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GROUP RINGS OF FINITE REPRESENTATION TYPE

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1. Introduction.

In this note, we wish to discuss Artinian group rings which are of finite representation type in the sense that they have only finitely many non-isomorphic indecomposable modules. Thus let A be a commutative, Artinian ring and let G be a finite group; we will consider the group ring AG. By Maschke's Theorem, we know that if A is a field whose characteristic is either zero or a prime which does not divide the order of G, then AG is semisimple and hence of finite representation type. If, on the other hand, A is a field whose characteristic p divides the order of G, then AG is of finite representation type if and only if the p-Sylow subgroup of G is cyclic. This was shown by D. G. Higman [6]. Later, Kasch, Kupisch and Kneser [9] and Janusz [7] gave more refined information about the number of indecomposable modules in this case. Janusz [8] determined the structure of the indecomposable modules in considerable detail.

Here, we will give necessary and sufficient conditions for AG to be of finite representation type, where A is an arbitrary commutative, Artinian ring.

2. The theorem.

Our approach will depend on two crucial facts:

i) *A commutative, Artinian ring is of finite representation type if and only if it is serial (i.e. each indecomposable projective module has a unique composition series).*

This is established in Colby [1]. It can also be deduced easily from Dickson and Kelly [3].

ii) *Any homomorphic image of a ring of finite representation type is also of finite representation type.*

This is easy to prove.

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Now, each commutative, Artinian ring A is a direct product $A_1 \times \ldots \times A_n$ of local, Artinian rings, and clearly we have

$$AG = A_1 G \times \ldots \times A_n G .$$

Since the identity elements of the $A_i G$ are central idempotents of AG, we see that an indecomposable AG-module is annihilated by all but one of the $A_i G$. Thus AG is of finite representation type if and only if each $A_i G$ is. Hence the problem is essentially solved once we prove

THEOREM. *Let A be a local, commutative, Artinian ring, and let G be a finite group. Then AG is of finite representation type if and only if*

a) *A is serial and*

b) *if the characteristic of $A/\mathrm{rad}\, A$ is a prime p which divides the order of G, then A is a field and the p-Sylow subgroup of G is cylic.*

PROOF. First, suppose that AG is of finite representation type. Since the augmentation map $\varepsilon : AG \to A$ is a surjection, A must be serial by i) and ii) above. Further, we have a surjection $AG \to kG$, where $k = A/\mathrm{rad}\, A$ hence, kG must be of finite representation type. It follows then that either the characteristic of k does not divide the order of G, or the characteristic p of k divides the order of G and the p-Sylow subgroup H of G is cyclic. In the latter case, we see by [4] that each AG-module is (G,H)-projective. That is, for each AG-module M, there is an AH-module N such that M is a direct summand of the induced module

$$N^G = AG \otimes_{AH} N .$$

Now, if AH is of infinite representation type, we see from [10] that for each $n \geqq 1$ we can find an indecomposable AH-module N whose length as an A-module is greater than n. But N is an AH-summand in N^G by [2, 63.6], so it follows by the Krull-Schmidt Theorem that some indecomposable AG-summand of N^G has A-length greater than n, whence AG is of infinite representation type. Thus, if AG if of finite type, so is AH. As in [5, Theorem 1.7], it is easy to see that AH is a commutative, local, Artinian ring with residue field k. Thus it suffices to show that if A is not a field and $H \neq \{1\}$, then AH is not serial. For this, we note that if I denotes $(\mathrm{rad}\, A) \cdot AH$ and Δ is the kernel of the augmentation map $\varepsilon : AH \to A$, then neither of these two ideals is contained in the other. For, if x is a non-zero element of $\mathrm{rad} A$, then $x \in I$; but $x \notin \Delta$, while if $h \in H$, $h \neq 1$, then $h-1$ is in Δ, but not in I.

Now let us suppose that a) and b) are satisfied. If the characteristic of k does not divide the order of G, then every AG-module is $(G, 1)$-projective, by [4]. Since A is uniserial, it has only a finite number of indecomposable modules $M_1, \ldots, M_n$, and all indecomposable AG-modules are among the direct summands of $M_1{}^G, \ldots, M_n{}^G$. By the Krull-Schmidt Theorem, AG is of finite representation type. If the characteristic p of k divides the order of G, $A = k$, and the p-Sylow subgroup of G is cyclic, then AG is of finite type, by Higman's Theorem. This completes the proof.

COROLLARY. *Let A be a commutative, Artinian ring and let G be a finite group. Write $A = A_1 \times \ldots \times A_n$, where each A_i is local. Then AG is of finite representation type if and only if*

a) *Each A_i is serial and*

b) *if the characteristic of $A_i/\mathrm{rad} A_i$ is a prime divisor p of the order of G, then A_i is a field and the p-Sylow subgroup of G is cyclic.*

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