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THE EXTREMAL CONVEX FUNCTIONS

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1. Introduction and summary.

We shall consider the convex cone $\boldsymbol{K}$ of finite continuous convex functions defined on a convex set K in R^2. A large class of extremal functions is identified and it is proved that the extremal functions are dense in $\boldsymbol{K}$.

Thus the results are very different from the results obtained for convex functions on intervals, where all the extremal functions are of the form $a \vee b$ for some affine functions a and b, see Blaschke and Pick [1].

It is easily seen that

$$\boldsymbol{K} \cap (-\boldsymbol{K}) = \boldsymbol{A} ,$$

where $\boldsymbol{A}$ denote the affine functions. Hence the cone $\boldsymbol{K}$ is not pointed and the usual definition of an extreme point has to be modified as follows:

Definition. Let f, g and h be elements of $\boldsymbol{K}$, then f is called extremal if for all g and h such that

$$f = \tfrac{1}{2}(g+h)$$

there exist a constant $\alpha > 0$ and an affine function a, such that

$$f = \alpha g + a .$$

We shall apply the following concepts from the theory of convex sets and functions, see Rockafellar [2].

A polyhedral set is a closed convex set which is the intersection of a finite number of halfspaces.

A polytope is a compact polyhedral set.

A face of a polyhedral set P is a subset $F \subset P$ with the property that

$$x \in F,\ y \in P,\ z \in P,\ x = \tfrac{1}{2}(y+z) \ \Rightarrow\ y \in F,\ z \in F .$$

The 0 dimensional faces are the extreme points or the vertices. The 1-dimensional faces are the edges. A polyhedral set is bounded by a finite number of edges and has a finite number of vertices. A polytope is the convex hull of its vertices.

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A family of polyhedral sets $P_1,\ldots,P_m$ is called a covering of the convex set K if

$$K \subset P_1 \cup \ldots \cup P_m$$

and

$$\operatorname{ri} P_i \cap \operatorname{ri} P_j = \emptyset, \quad i \neq j .$$

Here ri denotes the relative interior. The order of a vertex of a polyhedral set in the covering is the number of polyhedral sets which contains it.

Let $a_1,\ldots,a_n$ denote a family of affine functions, then

$$f = \max_{1 \leqq i \leqq n} a_i$$

is called a polyhedral function and it is seen that it is convex and continuous, and that the sets

$$P_i = \{f = a_i\}, \quad \dim P_i = 2 ,$$

give rise to a covering of R^2, and therefore of K, by polyhedral sets.

The class of extremal functions on R^2 which is found here can be described as polyhedral functions which induce a covering where the vertices are of order 3, see Theorem 1.

An alternative way of describing them is as follows: consider the cylinder

$$C = \{(x,\mu) : \mu \geqq 0, \ x \in \mathsf{R}\} .$$

Any affine function $a \in A$ can be thought of as cutting away from C the set

$$\{(x,\mu) : 0 \leqq \mu \leqq a(x), \ x \in \mathsf{R}^2\}$$

After having cut C by means of the affine functions $a_1,\ldots,a_n$ we are left with the epigraph of $f = 0 \vee a_1 \vee \ldots \vee a_n$

$$C_n = \{(x,\mu) : \mu \geqq f(x), \ x \in \mathsf{R}^2\} .$$

If each function a_m, $m = 1,2,\ldots,n$ is chosen such that it does not cut through an extreme point of C_{m-1} then f will be extreme, since then the vertices of the covering induced by f will have order 3.

The result that the functions thus constructed can approximate any continuous convex function f uniformly on a compact set is now rather obvious since the epigraph of f can be cut out of C by continuing the above procedure, each time avoing the vertices already created, see Theorem 2.

The actual proofs for convex sets in R^2 are more complicated since we need extra conditions to ensure that there are enough vertices inside K, see the lemma.

2. A combinatorial lemma.

LEMMA. *Let C be an open convex set of dimension 2. Let $P_1, \ldots, P_m$ be a covering of C with convex closed polyhedral sets of dimension 2 such that*

1) *Each polyhedral set has a vertex in C.*
2) *Any two vertices in C can be connected by a path of edges in C.*
3) *No vertex is in the relative interior of an edge.*
4) *Each vertex is of order 3.*

Let finally f be a continuous function which is affine on each P_i, $i = 1, \ldots, m$. Then if $f \equiv 0$ on two polyhedral sets with a common edge then $f \equiv 0$ on C.

PROOF. Let P_1 and P_2 have a common edge $L = P_1 \cap P_2$ and let $f = 0$ on $P_1 \cup P_2$. We want to prove that there is a third polyhedral set P_3 which has an edge L_1 in common with P_1 and an edge L_2 in common with P_2, then since f is affine on P_3 and 0 on L_1 and L_2, the condition 3) will ensure that $f = 0$ on P_3.

