ i$ where $i: A \to B'$. We let $i_1: A \to B$ and $i_2: B \to B'$, and let $f_1=f'\circ i_2$ and $g_1=g'\circ i_2$. Then since $B' \in \boldsymbol{A}(B)$ there are maps f_2 and g_2 on M and N, respectively, such that $f_2\circ f_1=g_2\circ g_1$, hence $f_2\circ f_1\circ i_1=g_2\circ g_1\circ i_1$. But

$$f_1\circ i_1 = (f'\circ i_2)\circ i_1 = f'\circ(i_2\circ i_1) = f'\circ i = f\,,$$

and similarly $g_1\circ i_1=g$. Thus $f_2\circ f=g_2\circ g$, and so $B' \in \boldsymbol{A}(A)$.

For later purposes, various characterizations of the relation "A' is an amalgamator of A" will be useful. So we formulate and prove:

THEOREM 1.3. *The following four are equivalent:*

(i) *A' is an amalgamator of A.*

(ii) *For any two existential sentences $\varphi(\boldsymbol{a})$ and $\psi(\boldsymbol{a})$ of $L(A)$ if each is consistent with $T+Dg(A')$, then $T+Dg(A)+\varphi(\boldsymbol{a})+\psi(\boldsymbol{a})$ is consistent.*

(iii) *Any existential sentence $\varphi(\boldsymbol{a})$ of $L(A)$ which is consistent with $T+Dg(A')$ is also consistent with $T+Dg(M)$ for any extension M of A'.*

(iv) *For any two universal formulas φ and ψ of L, any existential formula ζ, and any sequence $\boldsymbol{a}\in A$, if $A\models\zeta(\boldsymbol{a})$ and*

$$(*)\qquad T\vdash\forall\boldsymbol{x}(\zeta(\boldsymbol{x})\rightarrow\varphi(\boldsymbol{x})\vee\psi(\boldsymbol{x}))$$

then there is an existential formula ξ such that $A'\models\xi(\boldsymbol{a})$ and

$$(**)\qquad \text{either } T\vdash\forall\boldsymbol{x}(\xi(\boldsymbol{x})\rightarrow\varphi(\boldsymbol{x})) \text{ or } T\vdash\forall\boldsymbol{x}(\xi(\boldsymbol{x})\rightarrow\psi(\boldsymbol{x}))\,.$$

PROOF. We show the equivalence of (i) and (iv) only, since (ii) and (iii) can be shown similarly and much more simply to be equivalent to (i).

Assume that (i) is false. Hence there are two maps of A that can be factored through A' but cannot be amalgamated. By replacing isomorphic images whenever necessary, we can assume f, g, f' and g' in the above definition to be inclusion maps. Thus our assumption gives that there are extensions M and N of A' such that the inclusion maps of A into M and N cannot be amalgamated. This happens exactly when $T+Dg(M)+Dg(N)$ is inconsistent, where $Dg(M)$ and $Dg(N)$ are written in such a way that the constants common to them are exactly those that denote members of A. (Thus members of $A'-A$ are denoted differently in $Dg(M)$ and $Dg(N)$.) Hence there are three finite sets

$$\zeta'\subseteq Dg(A),\quad \varphi'\subseteq Dg(M)-Dg(A),\quad \psi'\subseteq Dg(N)-Dg(A)$$

such that $T\cup\{\zeta',\varphi',\psi'\}$ is inconsistent, or

$$T\vdash\zeta_1\rightarrow\neg\varphi_1\vee\neg\psi_1\,,$$

where ζ_1, φ_1 and ψ_1 are conjunctions of members of ζ', φ' and ψ', respectively. By replacing new constants by variables, quantifying universally, and using the predicate calculus, we have $\varphi,\psi\in\Pi_1$ and $\zeta\in\Sigma_1$ such that

$$T\vdash\forall\boldsymbol{x}(\zeta(\boldsymbol{x})\rightarrow\varphi(\boldsymbol{x})\vee\psi(\boldsymbol{x}))\,.$$

From the way in which constants were chosen in writing the diagrams, we know that the variables $\boldsymbol{x}$ are replacing constants of $L(A)$. Since ζ comes from $\zeta'\subseteq Dg(A)$ by conjunction and existential quantification, we have $A\models\zeta(\boldsymbol{a})$ for some $\boldsymbol{a}\in A$. Since φ comes from $\neg\varphi_1$ by universal quantification, and $\varphi'\subseteq Dg(M)$, we have $M\models\neg\varphi(\boldsymbol{a})$. Similarly $N\models$

$\neg\psi(\boldsymbol{a})$. Now take any $\xi \in \Sigma_1$ and assume $A' \models \xi(\boldsymbol{a})$. Then since $M \supseteq A'$, we have $M \models \xi(\boldsymbol{a})$. So, were it true that

$$T \vdash \forall \boldsymbol{x}(\xi(\boldsymbol{x}) \to \varphi(\boldsymbol{x})) ,$$

then $M \models \varphi(\boldsymbol{a})$, as M is a model of T. But this is impossible because $M \models \neg\varphi(\boldsymbol{a})$. Similarly,

$$T \nvdash \forall \boldsymbol{x}(\xi(\boldsymbol{x}) \to \psi(\boldsymbol{x})) .$$

Thus (iv) is false.

Conversely, assume (i), and take φ, ψ, $\boldsymbol{a}$, ζ as in (iv). We show first that either $T, Dg(A') \vdash \varphi(\boldsymbol{a})$ or $T, Dg(A') \vdash \psi(\boldsymbol{a})$. For, otherwise there are models M and N that are extensions of A', and $M \models \neg\varphi(\boldsymbol{a})$ and $N \models \neg\psi(\boldsymbol{a})$. The inclusion maps of A to M and to N are certainly factored through the amalgamator A'. Hence there are a model K and injections $f_1 : M \to K$ and $g_1 : N \to K$ that coincide on A. By taking isomorphic images, we may assume that f_1 and g_1 are the identity on A. Thus K is a model of $f_1(Dg(A'))$, $g_1(Dg(A'))$, $\neg\varphi(\boldsymbol{a})$ and $\neg\psi(\boldsymbol{a})$, as these are all existential sentences. On the other hand, since $A \models \zeta(\boldsymbol{a})$ and

$$T \vdash \forall \boldsymbol{x}(\zeta(\boldsymbol{x}) \to \varphi(\boldsymbol{x}) \vee \psi(\boldsymbol{x}))$$

by assumption, we have $K \models \zeta(\boldsymbol{a})$, hence $K \models \varphi(\boldsymbol{a})$ or $K \models \psi(\boldsymbol{a})$. This contradiction shows our claim above. So, for instance, assume that $T, Dg(A') \vdash \varphi(\boldsymbol{a})$. By taking the conjunction of an appropriate finite subset of $Dg(A')$, changing constants to variables and quantifying them, we have a $\xi \in \Sigma_1$ such that $A' \models \xi(\boldsymbol{a})$ and

$$T \vdash \forall \boldsymbol{x}(\xi(\boldsymbol{x}) \to \varphi(\boldsymbol{x})) .$$

REMARK. Condition (iv) reduces the question: "if N is an amalgamator of M", and hence "if M is amalgamative", to the question about the truth of sentences in $\Sigma_1(M)$ which is relatively easy to check. As an example of such application, we prove 1.2.1 in detail; and further on we dismiss similar proofs by saying "by condition (iv)", or some such phrase.

PROOF OF 1.2.1. Let $A' \in \boldsymbol{A}(A)$ and $B \supseteq A'$, and for given $\varphi, \psi \in \Pi_1$, $\zeta \in \Sigma_1$ and $\boldsymbol{a} \in M$ assume that $A \models \zeta(\boldsymbol{a})$ and (*) in 1.3, (iv). We are to find a $\xi \in \Sigma_1$ such that $B \models \xi(\boldsymbol{a})$ and (**). But since $A' \in \boldsymbol{A}(A)$ there is indeed a $\xi \in \Sigma_1$ such that $A' \models \xi(\boldsymbol{a})$ and (**). Since $B \supseteq A'$ and ξ is existential, we have $B \models \xi(\boldsymbol{a})$. Condition (**) naturally holds as it is about provability from T, and thus is independent of the models in question.

As immediate consequences of 1.3, we can characterize the class $\boldsymbol{A}$ in the infinitary language $L_{\omega_1, \omega}$.

PROPOSITION 1.4. *Given a theory T formulated in L, there is an infinitary sentence Φ in the vocabulary of L such that a model of T is amalgamative if and only if Φ is true in it.*

PROOF. For $\varphi, \psi \in \Pi_1$ and $\zeta \in \Sigma_1$ such that 1.3, (iv), (*) holds, let $S(\varphi, \psi, \zeta)$ be the set of those $\xi \in \Sigma_1$ whose free variables are included in those of φ, ψ and ζ, and for which 1.3, (iv), (**) holds. Let Φ be the infinitary sentence

$$\bigwedge\nolimits_{\varphi, \psi, \zeta} \forall \boldsymbol{x}(\zeta(\boldsymbol{x}) \to \bigvee S(\varphi, \psi, \zeta)) ,$$

where the initial infinite conjunction runs over all triples φ, ψ, ζ for which (*) holds. From the equivalence of (i) and (iv) of 1.3, we know that for any model M of T, $M \models \Phi$ exactly when M is its own amalgamator, that is, amalgamative by 1.1.1.

The following proposition will be proved in the manner of the proof of the Main Theorem of [3].

PROPOSITION 1.5. *Every model M has an amalgamator of power* $\mathrm{cd}(M) + \aleph_0$.

PROOF. Given a model A, let Δ be a set of $L(A)$ sentences such that $Dg(A) \subseteq \Delta \subseteq \Sigma_1(A)$, Δ is consistent with T, and is a maximal such set. (The existence of Δ follows from Zorn's lemma.) Let $A' \models T + \Delta$ and $\mathrm{cd}(A') = \mathrm{cd}(A) + \aleph_0$. We show 1.3, (iv) for A and A'. Take φ, ψ, ζ and $\boldsymbol{a}$ as in 1.3, (iv), and assume that $A \models \zeta(\boldsymbol{a})$ and (*). First we show that $T, \Delta \vdash \varphi(\boldsymbol{a})$ or $T, \Delta \vdash \psi(\boldsymbol{a})$. For otherwise, $T, \Delta \neg\varphi(\boldsymbol{a})$ and $T, \Delta \neg\psi(\boldsymbol{a})$ are consistent. Since $\neg\varphi(\boldsymbol{a}), \neg\psi(\boldsymbol{a}) \in \Sigma_1(A)$ and Δ is maximal, $\neg\varphi(\boldsymbol{a})$ and $\neg\psi(\boldsymbol{a})$ are both in Δ. Hence

$$A' \models \neg(\varphi(\boldsymbol{a}) \vee \psi(\boldsymbol{a})) .$$

Since $A' \supseteq A$, we have $A' \models \zeta(\boldsymbol{a})$, whence

$$A' \models \varphi(\boldsymbol{a}) \vee \psi(\boldsymbol{a})$$

by (*), causing a contradiction. Thus a ξ as required can be obtained from Δ.

