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ON EXPONENTIAL RECURRING SEQUENCES

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1.

A (polynomial) recurring sequence $\{z_n\}$ is an integral sequence satisfying

$$z_n = P(z_{n-1}, \ldots, z_{n-r})$$

for all $n \geqq r$, where P is a polynomial in r variables with integral coefficients. Every such sequence is periodic from some point on modulo any integer m. In this paper we look at the more general situation where P is a function containing iterated exponentials as well, and we prove that the sequences are still periodic modulo any m.

2.

To make things more precise, we introduce some notations. Let $\mathsf{N} = \{1,2,\ldots\}$ be the set of natural numbers and $\mathsf{N}_1 = \{2,3,\ldots\}$. We define a set $\mathfrak{F}$ of functions recursively as follows: $\mathfrak{F}$ contains the following *elementary functions*:

E1. $f(x_1,\ldots,x_n) = a, \quad a \in \mathsf{N}$;
E2. $f(x_1,\ldots,x_n) = x_i, \quad i = 1,2,\ldots,n$;
E2*. $f(x) = a^x, \quad a \in \mathsf{N}_1$.

The set $\mathfrak{F}$ is formed by the following *composition rules*:

C1. If $f,g \in \mathfrak{F}$, then $f+g$, $fg \in \mathfrak{F}$;
C2. If $f \in \mathfrak{F}$, then $x_i^f \in \mathfrak{F}$;
C2*. If $a \in N_1$ and $g \in \mathfrak{F}$, then $a^g \in \mathfrak{F}$;
C3. If $f(x_1,\ldots,x_n) \in \mathfrak{F}$, then

$$f(x_1,\ldots,x_{j-1},x_i,x_{j+1},\ldots,x_n) \in \mathfrak{F} \quad \text{for } i = 1,2,\ldots,n .$$

We see that every $f \in \mathfrak{F}$ may be expressed in the form

$$f = \textstyle\sum_k a_k \{\prod_l (q_{kl})^{f_{kl}} \prod_\lambda x_\lambda^{g_{k\lambda}}\} \tag{2.1}$$

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where the q_{kl}'s are primes (not necessarily distinct), $a_k \in \mathsf{N}$, $f_{kl} \in \mathfrak{F}$, $g_{k\lambda} \in \mathfrak{F}$, and the f_{kl}'s consist of a single term which is product of non-constant functions. Further, this representation is unique.

The subset of $\mathfrak{F}$ formed by choosing E1 and E2 as elementary functions and C1 and C3 as composition rules, is the set of all polynomials with positive integral coefficients. Let $\mathfrak{P}$ be the subset of $\mathfrak{F}$ formed by E1, E2*, C1, C2* and C3. For $f \in \mathfrak{P}$ we have $g_{k\lambda} \equiv 0$ in (2.1), and $f_{kl}(x)$ is either x_i for some i or is a product of functions from $\mathfrak{P}$.

An *exponential recurring sequence* $\{z_n\}$ is a sequence satisfying

$$z_n = F(z_{n-1},\ldots,z_{n-r}) \quad \text{for } n \geqq r\,, \tag{2.2}$$

where $F \in \mathfrak{F}$. If $F \in \mathfrak{P}$, then we call the sequence a *pure exponential recurring sequence.*

We prove the following theorems.

THEOREM 1. *Every exponential recurring sequence is periodic modulo any integer m.*

THEOREM 2. *Every pure exponential recurring sequence has period* 1 *modulo any integer m.*

3.

Before we go on to the proof of the theorems we define some further concepts.

Let φ be Euler's function. We define φ_k for $k \geqq 0$ and Φ by

$$\begin{aligned} \varphi_0(m) &= m && \text{for } m \in \mathsf{N}\,, \\ \varphi_k(m) &= \varphi(\varphi_{k-1}(m)) && \text{for } k \geqq 1,\ m \in \mathsf{N}\,, \\ \Phi(m) &= \operatorname{lcm}_{k\geqq 0}\{\varphi_k(m)\} && \text{for } m \in \mathsf{N}\,, \end{aligned}$$

where lcm denotes least common multiple. We note that if $p^\alpha \mid \Phi(m)$, then $p^\alpha \mid \varphi_k(m)$ for some k. Hence

$$\varphi(p^\alpha) \mid \varphi(\varphi_k(m)) = \varphi_{k+1}(m) \mid \Phi(m)\,.$$

For any $F \in \mathfrak{F}$ we define $\mathfrak{D}(F)$ as follows:

I. $F \in \mathfrak{D}(F)$.

II. If $f \in \mathfrak{D}(F)$ and we express f in the form (2.1), then

$$f_{kl},\, (q_{kl})^{f_{kl}},\, g_{k\lambda},\, x_\lambda^{g_{k\lambda}} \in \mathfrak{D}(F)$$

for all k, l, λ.