If $L \cap C$ contains no vertex of P_1 (or P_2) then L would bisect C, such that P_1 and P_2 would be on different sides of L. Now P_1 and P_2 each have a vertex in C by condition 1), and by 2) they can be connected by a path of edges inside C. This path must meet $L \cap C$ and hence $L \cap C$ does contain a vertex. Let therefore V_1 be a vertex in $L \cap C$. Since V_1 is of order 3 there is a third polyhedral set P_3 which meets P_1 and P_2 at V_1. This set clearly has a common edge with P_1 and with P_2 and by the above argument $f = 0$ on P_3.

This was the start of the induction and we can now prove that the conditions 1) through 4) are sufficient that the presented argument spreads to all of C.

Let us assume that $f = 0$ on

$$C_k = P_1 \cup \ldots \cup P_k, \quad (k \leq m),$$

which is connected and contains a vertex in the interior, namely V_1.

If $C \setminus C_k = \emptyset$ we have proved that $f = 0$ on C. Otherwise let $x \in C \setminus C_k$, and let P be a polyhedral set which contains x. Let V_2 be a vertex in $C \cap P$.

Now consider the connected path that leads from V_1 to V_2 inside C. Let V_3 be the first vertex on this path which lies on the boundary of C_k.

Notice that since V_3 is reached from inside C_k, the three edges that meet at V_3 all lie in C_k, but since V_3 is on the boundary of C_k and in C, there must be a polyhedral set P_{k+1} say, which is outside C_k and which meets the boundary of C_k at V_3 and at two edges. Hence since f is affine on P_{k+1} and 0 on C_k we find that f is 0 on P_{k+1} as well. This completes the induction and the proof of the lemma.

Let us remark, that in case $C = \mathsf{R}^2$ there are two cases possible. Either there are no vertices at all in which case the covering $(P_1,\ldots,P_m)$ consists of parallel strips covering the plane, or there is at least one vertex, in which case 1) and 2) are automatically satisfied.

Notice also that if the covering is induced by a convex function then 3) is automatically satisfied and hence we see that the most important condition is the fourth condition that each vertex should have order 3.

3. The extremal functions.

We can not identify all the extremal convex functions, but we can find so many that there are enough to prove the main result that they are dense in $\boldsymbol{K}$.

THEOREM 1. *A polyhedral convex function f is extremal in $\boldsymbol{K}$ if the covering of K given by f satisfies the conditions* 1), 2) *and* 4) *for* $C = \operatorname{int} K$.

PROOF. Let g and h be elements of $\boldsymbol{K}$ and let

$$f = \tfrac{1}{2}(g+h)\,.$$

Let $P_1,\ldots,P_m$ denote the covering induced by f which satisfies the conditions 1), 2) and 4). The condition 3) will then automatically be satisfied.

Clearly g and h must be affine on each of the polyhedral sets $P_1,\ldots,P_m$ and we shall assume that g and h are polyhedral functions.

Now let a, b and c denote affine functions, such that the functions

$$f_1 = f-a, \quad g_1 = g-b, \quad h_1 = h-c$$

all vanish on P_1. Then

$$f_1 = \tfrac{1}{2}(g_1+h_1)\,.$$

Let then $x_0 \in \operatorname{int} P_2 \cap \operatorname{int} K$, where P_2 has an edge in common with P_1.

If $g_1(x_0) = 0$ then g_1 is 0 on P_2 and P_1, and by the lemma $g_1 \equiv 0$ on K which proves that g is affine and hence that f is extremal.

Let us therefore assume that $g_1(x_0) > 0$, and by a similar argument that $h_1(x_0) > 0$.

Let us then define

$$f_2 = f_1/f_1(x_0), \quad g_2 = g_1/g_1(x_0), \quad h_2 = h_1/h_1(x_0) .$$

Then

$$f_2 = \alpha g_2 + (1-\alpha)h_2$$

where

$$0 < \alpha = g_1(x_0)/2f_1(x_0) < 1 .$$

Now we have that

$$f_2 = g_2 = h_2 = 0 \quad \text{on } P_1$$

and

$$f_2(x_0) = f_2(x_0) = h_2(x_0) = 1 ,$$

but then

$$f_2 = g_2 = h_2 \quad \text{on } P_2 .$$

If we apply the lemma to the piecewise affine function $f_2 - g_2$, we get that

$$f_2 = g_2 = h_2 \quad \text{on } K$$

which implies that f is extremal. This completes the proof of Theorem 1.

PROPOSITION. *If a and b are affine functions then a and $a \vee b$ are extremal functions. If c is an affine function such that the equations*

$$a(x) = b(x) = c(x)$$

have only one solution in $\operatorname{int} K$ *then $a \vee b \vee c$ is extremal.*

PROOF. It is easily seen that a is extremal, and that the construction in the proof of Theorem 1 will give that $a \vee b$ is extremal. The above condition on c ensures the existence of a vertex in $\operatorname{int} K$ and the covering induced by $a \vee b \vee c$ satisfies the conditions 1), 2), and 4) of the lemma.