PROPOSITIONS.

1.6.1. *Given an ascending chain of structures $\langle M_\alpha \rangle_{\alpha < \varkappa}$, if $M_{\alpha+1}$ is an amalgamator of M_α for each $\alpha < \varkappa$, then $M = \bigcup_{\alpha < \varkappa} M_\alpha$ is an amalgamative structure.*

1.6.2. *The union of an ascending chain of amalgamative structures is an amalgamative structure.*

1.6.3. *An existentially closed structure is amalgamative.*

PROOF OF 1.6.1. For any given $\boldsymbol{a} \in M$ and $\zeta \in \Sigma_1$ such that $M \models \zeta(\boldsymbol{a})$ and 1.3, (iv), (*) in terms of $T_\forall$, there must be an α such that $M_\alpha \models \zeta(\boldsymbol{a})$. Since $M_{\alpha+1} \in \boldsymbol{A}(M_\alpha)$ there is a $\xi \in \Sigma_1$ such that $M_{\alpha+1} \models \xi(\boldsymbol{a})$ and (**). But then $M \models \xi(\boldsymbol{a})$ also.

A proof of 1.6.2 can be obtained from 1.2.3 and 1.6.1.

PROOF OF 1.6.3. Given an existentially closed structure M, take $N \in \boldsymbol{A}(M)$. But since $N \models \varphi$ implies $M \models \varphi$ for all $N \supseteq M$ and all $\varphi \in \Sigma_1(M)$, we can conclude that $M \in \boldsymbol{A}(M)$ from 1.3, (iv).

THEOREM 1.7. *Every model M can be extended to an amalgamative model of power* $\mathrm{cd}(M) + \aleph_0$, *provided the theory is inductive.*

PROOF. From 1.5 we obtain an ascending chain of length ω of models of the same cardinality, each being an amalgamator of the previous members. The union is a model of the same cardinality, and is amalgamative by 1.6.1.

A kind of dual question to this result is whether every model includes an amalgamative submodel. The negative answer can be given by a simple modification of Kimura's example cited in 9.4 [4]. Let T_K be the theory of semigroups with four constants, say, 0, 1, 2, 3. The axioms of T_K say, besides that models are semigroups, that these four constants denote four different elements and the product of any two is always the element denoted by 0. Then the semigroup consisting exactly of these four elements is a prime model of T_K and is non-amalgamative, and hence has no amalgamative submodel.

Given a model M, and extension N of M is called an *existential extension*, in symbols $N \in \boldsymbol{E}(M)$, if for each $\varphi \in \Sigma_1(M)$, $M \models \varphi$ whenever $N \models \varphi$.

LEMMA 1.8. *A model M is amalgamative if it has an existential extension which is an amalgamator of M.*

PROOF. Let $N \in \boldsymbol{A}(M)$ and $N \in \boldsymbol{E}(M)$. For each $\boldsymbol{a} \in M$, $\zeta \in \Sigma_1$, $\varphi, \psi \in \Pi_1$ such that $M \models \zeta(\boldsymbol{a})$ and 1.3, (iv), (*), there is a $\xi \in \Sigma_1$ such that $N \models \xi(\boldsymbol{a})$ and 1.3, (iv), (**), since $N \in \boldsymbol{A}(M)$. But then $M \models \xi(\boldsymbol{a})$ because $N \in \boldsymbol{E}(M)$. Thus $M \in \boldsymbol{A}(M)$, or $M \in \boldsymbol{A}$.

PROPOSITIONS.

1.9.1. *An elementary submodel of an amalgamative model is amalgamative.*

1.9.2. *No reduced power of a structure is amalgamative, unless the structure is amalgamative.*

1.9.3. *A theory T has the amalgamative property if and only if the theory $T_{\forall\exists}$ has it. Here, $T_{\forall\exists}$ is the theory consisting of the sentences in Π_2 that are logical consequences of T.*

PROOF OF 1.9.1. Immediate from 1.8 and 1.2.3.

1.9.2. A map $f: A \to B$ and the canonical injection $i: A \to A^I/\mathscr{F}$ can be amalgamated by $f': A^I/\mathscr{F} \to B^I/\mathscr{F}$ and $j: B \to B^I/\mathscr{F}$, where f' is induced from the association of a function $\varrho: I \to A$ to $f \circ \varrho$. Consequently, if $A^I/\mathscr{F}$ is amalgamative, so is A.

1.9.3. We recall that for any system M, $M \models T_{\forall\exists}$ iff $N \models T$ for some $N \in \boldsymbol{E}(M)$. Thus the 'if' direction is trivial, and the 'only if' direction follows from 1.8.

It will be useful to grasp a model as the union of an ascending chain of "smaller" submodels. For a model M of uncountable power, it is easy to do so by using LST. In more detail, we call a sequence $\langle M_\alpha \rangle_{\alpha<\varkappa}$ a *ladder to M*, where $\mathrm{cd}(M) > \aleph_0$ and $\varkappa = \mathrm{cf}(\mathrm{cd}(M))$, if for all $\alpha \leqq \beta < \varkappa$, $M_\alpha \subseteqq M_\beta \subseteqq M$, $\mathrm{cd}(M_\alpha) < \mathrm{cd}(M)$, and $M = \bigcup_{\alpha<\varkappa} M_\alpha$. For a countable model the second condition is too much to ask, and so we require the theory to be convex. Recall that a theory is called convex when the intersection of two submodels of a model is again a model, unless it is empty, and that a convex theory is inductive [14], whence a finitely generated — which will be abbreviated as 'f.g.' — model makes sense. When M is a countable model of a convex theory, a sequence $\langle M_n \rangle_{n\in\omega}$ is called a *ladder to M* if, for all $m \leqq n < \omega$, $M_m \subseteqq M_n \subseteqq M$, M_n is f.g., and $M = \bigcup_{n\in\omega} M_n$.

The following is nothing but rewriting, in our context, the proof of [8], Lemma 2.5 and its modification.

LEMMA.

1.10.1. *Every uncountable model has a ladder to it. Moreover, any submodel of smaller power can be taken as the first term of the ladder.*

1.10.2. *Every countable model of a convex theory has a ladder to it. Moreover, any finitely generated submodel can be taken as the first term of the ladder.*

PROOF. 1.10.1. Given a model M, let $\varkappa$ be the cofinality of $\mathrm{cd}(M)$. Hence there is an increasing sequence $\langle S_\alpha \rangle_{\alpha<\varkappa}$ of infinite *sets* such that

$\mathrm{cd}(S_\alpha) < \mathrm{cd}(M)$ for each $\alpha < \varkappa$, $S_\lambda = \bigcup_{\alpha<\varkappa} S_\alpha$ for a limit ordinal $\lambda < \varkappa$, and $\bigcup_{\alpha<\varkappa} S_\alpha$ is the universe of M. We define a ladder $\langle M_\alpha \rangle_{\alpha<\varkappa}$ as follows: Take an $M_0 \prec M$ such that $S_0 \subseteqq M_0$ and $\mathrm{cd}(S_0) = \mathrm{cd}(M_0)$. Such M_0 exists by LST. Given M_β take an $M_{\beta+1}$ such that $M_{\beta+1} \prec M$, $M_{\beta+1} \supseteqq M_\beta \cup S_{\beta+1}$ and

$$\mathrm{cd}(M_{\beta+1}) = \max(\mathrm{cd}(M_\beta), \mathrm{cd}(S_{\beta+1})) .$$

Note that $M_{\beta+1}$ is of smaller power than M and $M_\beta \prec M_{\beta+1}$. For a limit ordinal $\lambda < \varkappa$, let $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$. M_λ is a submodel of M since it is the union of an elementary chain of submodels; $\mathrm{cd}(M_\lambda) < \mathrm{cd}(M)$ by the choice of $\varkappa$; and $M_\lambda \supseteqq S_\lambda$ because $S_\alpha \subseteqq M_\lambda$ for each $\alpha < \lambda$ and $S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$. Clearly M is $\bigcup_{\alpha<\lambda} M_\alpha$. We note that this ladder is an elementary chain. When a small submodel D is specified in advance, let S_0 be $|D|$ and let M_0 be D. For each $\alpha < \omega$, let $M_{\alpha+1}$ be M_α above. The construction of M_α is the same as above for $\alpha \geqq \omega$.

1.10.2. Let $\langle a_n \rangle_{n\in\omega}$ be an enumeration of the elements of the given model M. Let M_0 be the given f.g. submodel. Assume $M_n \subseteqq M$ is already constructed. If $M_k = M$, let $M_{k+1} = M_k$. If $M_k \neq M$, let a_n be the first element $\notin M_k$ and let M_{k+1} be the submodel generated by a_n and the finite generators of M_k. Clearly, $M = \bigcup_{k<\omega} M_k$.

Now we shall characterize an amalgamative model in terms of its smaller submodels. The first group is:

THEOREM.

1.11.1. *A model M is amalgamative if and only if each finite subset of $|M|$ is included in a countable and amalgamative submodel of M.*

1.11.2. *All submodels of a model M are amalgamative if (and only if) all countable submodels of M are amalgamative.*

PROOF OF 1.11.1. The 'only if' direction comes immediately from 1.9.1 and LST. To show the 'if' direction, we use 1.3, (iii). Hence let $N \supseteqq M$ and $\varphi(\boldsymbol{a}) \in \Sigma_1(M)$, and assume that $\varphi(\boldsymbol{a})$ is consistent with $T + Dg(M)$. By the assumption, there is a $B \subseteqq M$ such that $\boldsymbol{a} \in B$, $\mathrm{cd}(B) \leqq \aleph_0$ and $B \in \boldsymbol{A}$. Then certainly $\varphi(\boldsymbol{a})$ is consistent with $T + Dg(B)$, and hence with $T + Dg(N)$ because $B \subseteqq N$ and $B \in \boldsymbol{A}(B)$. Consequently, M is its own amalgamator, and so M is amalgamative.

1.11.2. That $M \in \boldsymbol{A}$ follows easily from the above, and the same argument works for any submodel N of M because a countable submodel of N is also a countable submodel of M.

For a convex theory, 'finite subset' in 1.11.1 and 'countable' in 1.11.2 can be replaced by 'finitely generated'. (Refer to the proof of 1.13.2, below.) But we do not know if 'countable' in 1.11.1 can be replaced by 'f.g.'. So we define a model of a convex theory as having the *finitely generated amalgamation property*, FGA in short, if each f.g. submodel is included in an f.g. and amalgamative submodel. We also say that a class of models has FGA when each member does. Note that a convex theory has AP if and only if each model has FGA.

THEOREM.

1.12.1. *An uncountable model is amalgamative if and only if it has a ladder consisting of amalgamative models.*

1.12.2. *Assume that the theory is convex. A countable model has* FGA *if and only if it has a ladder consisting of amalgamative models.*

PROOF. 1.12.1. The sufficiency is a special case of 1.6.2. The necessity comes from 1.9.1 and the proof of 1.10.1

1.12.2. The necessity follows from the definition of FGA referring to the manner of constructing ladders in the proof of 1.10.2. The sufficiency is an immediate consequence of the fact that a f.g. submodel must be included in a term of the given ladder.