III. If $F=F(x_1,\ldots,x_r)$, then the elementary functions defined by E2 (the projections) belong to $\mathfrak{D}(F)$ for $i=1,2,\ldots,r$.

For any $F\in\mathfrak{F}$ we define $h(F)$, the height of F, as follows:

$$\begin{aligned} h(a) &= h(x_i) = 0, \quad a\in\mathsf{N};\\ h(a^f) &= h(x_i{}^f) = h(f)+1 \quad \text{for } f\in\mathfrak{F} \text{ nonconstant};\\ h(f+g) &= h(fg) = \max\{h(f),h(g)\}\,. \end{aligned}$$

An example may clearify these concepts. If

$$F(x,y,z,u) = 6^{2y+3^{yz}}+z^y = 2^y2^y2^{3^{yz}}3^y3^y3^{3^{yz}}+z^y$$

then $\mathfrak{D}(F)$ consists of

$$F,x,y,z,u,2^y,2^{3^{yz}},3^y,3^{3^{yz}},3^{yz},yz,z^y\ ,$$

of heights 2, 0, 0, 0, 0, 1, 2, 1, 2, 1, 0, and 1 respectively.

Let $F=F(x_1,\ldots,x_r)=F(\boldsymbol{x})\in\mathfrak{F}$. Let

$$\Phi(m) = \textstyle\prod_i p_i^{\alpha_i}$$

be the product of $\Phi(m)$ as primepowers and put $\nu=\nu(m)=\max_i\{\alpha_i\}$. In the set N^r of r-dimensional vectors with elements from N we define a relation $\sim_F$, depending on F and m. It is easily seen to be an equivalence relation. We define

$$\boldsymbol{u} \sim_F \boldsymbol{v}$$

if and only if

I. $f(\boldsymbol{u})\equiv f(\boldsymbol{v}) \pmod{\Phi(m)}$ for all $f\in\mathfrak{D}(F)$.

II. If $f(\boldsymbol{u})\neq f(\boldsymbol{v})$ for some $f\in\mathfrak{D}(F)$, then $f(\boldsymbol{u})>\nu$ and $f(\boldsymbol{v})>\nu$ for this f.

4.

To prove theorem 1 we first prove two lemmas.

Lemma 1. *For each $F\in\mathfrak{F}$ the equivalence relation $\sim_F$ divides N^r into a finite number of equivalence classes.*

Proof. If d is the number of different functions in $\mathfrak{D}(F)$, then clause I divides N^r into at most $\Phi(m)^d$ classes and clause II divides each of these into at most $(\nu+1)^d$ classes. Hence there are at most $\{(\nu+1)\Phi(m)\}^d$ equivalence classes.

Lemma 2. *If $(u_1,\ldots,u_r)\sim_F(v_1,\ldots,v_r)$, then*

$$\big(F(u_1,\ldots,u_r),u_1,\ldots,u_{r-1}\big) \sim_F \big(F(v_1,\ldots,v_r),v_1,\ldots,v_{r-1}\big)\,.$$

PROOF. To simplify notations, we denote the vectors appearing in lemma 2 by $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{u}'$, and $\boldsymbol{v}'$ respectively, so that $u_1' = F(\boldsymbol{u})$ and $u_i' = u_{i-1}$ for $i > 1$ and similarly for $\boldsymbol{v}'$. We must show that the clauses I and II are satisfied by $\boldsymbol{u}'$ and $\boldsymbol{v}'$.

Clause I. We prove this by induction on $h(f)$. Assume $h(f) = 0$. Then $f(\boldsymbol{x})$ is a polynomial in $x_1, \ldots, x_r$. Since $\boldsymbol{u} \sim_F \boldsymbol{v}$ we have, by clause I, that

$$u_i' = u_{i-1} \equiv v_{i-1} = v_i' \pmod{\Phi(m)}, \quad i = 2, \ldots, r,$$
$$u_1' = F(\boldsymbol{u}) \equiv F(\boldsymbol{v}) = v_1' \pmod{\Phi(m)}.$$

Hence

$$f(\boldsymbol{u}') \equiv f(\boldsymbol{v}') \pmod{\Phi(m)}.$$

Now let $h(f) = h > 0$. We divide the induction step into three cases.