COROLLARY. *In the convex cone of finite continuous convex functions defined on* R^2, *the polyhedral functions which induce coverings with vertices of order* 3 *are extremal.*

In particular the functions a, $a \vee b$ and $a \vee b \vee c$ are extremal if the equations

$$a(x) = b(x) = c(x)$$

have only one solution.

PROOF. This follows from the remarks after the proof of the lemma, together with Theorem 1 and the proposition.

We shall now prove the main result.

THEOREM 2. *Any finite continuous convex function on the convex set K can be approximated uniformly on any convex compact subset of K by an extremal convex function.*

PROOF. Let $f \in \boldsymbol{K}$ be given as a finite continuous convex function on K. Let $K_1 \subset K$ be a compact convex set of dimension 2.

We shall prove that f can be approximated by modifying the function on K_1 a finite number of times in such a way that the final modification is an extremal function in $\boldsymbol{K}$, which differs less than ε from f on K_1.

1) The first modification is to approximate f by a polyhedral function

$$f_1 = \sup_{1 \leq i \leq n} a_i$$

as follows: For each $x \in K_1$ we find a subgradient a_x and determine a neighbourhood N_x of x, such that $f(y) < a_x(y) + \varepsilon/4$, $y \in N_x$.

By compactness we can pick out a finite number of neighbourhoods which cover K_1 and the corresponding subgradients provide us with the function f_1.

2) The next step consists in modifying f_1 such that the polyhedral covering induced by f_1 satisfies condition 1) of the lemma. Let therefore P be such a polyhedral set where f_1 is affine and such that

$$\operatorname{int} P \cap \operatorname{int} K_1 \neq \emptyset .$$

The set P need not have any vertices in $\operatorname{int} K_1$ but let us choose

$$x_1 \in \operatorname{int} P \cap \operatorname{int} K_1 .$$

Now choose three affine functions a, b and c, such that the equations

$$a(x) = b(x) = c(x)$$

only have the solution $x = x_1$. The function

$$f_1 + \delta(a \vee b \vee c)$$

will be a convex polyhedral function with a vertex at x_1. This function clearly induces a covering of K_1 with more polyhedral sets than before, but each new polyhedral set will have a vertex in $\operatorname{int} K_1$.

We then repeat this construction for each polyhedral set from f_1 which does not possess a vertex in $\operatorname{int} K_1$. The final modification f_2 will consist of f_1 plus a sum of simple extremal convex functions, and will have an induced covering satisfying the first condition of the lemma. For δ sufficiently small, $|f_2 - f_1| < \varepsilon/4$ on K_1.

3) The vertices of f_2 need not be connected but let x_1 and x_2 be any two vertices in $\operatorname{int} K_1$. Let a be an affine function such that

$$a(x_1) = a(x_2) = 0$$

and such that $a \not\equiv 0$.

The function

$$f_2 + \delta(0 \vee a)$$

is a convex polyhedral function whose covering of K_1 will contain some new polyhedral sets. Each new set, however, will have a vertex on the line determined by x_1 and x_2 inside $\operatorname{int} K_1$, and x_1 and x_2 can be connected by a path of edges in $\operatorname{int} K_1$, also lying on the line $[x_1, x_2]$. This procedure is repeated by replacing x_2 by any of the vertices from f_2, each time adding a simple extremal function. We end up with a function f_3 whose covering satisfies conditions 1) and 2) of the lemma. For δ sufficiently small we get $|f_2 - f_3| < \varepsilon/4$ on K_1.

4) The final step consists in ensuring that all vertices have order 3.

Assume V to be a vertex of order $s > 3$ and let a be a subgradient such that

$$f_3(x) > a(x), \quad x \neq V$$

$$f_3(x) = a(x), \quad x = V\,.$$

Then

$$f_3 \vee (a + \delta)$$

is a convex polyhedral function with the property that for δ sufficiently small the corresponding polyhedral covering will be changed only around V in such a way that V will be surrounded by a small polytope with s vertices each of order 3. Clearly the new polyhedral sets constructed this way still have vertices in $\operatorname{int} K_1$ and these can still be connected inside $\operatorname{int} K_1$.

This construction is repeated for each vertex of order > 3 each time taking the maximum of the function so far obtained and a suitable affine function. The final function will induce a covering with all the desired properties listed in the lemma.

For δ sufficiently small this function f_4 will lie within $\varepsilon/4$ of f_3 on K_1.

Thus we can apply Theorem 1 and we get that f_4 is an extremal function which differs less than ε from f on K_1 as was to be proved.

It is curious to notice that a piece of chalk exhibits the shape of an extremal convex function when it has been used on the blackboard for some time. Thus the answer was right at hand from the very beginning.

REFERENCES

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