PROPOSITIONS.

1.13.1. *A theory has the amalgamation property if [and only if] all countable models are amalgamative.*

1.13.2. *A convex theory has the amalgamation property if [and only if] all finitely generated models are amalgamative.*

PROOF. The first proposition follows from 1.11.2 and 1.1.4. For the second, we use 1.10.2 and 1.12.2 to conclude that every countable model has FGA, hence is in A. Thus the question is reduced to the previous case.

This proposition tells us that for the question of if the given T has AP, it suffices to watch only countable models. In contrast to this, the finite and the infinite models behave differently as to being amalgamative. For instance, let T be the theory that demands its model to be a semigroup and all models with more than five elements to be groups. There is a non-amalgamative model with four elements (Kimura's example), but each model with more than five elements is amalgamative due to the celebrated result of Neumann, [11]. On the other hand, when we consider the theory of infinite semigroups, its finite models are all amalgamative vacuously. But Kimura's example can be stretched easily to an infinite and non-amalgamative semigroup.

We know that there are enough amalgamative models when the theory is inductive (1.6), and that for some theory all models are amalgamative. Here we consider the question of "how many" non-amalgamative models there can be.

PROPOSITION.

1.14.1. *If M is an infinite and non-amalgamative model, then for any infinite cardinal $\varkappa$ there is a model N such that* $\mathrm{cd}(N)=\varkappa$, $N \subseteqq M$ *or* $M \subseteqq N$, *and no model between M and N is amalgamative.*

1.14.2. *For an algebraic theory (that is without relation symbols), there are enough non-amalgamative models, provided*

(a) *there is one non-amalgamative,*

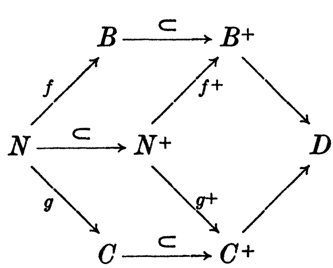
(b) *to each model a stable element can be adjoined unless there is one already, and*

(c) *the class of models is closed under the formation of direct product.*

(We call an element of a model *stable* if the singleton of that element is a subsystem of the model.)

PROOF. 1.14.1. If $\varkappa=\mathrm{cd}(M)$, take $N=M$. If $\varkappa>\mathrm{cd}(M)$, then take an N such that $\mathrm{cd}(N)=\varkappa$ and $N \succ M$ by the upward part of LST. Note that, $P \in \boldsymbol{E}(M)$ if P is between M and N. Thus by 1.8, $P \notin \boldsymbol{A}(M)$, hence $P \notin \boldsymbol{A}$ by 1.2.3. Assume, finally, $\varkappa<\mathrm{cd}(M)$. It follows from 1.11.1 that there is a finite set $S \subseteqq |M|$ such that no countable submodel including S is amalgamative. Let $P \prec M$ be of power $\aleph_0$ and include S, using LST. Then as above no model between P and M is amalgamative. Choose N of power $\varkappa$ among these.

1.14.2. First note that if $N \notin \boldsymbol{A}$ and N^+ is obtained from N by adjoining a stable element then $N^+ \notin \boldsymbol{A}$ also. For, suppose $f: N \to B$ and $g: N \to C$ could not be amalgamated. Then $f^+: N^+ \to B^+$ and $g^+: N^+ \to C^+$ can not be amalgamated, where f^+ and g^+ send the stable elements to the stable — adjoined or not — elements, because otherwise we shall have the following commutative diagram and so $N \in \boldsymbol{A}$.



Thus, using the assumptions (a) and (b) we may assume that there is an $N \notin \boldsymbol{A}$ that contains a stable element. We assume that a model M with a stable element is given. (Otherwise we adjoin one.) Suppose $f: N \to B$ and $g: N \to C$ can not be amalgamated. Let $f^+: N \times M \to B \times M$ be such that

$$f^+(\langle n,m\rangle) = \langle f(n),m\rangle ,$$

and define g^+ similarly. These f^+ and g^+ cannot be amalgamated, because we can embed, say, N to $N \times M$, as M has a stable element, in such a way as to obtain the above commutative diagram. Since the given M can be regarded as a submodel of $N \times M$ due to the presence of stable element in N, we can conclude the assertion.

Referring back to the example in connexion with 1.13, we notice here the assumption in 1.14.1 that $M \notin \boldsymbol{A}$ is infinite is necessary to obtain the conclusion. The assumptions in 1.14.2 are all satisfied in the classes of semigroups and of modular lattices, and so there are enough non amalgamatives in these classes. Indeed, (a) is fulfilled by Kimura's example and by a result of Grätzer–Lasker–Jonsson [7], and (c) is obviously met because these are equational classes. As to (b), we can adjoin a unit element to a semigroup, if necessary, while each element of a lattice is stable.

2.

In this section we consider the classes of universal and homogeneous models and related classes of models. For the convenience for later work, we modify the notions of universality and homogeneity so as to tie them to a given cardinality, taking suggestions from [6] and [20]. Whenever we consider the denumerable models in such a context, we automatically assume that the given theory is convex and regard a finitely generated model as having a small power. To smooth over the distinction between the countable and the uncountable cases, we introduce the *modified power* of a model M, in symbols $\mathrm{cd}^*(M)$, with the understanding that it is $\mathrm{cd}(M)$ when it is $> \aleph_0$, and $\mathrm{cd}^*(M) < \aleph_0$ exactly when M is finitely generated. (Hence, $\mathrm{cd}(M) \leqq \aleph_0$.)

We say a model M is *$\varkappa$-universal*, in symbols $M \in \boldsymbol{U}_\varkappa$, if any model N can be embedded in M, provided $\mathrm{cd}^*(N) < \varkappa$. A model M is said to be *$\varkappa$-homogeneous*, in symbols $M \in \boldsymbol{H}_\varkappa$, if $\mathrm{cd}(M) \geqq \varkappa$ and for any submodels $A \subseteq B \subseteq M$ such that $\mathrm{cd}^*(A) < \varkappa$ and $\mathrm{cd}^*(B) < \varkappa$ and any map $f: A \to M$ there is a map $g: B \to M$ such that $g \supseteq f$. (Thus our $\varkappa$-homogeneity coincides with local $\varkappa$-homogeneity of [6] and is in the spirit of homoiogeneity of [1].)

PROPOSITIONS.

2.1.1. *A $\varkappa$-universal model is also μ-universal for any $\mu \leqq \varkappa$.*

2.1.2. *A $\varkappa$-homogeneous model is also μ-homogeneous for any $\mu \leqq \varkappa$.*

2.1.3. *If a model is $\varkappa$-universal and $\varkappa$-homogeneous, then it is $\varkappa^+$-universal.*

PROOF. The first two propositions are obvious from the definitions.

2.1.3. Let $M \in \boldsymbol{U}_\varkappa$, $M \in \boldsymbol{H}_\varkappa$, and let B be of power $\varkappa$. Using 1.10, let $\langle B_\alpha \rangle_{\alpha < \mathrm{cf}(\varkappa)}$ be a ladder to B. We recall that $\mathrm{cd}^*(B_\alpha) < \varkappa$ for each $\alpha < \mathrm{cf}(\varkappa)$. By induction, we define an ascending chain of injections $f_\alpha : B_\alpha \to M$. Let f_0 be a map of B_0 to M, which exists because $M \in \boldsymbol{U}_\varkappa$. When f_α is already obtained, we pick a $g: B_{\alpha+1} \to M$. Since $g(B_\alpha)$ and $f_\alpha(B_\alpha)$ are isomorphic submodels of M and $M \in \boldsymbol{H}_\varkappa$, there is an $h: g(B_{\alpha+1}) \to M$ that extends this isomorphism. Let $f_{\alpha+1}$ be $h \circ g$. Then $f_{\alpha+1}$ maps $B_{\alpha+1}$ to M, extending f_α. For a limit ordinal λ, let $f_\lambda = \bigcup_{\alpha<\lambda} f_\lambda$. Then the union of all f_α, $\alpha < \mathrm{cf}(\varkappa)$, is a map of B to M.

Thus for a model of power $\varkappa$, if it is in our $\boldsymbol{U}_\varkappa$ and $\boldsymbol{H}_\varkappa$, it is universal in the sense of [8]. Also it will be shown later (2.14.5) that if $M \in \boldsymbol{H}_\varkappa$ and $\mathrm{cd}(M) = \varkappa$, then an isomorphism between submodels of power $< \varkappa$ can be extended to an automorphism of M. Thus our definition coincides with that in [9] and [20]. When the discussion applies to $\varkappa$-universality, etc., for an arbitrary $\varkappa$, or some fixed $\varkappa$ is considered throughout, we shall often omit the subscript and/or prefix "$\varkappa$-", to simplify the notation.

Using 2.1.3, one can show the following proposition by a similar argument as that in [21] or [2].

PROPOSITION 2.2. *If a model is $\varkappa$-universal and $\varkappa$-homogeneous for some $\varkappa$, then it is generic, hence existentially closed and amalgamative.*

Neither $\varkappa$-universality nor $\varkappa$-homogeneity alone is sufficient to ensure that the model even be amalgamative. To see this, let K be an enlargement to the cardinality $\varkappa$ of Kimura's example of a non-amalgamative semigroup; namely the semigroup with 0 on $\varkappa$ generators such that the product of any two elements is 0. This is not amalgamative as the argument in [4] works with little modification. On the other hand, K is $\varkappa$-homogeneous, because any one-one function of a subset of $|K|$ into $|K|$ can be regarded as an injection into K. Given a $\varkappa$-universal semigroup U, the direct product $U \times K$ is not amalgamative as was noted in connexion with 1.14.2, although it is still $\varkappa$-universal according to the definition.

Since our universal and homogeneous models are more inclusive than the usual ones, their existence follows from the usual assumptions, which include JE and AP. (For the existence question, see further 2.5 and 2.10.1 below.) On the other hand, the existence of even the weakest universal and homogeneous models entails these properties.

THEOREM.

2.3.1. *If there is a $\varkappa$-universal model for some $\varkappa$, then the theory has the joint embeddability property.*

2.3.2. *If there is a model that is $\varkappa$-universal and $\varkappa$-homogeneous for some $\varkappa$, then the theory also has the amalgamation property.*

PROOF. 2.3.1. For $\varkappa > \aleph_0$, the assertion can be shown as in [10, Theorem 4.1]. For $\varkappa = \aleph_0$, we use the standing convention that the theory is convex, as exemplified in the next proof.