CASE (A), $f = a^g$ where $a \in \mathsf{N}_1$ and $h(g) = h(f) - 1$. Let $p_i \mid \Phi(m)$.
Subcase (i), $p_i \nmid a$. By the induction hypothesis

$$g(\boldsymbol{u}') \equiv g(\boldsymbol{v}') \pmod{\Phi(m)}.$$

In particular

$$g(\boldsymbol{u}') \equiv g(\boldsymbol{v}') \pmod{\varphi(p_i^{\alpha_i})}.$$

Hence, by Euler's theorem

$$f(\boldsymbol{u}') = a^{g(\boldsymbol{u}')} \equiv a^{g(\boldsymbol{v}')} = f(\boldsymbol{v}') \pmod{p_i^{\alpha_i}}.$$

Subcase (ii), $p_i \mid a$. If $g(\boldsymbol{u}') \neq g(\boldsymbol{v}')$, then, by clause II,

$$g(\boldsymbol{u}') > \nu \geqq \alpha_i \quad \text{and} \quad g(\boldsymbol{v}') > \nu \geqq \alpha_i.$$

Hence

$$a^{g(\boldsymbol{u}')} \equiv a^{g(\boldsymbol{v}')} \equiv 0 \pmod{p_i^{\alpha_i}}.$$

Case (B), $f(\boldsymbol{x}) = x_j^{g(\boldsymbol{x})}$ where $h(g) = h(f) - 1$. If $p_i \nmid u_j'$, then, since

$$u_j' \equiv v_j' \pmod{\Phi(m)} \tag{4.1}$$

we proceed as in case (A), subcase (i). If $p_i \mid u_j'$, let $p_i^\beta \| u_j'$. If $\beta < \alpha_i$, then $p_i^\beta \| v_j'$ by (4.1) and we may go on as in case (A), subcase (ii). If $\beta \geqq \alpha_i$, then $p_i^{\alpha_i} \mid v_j'$ by (4.1) and hence

$$(u_j')^{g(\boldsymbol{u}')} \equiv (v_j')^{g(\boldsymbol{v}')} \equiv 0 \pmod{p_i^{\alpha_i}}.$$

Case (C), f is any function of height h. Then f is a sum of products of functions of the form considered in the cases (A) and (B). (Cf. (2.1).)
Hence

$$f(\boldsymbol{u}') \equiv f(\boldsymbol{v}') \pmod{p_i^{\alpha_i}}.$$

Since this congruence is true for all $p_i^{\alpha_i} \mid \Phi(m)$, it must hold modulo $\Phi(m)$ as well.

Clause II. This is also proved by induction on $h(f)$. $h(f)=0$. Then $f(x)$ is a polynomial. Suppose $f(u') \neq f(v')$. Then $u_i' \neq v_i'$ for at least one i, such that x_i appears in the polynomial $f(x)$. Then, by clause I, $u_i' > \nu$ and $v_i' > \nu$. Hence $f(u') \geqq u_i' > \nu$ and $f(v') \geqq v_i' > \nu$.

$h(f)=h>0$. Case (A), $f=a^g$ where $a \in N_1$ and $h(g)=h(f)-1$. If $f(u') \neq f(v')$, then $g(u') \neq g(v')$. By the induction hypothesis $g(u') > \nu$ and $g(v') > \nu$. Hence

$$f(u') = a^{g(u')} > a^\nu > \nu$$

and $f(v') > \nu$ similarly.

Case (B), $f(x)=x_j^{g(x)}$. If $u_j'=1$, then $u_j' \leqq \nu$. Hence, by clause II, $u_j'=v_j'=1$ and so $f(u')=f(v')$. If $u_j'>1$ and $f(u') \neq f(v')$, then either $u_j' \neq v_j'$ or $g(u') \neq g(v')$. Hence either $u_j' > \nu$ and $v_j' > \nu$ or $g(u') > \nu$ and $g(v') > \nu$. In either case $f(u') > \nu$ and $f(v') > \nu$.

Case (C), f is any function of height h. Then f is sum of products of functions f_i of the form considered in cases (A) and (B). If $f(u') \neq f(v')$, then $f_i(u') \neq f_i(v')$ for at least one i. Hence

$$f(u') \geqq f_i(u') > \nu$$

and $f(v') > \nu$ similarly. This completes the proof of lemma 2.