2.3.2. We prove the more complex case that $\varkappa = \aleph_0$. By 1.13.2 and 1.3. (ii), it suffices to show that for every finitely generated model M and $\exists \boldsymbol{x}\varphi(\boldsymbol{x})$, $\exists \boldsymbol{y}\psi(\boldsymbol{y}) \in \Sigma_1(M)$ if each is satisfied in extensions P and Q of M, respectively, then their conjunction is consistent with $T + Dg(M)$. We may assume P to be finitely generated, because the submodel generated by the generators of M and those $\boldsymbol{a} \in P$ such that $P \models \varphi(\boldsymbol{a})$ is a model of $\exists \boldsymbol{x}\varphi(\boldsymbol{x})$. Similarly, Q can be assumed to be finitely generated. Let $U \in \boldsymbol{UH}$, (the subscript $\aleph_0$ is omitted) which exists by assumption. Then P and Q are embeddable into U. By identifying isomorphic copies, we may assume $U \supseteqq P$ and $g: Q \to U$ is an embedding. Then $g^{-1} \restriction g(M)$ is a map of $g(M)$ to U. Since $g(M) \subseteqq U$, $\mathrm{cd}^*(g(M)) < \aleph_0$, and $\mathrm{cd}^*(g(Q)) < \aleph_0$, this map can be extended to an $h: g(Q) \to U$. Thus $h \circ g$ maps Q to U and

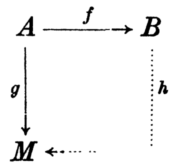
$$(h \circ g) \restriction M = \mathrm{id}_M .$$

Therefore U is a model of the above sentences as well as $T + Dg(M)$.

The contrapositive of this theorem says that for those theories that lack AP, no universal model can ever be extended to a homogeneous model even in our sense, much less in the sense of [9]. So, for instance, there is no universal and homogeneous model in classes of semigroups, rings, or modular lattices. As we shall see below, 2.9.1 and 2.10.1 in combination with 2.5, if an inductive theory has JE (JE and AP) then there are enough $\varkappa$-universals ($\varkappa$-universal and $\varkappa$-homogeneouses) for any $\varkappa$. Thus, 2.3 implies that if $\boldsymbol{U}_\varkappa$ ($\boldsymbol{UH}$) is non-empty for one $\varkappa$, then there are enough $\boldsymbol{U}_\varkappa$ ($\boldsymbol{UH}$) for all $\varkappa$.

The above theorem shows that when we do not assume AP, we should look for a weaker notion than universal-homogeneity. Note that if M

is such a model of cardinality $\varkappa$, M has the following injective property: Given injections $f: A \to B$ and $g: A \to M$, if $\mathrm{cd}^*(A) < \varkappa$ and $\mathrm{cd}^*(B) < \varkappa$ then there is an injection $h: A \to M$ such that $g = h \circ f$.



(And this can be taken as the definition of universal-homogeneity if A is allowed to be empty.) A possible weakening of the notion would be to require that A have some additional properties; we find being amalgamative a suitable one. (Another possible weakening is to restrict maps f, g and h; this approach is extensively investigated by Simmons in [18].) Thus, we say that a model M is *$\varkappa$-objective*, in symbols $M \in \boldsymbol{O}_\varkappa$, if

(a) $\mathrm{cd}(M) \geqq \varkappa$,

(b) $M \in \boldsymbol{A}$, and

(c) for any $f: A \to B$ and $g: A \to M$ there is an $h: B \to M$ such that $g = h \circ f$, provided $\mathrm{cd}^*(A) < \varkappa$, $\mathrm{cd}^*(B) < \varkappa$ and $A \in \boldsymbol{A}$.

(We call the data A, B, F, g a *$\varkappa$-quadruple* for brevity. Often f is an inclusion map, and we call A, B, g a *$\varkappa$-triple.*) A similar consideration on $\varkappa$-homogeneity leads to $\varkappa$-subjectivity. A model M is called *$\varkappa$-subjective*, in symbols $M \in \boldsymbol{S}_\varkappa$, if (a), (b) above, and

(c′) for any submodels $A \subseteqq B \subseteqq M$ and any map $f: A \to M$, f can be extended to a map $h: B \to M$, provided $\mathrm{cd}^*(A) < \varkappa$, $\mathrm{cd}^*(B) < \varkappa$ and $A \in \boldsymbol{A}$.

(The data A, B, f is called a *$\varkappa$-triad.*) Obviously $\boldsymbol{O}_\varkappa \subseteqq \boldsymbol{O}_\mu$ and $\boldsymbol{S}_\varkappa \subseteqq \boldsymbol{S}_\mu$ if $\mu \leqq \varkappa$.

Apology and Explanation. It is freely admitted that "objective" and "subjective" are chosen for a bit of fun. A more standard way may be to use phrases like "weakly universal-homogeneous" and "weakly homogeneous although amalgamative", in view of later results. But these are awfully long and yet do not really indicate what is weakened and how. Since our starting point was the injective property of universal-homogeneous models, some kind of "jective" is called for. Our objective models take into consideration all models of smaller cardinalities, while subjective models concern submodels only. Thus certainly the first is more objective and the second more subjective!

Although the above modification and Simmons' in [18] started from different view points, there is some connexion. Indeed, $\boldsymbol{O}_\varkappa$ and $\varkappa - WJ_0$

in [18] are the same class of models. We recall that a model M was defined to be in $\varkappa - WJ_0$, where $\varkappa > \aleph_0$ (in our context), if for any injections $f: A \to B$ and $g: A \to M$ there is an injection $h: B \to M$ such that $g = h \circ f$, provided $\mathrm{cd}(B) < \varkappa$ and g preserves the truth of sentences in $\Sigma_1(A)$. (In Simmons' terminology, g is $\prec_1$-like.) Note that such a model is of power $\geqq \varkappa$, and is existentially closed hence in $\boldsymbol{A}$. Naturally, we can extend the definition to include the case $\varkappa = \aleph_0$ by requiring A and B to be finitely generated, $M \in \boldsymbol{A}$, and $\mathrm{cd}(M) \geqq \aleph_0$. (These last two conditions do not seem to follow from the first, unlike the case $\varkappa > \aleph_0$.)

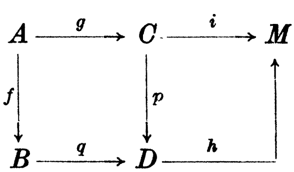
PROPOSITIONS.

2.4.1. *For any* $\varkappa$, *a* $\varkappa$*-objective model is in* $\varkappa - WJ_0$.

2.4.2. *A model in* $\varkappa$-WJ_0 *is* $\varkappa$*-objective, if* $\varkappa > \aleph_0$.

PROOF. 2.4.1. Let $M \in \boldsymbol{O}_\varkappa$, $f: A \to B$ an injection, $g: A \to M$ a $\prec_1$-like injection, and $\mathrm{cd}^*(A)$, $\mathrm{cd}^*(B) < \varkappa$. Then, as in the proof of 1.8, we have $A \in \boldsymbol{A}$. Hence by (c), there is an $h: B \to M$ such that $g = h \circ f$. So M is in $\varkappa - WJ_0$.

2.4.2. If M is in $\varkappa - WJ_0$ then M has (a) and (b), as noted above. Given $f: A \to B$ and $g: A \to M$ where $\mathrm{cd}(B) < \varkappa$ and $A \in \boldsymbol{A}$, take a $C \prec M$ such that $C \supseteqq f(A)$ and $\mathrm{cd}(C) < \varkappa$. Since $A \in \boldsymbol{A}$, there are $p: C \to D$ and $q: B \to D$ such that $p \circ g = q \circ f$, and $\mathrm{cd}(D) < \varkappa$. Hence there is an $h: D \to M$ such that $h \circ p = i: C \to M$. Then observe that $g = i \circ g = (h \circ q) \circ f$.



Thus many results in [18] can be used for the investigation of the $\varkappa$-objective models, although we shall not always avoid duplications, as we are interested in developing results about $\boldsymbol{O}_\varkappa$ and $\boldsymbol{S}_\varkappa$ in parallel. However, we make good use of Simmons' work for the following:

THEOREM 2.5. (Essentially Simmons) *Assume that the theory is inductive. Then for each cardinality* $\varkappa$, *an infinite model* M *of power* μ *can be extended to a* $\varkappa$*-objective model of power* $\mu^*\varkappa$. *Here,*

$$\mu^*\varkappa \text{ is } \begin{cases} \sum_{\alpha<\varkappa}\mu^\alpha, & \text{if } \varkappa \text{ is regular and } > \aleph_0, \\ \mu^\varkappa, & \text{if } \varkappa \text{ is singular or } \varkappa = \aleph_0. \end{cases}$$

Indeed, for the case $\varkappa > \aleph_0$, the theorem comes directly from [18, Theorem 2.8], in conjunction with 2.4.2. For the case $\varkappa = \aleph_0$, it follows from the existence of $\aleph_1$-objective models, since $\boldsymbol{O}_\varkappa \subseteqq \boldsymbol{O}_\mu$ if $\varkappa \geqq \mu$. Since $\aleph_0$ is regular, it is worth considering why $\mu^*\aleph_0$ is not $\mu = \sum_{n\in\omega}\mu^n$ As we shall see in 2.10.1, if the theory has JE and AP, then $\boldsymbol{O}_\varkappa = \boldsymbol{U}_\varkappa \cap \boldsymbol{H}_\varkappa$ for all $\varkappa \geqq \aleph_0$. Hence, if there are more than countably many isomorphism types among finitely generated models, then $\aleph_0$-objective models must be of power $> \aleph_0$. So, in particular, for the case $\mu = \aleph_0$, $\mu^*\aleph_0$ must be $> \mu$.

We shall make a few easy observations about relations among $\boldsymbol{O}$, $\boldsymbol{U}$, etc. For this purpose, it is convenient to recall Simmons' class $\varkappa - WI_0$. A model M is in $\varkappa - WI_0$ exactly when for any maps $f: A \to B$ and $g: A \to M$ there is a map $h: B \to M$ such that $g = h \circ f$, provided $\mathrm{cd}^*(A) < \varkappa$ and $\mathrm{cd}^*(B) < \varkappa$. In a way parallel to the class $\varkappa - WJ_0$, we can show that M is of power $\geqq \varkappa$ and $M \in \boldsymbol{A}$ if $M \in \varkappa - WI_0$ and $\varkappa > \aleph_0$. For the case $\varkappa = \aleph_0$ we build these properties into the definition of $\aleph_0 - WI_0$. Since this is the only class we borrow from [18], we denote it as $\boldsymbol{I}_\varkappa$ and refer to its members simply as *$\varkappa$-injective*.

PROPOSITIONS.

2.6.1. *If a model is $\varkappa$-universal and $\varkappa$-homogeneous then it is $\varkappa$-injective.*

2.6.2. *If a model is $\varkappa$-universal and $\varkappa$-homogeneous then it is $\varkappa$-subjective.*

2.6.3. *If a model is $\varkappa$-injective then it is $\varkappa$-objective.*

2.6.4. *If a model is $\varkappa$-injective then it is $\varkappa$-homogeneous and amalgamative.*

2.6.5. *If a model is $\varkappa$-universal and $\varkappa$-subjective then it is $\varkappa$-objective.*

2.6.6. *If a model is $\varkappa$-objective then it is $\varkappa$-subjective.*

2.6.7. *If a model is $\varkappa$-homogeneous and amalgamative then it is $\varkappa$-subjective.*

PROOF. The last proposition is a trivial consequence of definitions, and so 2.6.2 follows from 2.2. Similarly, 2.6.3, 2.6.4, and 2.6.6 follows easily from definitions. To prove 2.6.1, let $M \in \boldsymbol{UH}$, $f: A \to B$, $g: A \to M$, and $\mathrm{cd}^*(A)$, $\mathrm{cd}^*(B) < \varkappa$. Since $M \in \boldsymbol{U}$, there is a map $p: B \to M$. Let $B' = p(B)$ and $A' = p \circ f(A)$. Note that $A' \subseteqq B' \subseteqq M$, and that $g \circ f^{-1} \circ p^{-1} \restriction A'$ is a map $q: A' \to M$. Since $M \in \boldsymbol{H}_\varkappa$, there is an $r \supseteqq q$ which is a map of B' to M. Since $g = q \circ p \circ f$, $h = r \circ p$ is a map of B to M such that $g = h \circ f$. We can prove 2.6.5 in a similar fashion.

The following diagram summarizes relations among various classes. We use '$\to$' in place of '$\subseteqq$'.

$$\begin{array}{ccccc} \boldsymbol{UH} & \longrightarrow & \boldsymbol{I} & \longrightarrow & \boldsymbol{HA} \\ \downarrow & & \downarrow & & \downarrow \\ \boldsymbol{US} & \longrightarrow & \boldsymbol{O} & \longrightarrow & \boldsymbol{S} \end{array}$$

Notice that 2.6.1 is a little stronger than [18, Theorem 6.6] because our $\varkappa$-homogeneity is weaker than that in [18]. Of course the proofs are essentially the same. We also remark that $\boldsymbol{H}$ and $\boldsymbol{S}$ are incomparable classes. Already we have seen that $\boldsymbol{H} \nsubseteq \boldsymbol{A}$ in some cases, hence $\boldsymbol{H} \nsubseteq \boldsymbol{S}$. On the other hand, consider the class $\boldsymbol{R}$ of rings. The ring consisting of 0 and 1 is a prime model and is amalgamative, and $\boldsymbol{R}$ has JE but lacks AP. By the next proposition 2.7.2, the subclass $\boldsymbol{O}$ of $\boldsymbol{R}$ coincides with $\boldsymbol{US}$. Thus $\boldsymbol{O}$ and $\boldsymbol{H}$ are disjoint by 2.3.2, and hence $\boldsymbol{S} \nsubseteq \boldsymbol{H}$ as $\boldsymbol{O} \subseteq \boldsymbol{S}$.

Now we consider the influence of the theory having prime models, AP, etc., on the above relations among various classes.

PROPOSITIONS.

2.7.1. *If the theory has a prime model, then a model is $\varkappa$-universal and $\varkappa$-homogeneous exactly when it is $\varkappa$-injective.*

2.7.2. *If the theory has a prime model which is amalgamative, then a model is $\varkappa$-universal and $\varkappa$-subjective exactly when it is $\varkappa$-objective. Hence, if the theory is inductive in addition, it has the joint embeddability property.*

PROOF OF 2.7.2. (2.7.1 can be shown similarly.) Since $\boldsymbol{US} \subseteq \boldsymbol{O} \subseteq \boldsymbol{S}$ by 2.6.5 and 2.6.6, to show the first half it suffices to show $\boldsymbol{O} \subseteq \boldsymbol{U}$. Let $M \in \boldsymbol{O}$, and let $A \in \boldsymbol{A}$ be the given prime model. Note that $\mathrm{cd}^*(A) < \varkappa$. Take an arbitrary model B such that $\mathrm{cd}^*(B) < \varkappa$. Then there are maps $f\colon A \to B$ and $g\colon A \to M$ as A is prime. Since $M \in \boldsymbol{O}$ and $A \in \boldsymbol{A}$, there is an $h\colon B \to M$ such that $g = h \circ f$. Thus $M \in \boldsymbol{U}$. The second half follows from 2.5 and 2.3.1.

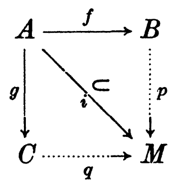
PROPOSITIONS.

2.8.1. *If the theory has the amalgamation property, then a model is $\varkappa$-objective exactly when it is $\varkappa$-injective, for any $\varkappa$.*

2.8.2. *An inductive theory has the amalgamation property, if the classes $\boldsymbol{I}_\varkappa$ and $\boldsymbol{O}_\varkappa$ coincide for some $\varkappa$.*

PROOF. 2.8.1. That $\boldsymbol{I} \subseteq \boldsymbol{O}$ is already shown in 2.6.3. The other implication is trivial since every model is amalgamative.

2.8.2. From 2.5, we know every model can be extended to a member of $\boldsymbol{O}$, hence of $\boldsymbol{I}$ by the assumption. By 1.13, it suffices to show that $f: A \to B$ and $g: A \to C$ can always be amalgamated where A, B, and C are all finitely generated, in case the theory is convex. (For the non-convex case, a similar consideration taking countable models works equally well.) Let $M \supseteq A$ be in $\boldsymbol{I}$. Let i be the inclusion map of A into M. Since $M \in \boldsymbol{I}$, there are $p: B \to M$ and $q: C \to M$ such that $p \circ f = i = g \circ q$ amalgamating the given f and g.



We note here that 2.8.1 is already in Theorem 6.3, [18], and 2.8.2 gives a partial converse to it.

PROPOSITIONS.

2.9.1. *If a theory has the joint embeddability property, then a model is $\varkappa$-objective if and only if it is $\varkappa$-universal and $\varkappa$-subjective.* (FGA *is assumed on the $\varkappa$-objective models, when* $\varkappa = \aleph_0$.)

2.9.2. *An inductive theory has the joint embeddability property, if the class $\boldsymbol{O}_\varkappa$ is the intersection of $\boldsymbol{U}_\varkappa$ and $\boldsymbol{S}_\varkappa$, for some $\varkappa \geqq \aleph_0$.*

PROOF. 2.9.1. By 2.6.2 and 2.6.3, it suffices to show $\boldsymbol{O} \subseteq \boldsymbol{U}$. Take $M \in \boldsymbol{O}$. By using 1.9.1 or FGA, take an $A \subseteq M$ such that $A \in \boldsymbol{A}$ and $\mathrm{cd}^*(A) < \varkappa$, and take a model N of modified power $< \varkappa$. By JE, there are B and $f: A \to B$ and $g: N \to B$, and we can assume $\mathrm{cd}^*(B) < \varkappa$. Then A, B, f, i constitute a $\varkappa$-quadruple of M where $i: A \subsetneqq M$. Hence there is an $h: B \to M$. So $h \circ g$ is an injection of N into M showing that M is $\varkappa$-universal.

2.9.2. By 2.5, $\boldsymbol{O}$ is not empty, a fortiori nor is $\boldsymbol{U}$ by the given assumptions. Thus 2.3.1 concludes the proof.

PROPOSITIONS.

2.10.1. *If a theory has the joint embeddability property and the amalgamation property, then a model is $\varkappa$-objective if and only if it is $\varkappa$-universal and $\varkappa$-homogeneous, for any $\varkappa$.*

2.10.2. *An inductive theory has the joint embeddability property and the amalgamation property, if the class $\boldsymbol{O}_\varkappa$ is the intersection of $\boldsymbol{U}_\varkappa$ and $\boldsymbol{H}_\varkappa$.*

PROOF. 2.10.1. Since AP implies that every model has FGA and $\boldsymbol{H}=\boldsymbol{S}$, we have $\boldsymbol{UH}=\boldsymbol{US}=\boldsymbol{O}$ by 2.9.1.

2.10.2. As in the proof of 2.9.2, the non-emptiness of $\boldsymbol{UH}$ implies AP and JE by 2.3.

The above four groups of propositions may be summarized schematically as follows:

$$\boldsymbol{P} \neq \emptyset \Rightarrow \boldsymbol{I} = \boldsymbol{UH}, \qquad \boldsymbol{PA} \neq \emptyset \Rightarrow \boldsymbol{O} = \boldsymbol{US},$$
$$\text{AP} \Leftrightarrow \boldsymbol{O} = \boldsymbol{I}, \quad \text{JE} \Leftrightarrow \boldsymbol{O} = \boldsymbol{US}, \quad \text{AP \& JE} \Leftrightarrow \boldsymbol{O} = \boldsymbol{UH}.$$

Here $\boldsymbol{P}$ is the class of prime models. For the '$\Leftarrow$' direction, the theory is assumed to be inductive, and when $\varkappa=\aleph_0$ FGA is assumed here and there. Similarly we can show that

$$\text{AP} \Rightarrow (\boldsymbol{HA}=\boldsymbol{S} \ \& \ \boldsymbol{UH}=\boldsymbol{US}) \quad \text{and} \quad \text{JE} \Rightarrow \boldsymbol{UH} = \boldsymbol{I}.$$

As to the converse direction, if the theory is inductive then

$$(\text{JE} \ \& \ \boldsymbol{US}=\boldsymbol{UH}) \Rightarrow \text{AP} \quad \text{and} \quad (\text{AP} \ \& \ \boldsymbol{I}=\boldsymbol{UH}) \Rightarrow \text{JE}.$$

The influence of the existence of the model companion will be considered in the next section.

The class $\boldsymbol{I}$ has a close relation with AP as shown in 2.8. As a matter of fact, it can be used to give an algebraic characterization of AP for an inductive theory.

THEOREM 2.11. *An inductive theory has the amalgamation property if and only if the class of $\varkappa$-inductive models is cofinal in the class of models.*

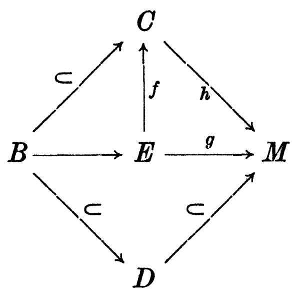
PROOF. The 'only if' direction follows from 2.5 and 2.8.1. The 'if' direction holds for any theory, and is an obvious consequence of the fact that a submodel of an injective model is amalgamative. To show this, we use 1.11.2 and the accompanying comment about convex theories. So let $M \in \boldsymbol{I}$, $A \subseteqq M$ and $\mathrm{cd}(A) \leqq \aleph_0$ ($\mathrm{cd}^*(A) < \aleph_0$ for a convex theory); we are to show that $A \in \boldsymbol{A}$. For this, we use 1.3, (ii). Thus assume $\varphi(\boldsymbol{a}), \psi(\boldsymbol{a}) \in \Sigma_1(A)$ and each is consistent with $T+Dg(A)$. Let B and C be models ot $T+Dg(A)+\varphi(\boldsymbol{a})$ and $T+Dg(A)+\psi(\boldsymbol{a})$, respectively. We can take B and C to be countable (f.g. for a convex theory). Since $M \in \boldsymbol{I}$ and $\varkappa > \aleph_0$ ($\varkappa \geqq \aleph_0$ for a convex theory), B and C can be embedded into M keeping A fixed. Thus

$$M \models T+Dg(A)+\varphi(\boldsymbol{a})+\psi(\boldsymbol{a}).$$

A $\varkappa$-objective model is a rather large, amalgamative model in the sense that any other model of power $<\varkappa$ can be embedded into M once it shares an amalgamative model with M. Interestingly, this largeness excludes the existence of small amalgamative models. Given a model B, call its extension C an *amalgamative envelope* of B if $C \in \boldsymbol{A}$ and for any model $D \in \boldsymbol{A}$ any injection $f: B \to D$ can be extended to an injection of C to D. If the given B itself is in $\boldsymbol{A}$ then B is its own amalgamative envelope, and so an amalgamative envelope is a small extension to a member of $\boldsymbol{A}$.