Put $z_n = (z_{n-1}, \ldots, z_{n-r})$. By lemma 1 there exist n_1 and n_2 such that $n_1 < n_2$ and $z_{n_2} \sim_F z_{n_1}$. By lemma 2 and (2.2) we have $z_{n_2+1} \sim_F z_{n_1+1}$, and applying lemma 2 repeatedly we obtain $z_{n_2+k} \sim_F z_{n_1+k}$ for all $k \geqq 0$. In particular (putting $\mu = n_2 - n_1$) we get

$$z_{n+\mu} \equiv z_n \pmod{m}$$

for all $n \geqq n_1 - r$. This is theorem 1.

5.

To prove theorem 2 we need two more lemmas.

LEMMA 3. *If $F \in \mathfrak{P}$ is nonconstant and $\{z_n\}$ satisfies* (2.2) *then $f(z_n) \to \infty$ when $n \to \infty$ for all nonconstant $f \in \mathfrak{D}(F)$.*

PROOF. The proof is by induction on $h(f)$. First we prove that $z_n \to \infty$ when $n \to \infty$.

By (2.1)

$$F(x) \geqq (q_{11})^{f_{11}(x)} > f_{11}(x)$$

where $h(f_{11}) < h(F)$. Applying the same procedure to f_{11} we find a f'_{11} such that $f_{11}(x) > f'_{11}(x)$ and $h(f'_{11}) < h(f_{11})$. Applying the procedure repeatedly a finite number of times we arrive at a function of height 0, i.e.

$$F(x) > x_i$$

for some fixed i. By (2.2) we have

$$z_n > z_{n-i} \quad \text{for all } n \geqq r .$$

Hence $z_{n+ki} \geqq z_n + k$, that is $z_n \to \infty$ when $n \to \infty$. If $h(f) = 0$ and f is nonconstant then $f(x) \geqq x_i$ for some i. Hence $f(z_n) \geqq z_{n-i} \to \infty$ when $n \to \infty$. If $h(f) = h > 0$, then

$$f(x) \geqq (q_{11})^{f_{11}(x)} > f_{11}(x)$$

where $h(f_{11}) < h(f)$. By the induction hypothesis $f_{11}(z_n) \to \infty$, hence $f(z_n) \to \infty$.

Lemma 4. *For all primepowers p^α and all $f \in \mathfrak{D}(F)$ we have*

$$(5.1) \qquad f(z_{n+1}) \equiv f(z_n) \ (\operatorname{mod} \Phi(p^\alpha)) \quad \text{for } n \gg 0 .$$

Proof. We prove lemma 4 by induction. Since $\Phi(1) = 1$, (5.1) is true when $\alpha = 0$. Our induction hypothesis is that (5.1) is true for all powers of all primes less then p and also for p^β when $\beta < \alpha$. We prove that it is true for p^α. If f is nonconstant, we have

$$f(z_n) = \textstyle\sum_k a_k \prod_l (q_{kl})^{f_{kl}(z_n)} .$$

Fix k (we look at one term at a time). If $p = q_{kl}$ for some l, then

$$(q_{kl})^{f_{kl}(z_n)} \equiv 0 \ (\operatorname{mod} p^\alpha) \quad \text{for } n \gg 0$$

by lemma 3. If $p \neq q_{kl}$, then

$$(q_{kl})^{f_{kl}(z_{n+1})} \equiv (q_{kl})^{f_{kl}(z_n)} \ (\operatorname{mod} p^\alpha)$$

by Euler's theorem since

$$f_{kl}(z_{n+1}) \equiv f_{kl}(z_n) \ (\operatorname{mod} \varphi(p^\alpha)) \quad \text{for } n \gg 0$$

by the induction hypothesis. Hence

$$f(z_{n+1}) \equiv f(z_n) \ (\operatorname{mod} p^\alpha) \quad \text{for } n \gg 0 .$$

Further

$$\Phi(p^\alpha) = p^\alpha \textstyle\prod_{q_j < p} q_j^{\beta_j} .$$

By the induction hypotehsis

$$f(z_{n+1}) \equiv f(z_n) \ (\operatorname{mod} \Pi q_j^{\beta_j}) \quad \text{for } n \gg 0 .$$

Hence

$$f(z_{n+1}) \equiv f(z_n) \pmod{\Phi(p^\alpha)} \quad \text{for } n \gg 0 .$$

This completes the proof of lemma 4.

To prove theorem 2, fix $m = \prod p_i^{\gamma_i}$. By lemma 4 we have

$$z_{n+1} \equiv z_n \pmod{p_i^{\gamma_i}} \quad \text{for } n \gg 0 .$$

Hence

$$z_{n+1} \equiv z_n \pmod{m} \quad \text{for } n \gg 0 .$$

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