THEOREM 2.12. *If the theory is inductive, no non-amalgamative model possesses an amalgamative envelope.*

PROOF. Given $B \notin \boldsymbol{A}$, we know from 1.3, (iii) that there are $C \supseteqq B$, $D \supseteqq B$, and $\varphi = \varphi(\boldsymbol{b}) \in \Sigma_1(B)$ such that $C \models \varphi$ and $T + Dg(D) \vdash \neg\varphi$. By 1.7 we may assume $C \in \boldsymbol{A}$, and by 2.5 we can extend D to an $M \in \boldsymbol{O}_\varkappa$ with $\varkappa > \mathrm{cd}(C) + \aleph_0$. Had B an amalgamative envelope E, $\mathrm{cd}(E) \leqq \mathrm{cd}(B) + \aleph_0$. So there should be $f: E \to C$ and $g: E \to M$ such that $f \restriction B$ and $g \restriction B$ are both inclusion maps. Further, E, C, f, g, constitute a $\varkappa$-quadruple of M by the choice of $\varkappa$. Hence there is an $h: C \to M$ such that $g = h \circ f$. Since $f \restriction B$ and $g \restriction B$ are inclusions, so is $h \restriction B$. Thus, $\varphi(h(\boldsymbol{b})) = \varphi(\boldsymbol{b}) = \varphi$. Since $C \models \varphi$ and this truth is preserved by the injection h, we have $M \models \varphi$. On the other hand, as $M \models T + Dg(D)$ and $T + Dg(D) \vdash \neg\varphi$, it must be the case that $M \models \neg\varphi$. This contradiction with $M \models \varphi$ shows the non-existence of the amalgamative envelope E.



By the standard back and forth argument, we can show that maps of small submodels of objective models, for instance, can be extended to larger submodels.

THEOREM 2.13. *Given models M and submodels $A \subseteqq B \subseteqq M$ and $C \subseteqq N$ such that* $\mathrm{cd}^*(A) < \varkappa$, $\mathrm{cd}(B) = \mathrm{cd}(C) = \varkappa$, *any map $f: A \to M$ can be extended to a map g such that*

$$B \subseteqq \mathrm{dm}(g) \subseteqq M \quad \textit{and} \quad C \subseteqq \mathrm{rg}(g) \subseteqq N\,,$$

provided that any one of the following conditions is satisfied:

2.13.1. *M and N are both $\varkappa$-injective.*

2.13.2. *M and N are both $\varkappa$-objective (and have* FGA *when $\varkappa = \aleph_0$), and A is amalgamative.*

2.13.3. *$M = N$ and is $\varkappa$-homogeneous.*

2.13.4. *$M = N$ and is $\varkappa$-subjective (and has* FGA *when $\varkappa = \aleph_0$), and A is amalgamative.*

PROOF. Since all proofs are parallel, we prove only 2.13.2 in detail. Let $\langle P_\alpha\rangle_{\alpha<\mu}$ and $\langle Q_\alpha\rangle_{\alpha<\mu}$ be ladders to B and C, where $\mu = \mathrm{cf}(\varkappa)$. We are to construct other ladders $\langle B_\alpha\rangle$ and $\langle C_\alpha\rangle$, and two ascending sequences of maps $\langle f_\alpha\rangle$ and $\langle g_\alpha\rangle$ such that

(*) $B_\alpha \supseteqq P_\alpha$, $C_\alpha \supseteqq Q_\alpha$, $B_\alpha \in \boldsymbol{A}$, $C_\alpha \in \boldsymbol{A}$, $f_\alpha\colon B_\alpha \to C_\alpha$, $g_\alpha\colon C_\alpha \to B_{\alpha+1}$ and $f \subseteqq f_\alpha \subseteqq g_\alpha^{-1} \subseteqq f_{\alpha+1}$ for all $\alpha < \mu$.

Let B_0 be $A \cup P_0 \subseteqq B_0 \subseteqq M$, $B_0 \in \boldsymbol{A}$ and $\mathrm{cd}^*(B_0) < \varkappa$ by using LST or FGA. Then A, B_0, f is a $\varkappa$-triple of N, hence there is an $f_0\colon B_0 \to N$ extending f. Let C_0 be

$$f_0(B_0) \cup Q_0 \subseteqq C_0 \subseteqq N, \quad C_0 \in \boldsymbol{A}$$

and $\mathrm{cd}^*(C_0) < \varkappa$. Then $f_0(B_0), C_0, f_0^{-1}$ is a $\varkappa$-triple of M, hence there is a $g_0\colon C_0 \to M$ extending f_0^{-1}.

$$\begin{array}{ccccccc}
A & \xrightarrow{\subset} & B_0 & \xrightarrow{\subset} & g_0(C_0) & \qquad & \text{in } M \\
\Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f_0} & & \Big\uparrow{\scriptstyle g_0} & & \\
f(A) & \xrightarrow{\subset} & f_0(B_0) & \xrightarrow{\subset} & C_0 & \qquad & \text{in } N
\end{array}$$

The construction of $B_{\alpha+1}$, $C_{\alpha+1}$, $f_{\alpha+1}$ and $g_{\alpha+1}$ from the previous data goes similarly by using $g_\alpha(C_\alpha)$, and g_α^{-1} in place of A and f. For a limit ordinal $\lambda \leqq \mu$, we let B_λ, C_λ, f_λ and g_λ simply be the unions of the previous data. The requirement (*) is obviously met at each stage of construction, and f_μ is a map of $\bigcup_{\alpha<\mu} B_\alpha$ onto $\bigcup_{\alpha<\mu} C_\alpha$ which include B and C, respectively.

When M and N are injective or homogeneous, the inductive construction can be performed without assuming B_α and C_α to be in $\boldsymbol{A}$, and thus the hypothesis $A \in \boldsymbol{A}$ is unnecessary in 2.13.1 and 2.13.3.

COROLLARY.

2.14.1 (2.14.2). *There is at most one $\varkappa$-homogeneous ($\varkappa$-surjective) and $\varkappa$-universal model of power $\varkappa$ (that has* FGA *when $\varkappa=\aleph_0$), up to isomorphism.*

2.14.3 (2.14.4). *Let M and N be in $\boldsymbol{I}_\varkappa$ ($\boldsymbol{O}_\varkappa$) of power $\varkappa$ and $A \subseteqq M$. Then a map $f: A \to N$ can be extended to an isomorphism of M onto N, provided that* $\mathrm{cd}^*(A) < \varkappa$ *(and $A \in \boldsymbol{A}$, and M and N have* FGA *in case $\varkappa=\aleph_0$.)*

2.14.5 (2.14.6). *An isomorphism between two submodels of a $\varkappa$-homogeneous ($\varkappa$-subjective) model M of power $\varkappa$ can be extended to an automorphism of M, if the submodels are of modified power $<\varkappa$ (and amalgamative. Also, M is assumed to have* FGA, *if $\varkappa=\aleph_0$).*

PROOF. The last four assertions follow immediately from 2.13 by taking $B=M$ and $C=N$.

2.14.1 (2.14.2). Let M and N be of power $\varkappa$ and in $\boldsymbol{UH}$ ($\boldsymbol{US}$). Take an $A \subseteqq M$ such that $\mathrm{cd}^*(A) < \varkappa$ (and $A \in \boldsymbol{A}$). Then there is an $f: A \to N$ as $N \in \boldsymbol{U}$. Since M and N are in $\boldsymbol{I}$ (in $\boldsymbol{O}$) by 2.6.1 (2.6.5), this f can be extended to an isomorphism by 2.14.3 (2.14.4).

We remark here that 2.14.3 is Theorem 5.3 of [18], and that 2.14.4 slightly extends Theorem 5.1 in that A_0 and B_0 there need not be elementary submodels of M and N — to be in $\boldsymbol{A}$ is sufficient.

Another use of ladders is to loosen the restriction of cardinality $<\varkappa$ to $<\varkappa^+$, here and there. One instance is already given in 2.1.3. The second is that if $M \in \boldsymbol{U}_\varkappa \cap \boldsymbol{S}_\varkappa$ and $N \in \boldsymbol{A}$ is of power $<\varkappa^+$ then N is embeddable into M. Hence, in particular, if the theory is inductive then M is $\varkappa^+$-universal, due to 1.7 — FGA is assumed on N in case $\varkappa=\aleph_0$. The third is that in the definition of $\varkappa$-objective models the condition '$\mathrm{cd}^*(B)$' $<\varkappa$' can be changed to '$\mathrm{cd}^*(B) < \varkappa^+$ and $B \in \boldsymbol{A}$ (and B has FGA when $\varkappa=\aleph_0$.)'. (Of course, $B \in \boldsymbol{A}$ is unnecessary if the theory is inductive.) This answers partially to the question raised by Simmons in connexion with his Theorem 5.7, [18]. The possibility of similar changes in the definitions of $\boldsymbol{H}_\varkappa$ and $\boldsymbol{S}_\varkappa$ are already shown by 2.13.3 and 2.13.4.

Now we consider the preservation of the truth of sentences by maps between models belonging to various classes.

THEOREM 2.15. *Let M and N be two models, A a submodel of M with* $\mathrm{cd}^*(A) < \varkappa$, *and f a map of A into N. For any sentence $\varphi(\boldsymbol{a})$ in the language of $L(A)$, $\varphi(\boldsymbol{a})$ holds in M if and only if $\varphi(f(\boldsymbol{a}))$ holds in N, under any one of the following conditions.*

2.15.1–2.15.4. *M and N are $\varkappa$-objective, and A is amalgamative unless*

M and N are both $\varkappa$-injective. When $\varkappa = \aleph_0$, M (N) is assumed to have FGA *unless it is $\varkappa$-injective.*

2.15.5–2.15.8. *M is $\varkappa$-objective and is a submodel of N, N is either $\varkappa$-homogeneous or $\varkappa$-subjective, and A is amalgamative unless M is k-injective and N is $\varkappa$-homogeneous. When $\varkappa = \aleph_0$,* FGA *is assumed on M (N) unless it is $\varkappa$-injective ($\varkappa$-homogeneous).*

2.15.9 (2.15.10). *M and N are the same and $\varkappa$-homogeneous ($\varkappa$-subjective and A is amalgamative. Further, when $\varkappa = \aleph_0$,* FGA *is assumed on $M = N$.)*

NOTE. The following table summarises various assumptions.

	M (with FGA?)			N (with FGA?)		$A \in \boldsymbol{A}$?
1.	$\boldsymbol{I}$			$\boldsymbol{I}$		
2.	$\boldsymbol{I}$			$\boldsymbol{O}$	yes	yes
3.	$\boldsymbol{O}$	yes		$\boldsymbol{I}$		yes
4.	$\boldsymbol{O}$	yes		$\boldsymbol{O}$	yes	yes
5.	$\boldsymbol{I}$		$\subseteq$	$\boldsymbol{H}$		
6.	$\boldsymbol{I}$		$\subseteq$	$\boldsymbol{S}$	yes	yes
7.	$\boldsymbol{O}$	yes	$\subseteq$	$\boldsymbol{H}$		yes
8.	$\boldsymbol{O}$	yes	$\subseteq$	$\boldsymbol{S}$	yes	yes
9.	$\boldsymbol{H}$		$=$	$\boldsymbol{H}$		
10.	$\boldsymbol{S}$	yes	$=$	$\boldsymbol{S}$	yes	yes

'Yes' means the condition in question is assumed (when $\varkappa = \aleph_0$), and the blank implies that it is not.

PROOF. We use induction on the formation of $\varphi(\boldsymbol{a})$. If $\varphi(\boldsymbol{a})$ is atomic or its major connective is Boolean, the aasertion follows from the definition of an injection and the induction hypothesis. Thus, we assume $\varphi(\boldsymbol{a})$ is $\exists x\psi(\boldsymbol{a}, x)$. We prove only the fourth, the seventh, and the ninth cases, as typical examples.

2.15.4. If $M \models \varphi(\boldsymbol{a})$ then there is an $m \in M$ such that $M \models \psi(\boldsymbol{a}, m)$. Take a $B \subseteq M$ such that $A \subseteq B$, $m \in B$, $B \in \boldsymbol{A}$ and $\mathrm{cd}^*(B) < \varkappa$, using 1.9.1 or FGA of M. Since $N \in \boldsymbol{O}$ and $A \in \boldsymbol{A}$, $f: A \to N$ can be extended to a $g: B \to N$. Then by the induction hypothesis, $N \models \psi(g(\boldsymbol{a}), g(m))$, hence $N \models \varphi(f(\boldsymbol{a}))$ as $g \restriction A = f$. The converse direction can be shown in the same way by starting from f^{-1} in place of f.

2.15.7. Assume $M \models \varphi(\boldsymbol{a})$. Take a B as above. Since $M \subseteq N \in \boldsymbol{H}$, we can extend f to a $g: B \to N$. Thus $N \models \varphi(f(\boldsymbol{a}))$ as above. Assume, conversely, $N \models \psi(f(\boldsymbol{a}), n)$ for some $n \in N$. Take a $B \subseteq N$ such that $f(A) \subseteq B$, $n \in B$

and $\mathrm{cd}^*(B) < \varkappa$. Since $M \in \boldsymbol{O}$ and $f(A) \in \boldsymbol{A}$, there is a $g: B \to M$ extending f^{-1}. Extend $g(B)$ to a $C \subseteq M$ such that $C \in \boldsymbol{A}$ and $\mathrm{cd}^*(C) < \varkappa$ applying 1.9.1 or FGA of M. Since $N \in \boldsymbol{H}$, g^{-1} can be extended to an $h: C \to N$. The induction hypothesis applies to these C, h and $\psi(\boldsymbol{a}, g(n))$ as $f \subseteq g^{-1} \subseteq h$. Hence, $M \models \psi(\boldsymbol{a}, g(n))$ or $M \models \varphi(\boldsymbol{a})$, because $N \models \psi\big(h(\boldsymbol{a}), h(g(n))\big)$.

2.15.9. Assume $M \models \psi(\boldsymbol{a}, m)$ for some $m \in M$. Extend A to a model $B \subseteq M$ such that $A \subseteq B$, $m \in B$ and $\mathrm{cd}^*(B) < \varkappa$. Since $M = N \in \boldsymbol{H}$, we can extend f to a $g: B \to N$. By the induction hypothesis, $N \models \psi(g(\boldsymbol{a}), g(m))$. Thus $N \models \exists x \psi(f(\boldsymbol{a}), x)$ as $g \restriction A = f$. The converse direction can be proved in the same way by using f^{-1}.

Corollary.

2.16.1. *A map between two $\varkappa$-objective models is elementary.* (*When $\varkappa = \aleph_0$, both models are assumed to have* FGA.)

2.16.2. *Given a $\varkappa$-objective model M and its extension N which is either $\varkappa$-subjective or $\varkappa$-homogeneous, then M is an elementary submodel of N. When $\varkappa = \aleph_0$, M (N) is assumed to have* FGA *unless it is $\varkappa$-injective ($\varkappa$-homogeneous).*

Proof. 2.16.1. Let $g: M \to N$ be the given map where $M, N \in \boldsymbol{O}$. Given $\varphi(\boldsymbol{a}) \in L(M)$, take an $A \subseteq M$ such that $\boldsymbol{a} \in A$, $\mathrm{cd}^*(A) < \varkappa$, and $A \in \boldsymbol{A}$. Taking $g \restriction A$ as f, we have $M \models \varphi(\boldsymbol{a})$ iff $N \models \varphi(g(\boldsymbol{a}))$ from the fourth case of the theorem. Thus g is elementary. 2.16.2 can be shown similarly.

3.

We recall a few facts about infinite forcing. This notion is formulated with respect to a given class of relational systems; that is, given two classes $\boldsymbol{C}$ and $\boldsymbol{K}$, a system $M \in \boldsymbol{C} \cap \boldsymbol{K}$ and a sentence φ in $L(M)$, φ may be forced in M with respect to $\boldsymbol{C}$ but not with respect to $\boldsymbol{K}$. However if $\boldsymbol{C}$ and $\boldsymbol{K}$ are cofinal in each other, a sentence is forced in M with respect to $\boldsymbol{C}$ if and only if it is with respect to $\boldsymbol{K}$. Thus, when we are considering the class of all models of a theory and the class of all substructures, we can say unambiguously that a sentence is forced in a model.

The class $\boldsymbol{G}$ of generic structures is the unique subclass of all structures that satisfies:

(a) Every structure can be extended to a member of $\boldsymbol{G}$,
(b) if $M \subseteq N$ and $M, N \in \boldsymbol{G}$ then $M \prec N$,
(c) if $M \prec N$ and $N \in \boldsymbol{G}$ then $M \in \boldsymbol{G}$.

Thus, if the theory T has the model companion T^*, then $\boldsymbol{G}$ is exactly the models of T^*; and conversely if G is the class of all models of some theory T_1 then T_1 is the model companion of T. Also, if the theory is inductive, all generic structures are models. But, in general, a generic structure may not be a model. However, we have a weaker result about the class of models: For any infinite model M there is a model $N \supseteq M$ such that $\mathrm{cd}(N) = \mathrm{cd}(M)$, and for each sentence $\varphi \in L(M)$ either φ or $\neg\varphi$ is forced in N. (We say that N is *generic over* M.) The reason is simple: M can be extended to a generic structure G and G is a substructure of a model N. By using (c) above and LST, we may assume that M, G and N are of the same cardinality. Naturally, if N is generic over M then any extension of N is generic over M; if $M \subseteq M_1$ and N is generic over M_1 then N is generic over M; and, if $f\colon N \cong N'$ where f is the identity on M then N' is generic over M iff N is. We also note that a model is generic if and only if it is generic over each of its countable submodels (f.g. submodels for a convex theory).

Simmons introduced in [19] the operation $\mathscr{G}$ on classes of models. Given a class $\boldsymbol{K}$, let $M \in \mathscr{G}(\boldsymbol{K})$ exactly when for each $N \in \boldsymbol{K}$ if $M \subseteq N$ then $M \prec N$. To exclude freak cases, we assume $\boldsymbol{K}$ to be cofinal when it appears in the context of $\mathscr{G}(\boldsymbol{K})$. Also we introduce the operation Ω with the definition that $M \in \Omega(\boldsymbol{K})$ if and only if $M \prec N$ for some $N \in \boldsymbol{K}$. Obviously, Ω is idempotent and monotone. It is shown in [19] that $\mathscr{G}(\boldsymbol{K}) = \mathscr{G}(\Omega(\boldsymbol{K})) \subseteq \Omega(\boldsymbol{K})$. We note that 2.16.2 can be reformulated as:

(**) If a cofinal class $\boldsymbol{K}$ is included in $\boldsymbol{S}_\varkappa \cup \boldsymbol{H}_\varkappa$ then $\mathscr{G}(\boldsymbol{K})$ includes $\boldsymbol{O}_\varkappa$. (When $\varkappa = \aleph_0$, FGA is assumed on $\boldsymbol{S}_\varkappa$.)

PROPOSITION 3.1. *Assume that* $\boldsymbol{K}$ *is a cofinal class,* $M, N \in \mathscr{G}(\boldsymbol{K})$, *and* $M \subseteq N$. *Then* M *is an elementary substructure of* N.

PROOF. There is a $K \in \boldsymbol{K}$ such that $N \subseteq K$, because $\boldsymbol{K}$ is cofinal. Then $M \prec K$ and $N \prec K$, because $M, N \in \mathscr{G}(\boldsymbol{K})$. Thus $M \prec N$.

THEOREM 3.2. *Assume that* $\boldsymbol{O}_\varkappa$ *is a cofinal class.* (*When* $\varkappa = \aleph_0$, FGA *is assumed on* $\boldsymbol{S}_\varkappa$.)

3.2.1. *For every cofinal subclass* $\boldsymbol{K}$ *of* $\boldsymbol{S}_\varkappa \cup \boldsymbol{H}_\varkappa$, $\Omega(\mathscr{G}(K))$ *is the class of generic models.*

3.2.2. *For a cofinal subclass* $\boldsymbol{K}$ *of* $\boldsymbol{O}_\varkappa$, $\Omega(\boldsymbol{K})$ *is the class of generic models. In particular, if there is a* $\varkappa$*-universal and* $\varkappa$*-homogeneous model, then a model is generic exactly when it is an elementary submodel of a member of* $\boldsymbol{U}_\varkappa \cap \boldsymbol{H}_\varkappa$.

3.2.3. *The theory of the class* $\boldsymbol{O}_\varkappa$ *is the forcing companion.*

PROOF. 3.2.1. By (**), $\mathscr{G}(\boldsymbol{K})$ has (a) of $\boldsymbol{G}$; and by 3.1, it has (b). By [19, Theorem 3.5], $\Omega(\mathscr{G}(\boldsymbol{K}))$ has all three properties and hence it is $\boldsymbol{G}$.

3.2.2. By (**) and 2.6.6, $\boldsymbol{O} \subseteq \mathscr{G}(\boldsymbol{K})$. Thus by the nature of $\mathscr{G}$ and Ω, we have

$$\Omega(\boldsymbol{O}) \subseteq \Omega(\mathscr{G}(\boldsymbol{K})) \subseteq \Omega(\Omega(\boldsymbol{K})) = \Omega(\boldsymbol{K}) \subseteq \Omega(\boldsymbol{O}) .$$

Hence these are all identical with $\Omega(\mathscr{G}(\boldsymbol{K}))$, which is $\boldsymbol{G}$ by 3.2.1. If $\boldsymbol{UH}$ is not empty, then the theory has JE and AP by 2.3. Thus $\boldsymbol{UH} = \boldsymbol{O}$ by 2.10.1. So the second part follows from the first by taking $\boldsymbol{UH}$ as $\boldsymbol{K}$.

3.2.3. By the above, $\mathrm{Th}(\boldsymbol{O}) = \mathrm{Th}(\boldsymbol{G})$ which is the forcing companion by definition.

The above gives a "global" characterization of the class of generic models, while the following gives some "local" information.

THEOREM.

3.3.1. *An elementary submodel of a $\varkappa$-objective model M is generic. (When $\varkappa = \aleph_0$,* FGA *is assumed on M.)*

3.3.2. *If M is $\varkappa$-universal and $\varkappa$-subjective, then every generic model of power $\leqq \varkappa$ is isomorphic to an elementary submodel of M.*

PROOF. 3.3.1. It suffices to show $M \in \boldsymbol{G}$, because of (c) of $\boldsymbol{G}$. Take an $A \subseteq M$ such that $\mathrm{cd}^*(A) < \min(\varkappa, \aleph_1)$ and $A \in \boldsymbol{A}$ by the given assumption or by LST and $\boldsymbol{O} \subseteq \boldsymbol{A}$. Let $B \supseteq A$ be countable and generic over A. Then either by the definition of $\boldsymbol{O}$ when $\varkappa > \aleph_0$ or by a remark after 2.14 and $B \in \boldsymbol{G} \subseteq \boldsymbol{A}$ when $\varkappa = \aleph_0$, there is an $h\colon B \to M$ extending the inclusion $i\colon A \to M$. Thus $h(B)$, hence $M \supseteq h(B)$ also, are generic over A. We can conclude that $M \in \boldsymbol{G}$ because M is generic over each of its countable (when $\varkappa > \aleph_0$) or finitely generated (when $\varkappa = \aleph_0$) submodels.

3.3.2. Take a $G \in \boldsymbol{G}$ such that $\mathrm{cd}(G) \leqq \varkappa$. Since $\boldsymbol{G} \subseteq \boldsymbol{A}$, we have a map $f\colon G \to M$ by a remark after 2.14. By taking isomorphic copies, we may take f to be an inclusion map. Because $M \in \boldsymbol{G}$ by virtue of 2.6.5 and 3.3.1, we have $G \prec M$ by (b) of $\boldsymbol{G}$.

In 3.3.2, the assumption $M \in \boldsymbol{US}$ can not be weakened to $M \in \boldsymbol{O}$ meaningfully. For, if there is one such objective model M, then the theory has JE and hence $\boldsymbol{US} = \boldsymbol{O}$ by 2.9.1. Indeed, any two existential sentences of L, each being consistent with T, are true in generic structures (not necessarily models of T) due to (a) of $\boldsymbol{G}$. These can be embedded in M, hence the sentences are jointly consistent. Thus the theory has JE.

In the previous section, the relation among $\boldsymbol{O}$ and $\boldsymbol{U}$, etc., was studied.

Now we turn our attention to the relation between $\boldsymbol{O}$ and saturated models.

THEOREM.

3.4.1 (Simmons). *If the given theory has the model companion, then all $\varkappa$-objective models are $\varkappa$-saturated.*

3.4.2. *If there is a $\varkappa$-universal and $\varkappa$-subjective model which is $\varkappa$-saturated, then the theory has the model companion.*

PROOF. The first proposition follows immediately from [18, Theorem 4.7], in view of 2.4. To show the second, assume $M \in \boldsymbol{US}$ and $\varkappa$-saturated. So the theory has JE as $\boldsymbol{U} \neq \emptyset$. By 2.6.5 and 3.3.1, we have $M \in \boldsymbol{G}$, hence M is existentially closed. Since M is $\varkappa$-saturated, it is *-saturated in the sense of [17]. Hence, from Theorem 5.3 there, we can conclude that the model companion exists.

As a corollary, it follows from 2.6.5 that $\boldsymbol{US}$ is either included in the class of saturated models or disjoint from it, according as the model companion exists or not. E. R. Fisher kindly informed me that $\boldsymbol{US}$ can not be replaced by $\boldsymbol{O}$ in 3.4.2.

Now we study the nature of amalgamative models in terms of cofinal classes of structures. Given a class $\boldsymbol{K}$ and a model M, we say that $\boldsymbol{K}$ is *n-equivalent over* M when, for any C and D in $\boldsymbol{K}$ and any sentence $\varphi \in \Sigma_n(M)$, if C and D are extensions of M then $C \models \varphi$ iff $D \models \varphi$. A class $\boldsymbol{K}$ is said to be an *n-class* if it is n-equivalent over each member of $\boldsymbol{K}$. Note that if $m \leqq n$, then "n-equivalence" implies "m-equivalence", hence an n-class is an m-class also.

LEMMA.

3.5.1. *If a model M is amalgamative, then each cofinal class $\boldsymbol{K}$ of structures is n-equivalent over M whenever it is an n-class and is closed under isomorphism.*

3.5.2. *If a model M is not amalgamative, then no cofinal class $\boldsymbol{K}$ of structures is n-equivalent over M, for any $n \geqq 1$. Further, if $\boldsymbol{K}$ is cofinal in power, in the sense that each infinite structure can be extended to a member of $\boldsymbol{K}$ of the same power, then there are structures C and D in $\boldsymbol{K}$ of power* $\mathrm{cd}(M)+\aleph_0$ *and a sentence $\varphi \in \Sigma_1(M)$ such that $C \models \varphi$ but $D \models \neg\varphi$.*

PROOF OF 3.5.1. Let $C,D \in \boldsymbol{K}$ be extensions of M. Since $M \in \boldsymbol{A}$ and $\boldsymbol{K}$ is cofinal, there is an $N \in \boldsymbol{K}$ and $f\colon C \to N$ and $g\colon D \to N$ such that f and g coincide on M. Since $\boldsymbol{K}$ is an n-class and closed under isomorphism, for each $\varphi \in \Sigma_n(M)$, $C \models \varphi$ iff $N \models f(\varphi)$, and $D \models \varphi$ iff $N \models g(\varphi)$. But $f(\varphi)$ and $g(\varphi)$ are indeed the same formula because all new constants are taken from M. Thus $C \models \varphi$ iff $D \models \varphi$.

3.5.2. We prove the second assertion. Since $M \notin \boldsymbol{A}$, by 1.3, (ii), there are sentences $\varphi, \psi \in \Sigma_1(M)$ such that each of $T + Dg(M) + \varphi$ and $T + Dg(M) + \psi$ has a model, but $T + Dg(M) + \varphi + \psi$ has not. These models can be taken to be of power $\mathrm{cd}(M) + \aleph_0$. By assumption these can be extended to $C, D \in \boldsymbol{K}$ of the same power. Since φ and ψ are existential, $C \models \varphi$ and $D \models \psi$. Were $D \models \varphi$ also, then $T + Dg(M) + \varphi + \psi$ would have a model contrary to the assumption, since D is a substructure of a model. Thus $C \models \varphi$ and $D \models \neg\varphi$.

The first part of the lemma is an immediate consequence of the second.

As an application, we have a characterization of amalgamative models. Here the classes $\boldsymbol{O}$, $\boldsymbol{G}$ and $\boldsymbol{E}$ are taken to be classes of structures, and hence are cofinal classes.

THEOREM 3.6. *The following are equivalent*:

(a) *M is amalgamative.*
(b) *The class $\boldsymbol{O}$ is n-equivalent over M, where $n \geqq 1$. (When $\varkappa = \aleph_0$,* FGA *is assumed on $\boldsymbol{O}_\varkappa$.)*
(c) *Each cofinal subclass of $\boldsymbol{G}$ is n-equivalent over M, where $n \geqq 1$.*
(d) *Each cofinal subclass of $\boldsymbol{E}$ is 1-equivalent over M.*

PROOF. Equivalence of (a), (c) and (d) are immediate from 3.5, because the class of generic structures is an ω-class, and that of existentially closed structures is a 1-class. Since $\boldsymbol{O}$ is a cofinal subclass of $\boldsymbol{G}$ by 2.5 and 3.3.1 considered in terms of $T_\forall$, it is an ω-class. Thus (a) and (b) are equivalent also.

Since a structure is in $\boldsymbol{A}$ iff it is pregeneric (cf. [23, Theorem 2]), the above result extends Theorem 6.2 of [15]. Using condition (d), we can show a sort of converse to 2.13.2, also.

THEOREM 3.7. *A structure M of power $< \varkappa$ is amalgamative if and only if for structures A, B, C, and D such that $M \subseteq C \subseteq A$, $M \subseteq D \subseteq B$, A and B are $\varkappa$-objective, and* $\mathrm{cd}(C) = \mathrm{cd}(D) = \varkappa$, *there is a map f such that*

$$C \subseteq \mathrm{dm}(f) \subseteq A, \quad D \subseteq \mathrm{rg}(f) \subseteq B$$

and f is the identity on M. (When $\varkappa = \aleph_0$, $\boldsymbol{O}_\varkappa$ is assumed to have FGA.)

PROOF. The 'only if' part is a special case of 2.13.2. To show the other direction, assume $M \notin \boldsymbol{A}$. From 3.6, we know that there are $C, D \in \boldsymbol{E}$ and $\varphi(\boldsymbol{m}) \in \Sigma_1(M)$ such that $C \supseteqq M$, $D \supseteqq M$, and $C \models \varphi(\boldsymbol{m})$ but $D \models \neg\varphi(\boldsymbol{m})$. By the cofinality in power of $\boldsymbol{E}$ and LST, we may assume $\mathrm{cd}(C) = \mathrm{cd}(D) = \varkappa$. Applying 2.5 in terms of $T_\forall$, we have $A, B \in \boldsymbol{O}$, including C and D, respectively. As $\varphi(\boldsymbol{m}) \in \Sigma_1(M)$ and $M \subseteqq C \subseteqq A$, we have $A \models \varphi(\boldsymbol{m})$. As $D \in \boldsymbol{E}$ and $D \subseteqq B$, we have $B \models \neg\varphi(\boldsymbol{m})$. Take a structure P such that $M \subseteqq P \subseteqq B$, $P \in \boldsymbol{A}$ and $\mathrm{cd}^*(P) < \varkappa$, by using LST or FGA on B. Were there a map f as in the statement of the theorem, let g be $f \upharpoonright P$. Then, by 2.15.4 we have $B \models \varphi(g(\boldsymbol{m}))$ because $A \models \varphi(\boldsymbol{m})$. But $g \upharpoonright M = f \upharpoonright M = \mathrm{id}_M$. So, $B \models \varphi(\boldsymbol{m})$, which contradicts $B \models \neg\varphi(\boldsymbol{m})$.

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