MATH. SCAND. 34 (1974), 84–90

HOLOMORPHICALLY CLOSED ALGEBRAS OF ANALYTIC FUNCTIONS

ROBERT GEORGE BLUMENTHAL

Synopsis.

Closed subalgebras of the disc algebra which are generated by functions with smooth boundary values are considered. Each such algebra A is shown to be holomorphically closed, i.e. to contain each function which is locally A-approximable on the disc. This result is used to obtain a generalization of an approximation theorem of John Wermer.

1.

Let $\mathscr{A}$ denote the disc algebra, i.e. the algebra of all continuous functions defined on the closed unit disc D in C which are analytic in the interior D° of D. A function $f \in \mathscr{A}$ is *smooth* if its restriction to the unit circle T is continuously differentiable. Our object is to prove the following generalization of a result of Wermer [7, Theorem 1.2] and [8, Lemma 3.2].

THEOREM 1. *Suppose that*

(1) *A is a closed, point-separating subalgebra of $\mathscr{A}$ such that $1 \in A$ which contains a dense subalgebra of smooth functions.*

(2) *For each $z \in \mathsf{T}$ there exists a smooth function in A whose derivative at z is non-zero.*

Then there exists a polynomial $g(z)=\prod_{i=1}^{n}(z-z_i)^{N_i}$, where $z_i \in D^\circ$, $i=1,\ldots,n$, such that A contains the ideal $g\mathscr{A}$ of $\mathscr{A}$.

Wermer's result yields the same conclusion but requires the stronger assumption that the algebra in question contains a dense subalgebra generated by finitely many functions, each of which is analytic on a neighborhood of D. A related but more general theorem was obtained by Bishop [1, Corollary 1] who considered the restrictions of algebras of analytic functions defined on Riemann surfaces to compact subsets.

Received March 31, 1973.

Theorem 1 is actually a consequence of a result (Theorem 2) for algebras A which satisfy only the first of the above two conditions. A function $f \in C(D)$ is called *locally A-approximable* on D [6, II. 14.8] (or *A-holomorphic* on D [3, III.9]) if there exists an open covering $\{U_\gamma\}$ of D such that for each γ $f | U_\gamma \in (A | U_\gamma)^-$, where $(A | U_\gamma)^-$ denotes the uniform closure on U_γ of the restriction algebra $A | U_\gamma$. A is *holomorphically closed* on D if it contains each function which is locally A-approximable on D.

THEOREM 2. *If A satisfies* (1), *then A is holomorphically closed on D.*

Our proofs are strongly inspired by Bishop's, but make free use of results from the theory of uniform algebras (see e.g. [3], [6]). An essential ingredient of the proof of Theorem 2 is the following recent result of Björk [2, Theorem 2.1].

BJÖRK'S THEOREM. *If A satisfies* (1), *then the maximal ideal space of A is equal to D.*

Björk's Theorem and a basic result [3, II.6.1], [6, I.9.11] imply that each subset K of D such that

$$K = \{z \in D : \ |f(z)| \leqq \|f\|_K \text{ for each } f \in A\}$$

is the maximal ideal space of the restriction algebra $(A | K)^-$; such subsets K are called *A-convex.*

A theorem related to Theorem 2 is mentioned in a recent article by Gamelin [4]. B. S. Lund [5] has recently proved that algebras which satisfy the conclusion of Theorem 1 are generated by a finite number of polynomials.

2.

We assume in this section that A satisfies (1). We begin by stating two lemmas which are implicit in Bishop's argument (see [1, the proof of Lemma 5]).

LEMMA 1. *Let K be a compact subset of $\mathbf{C}$, let $z \in K$ lie in the closure of the unbounded component of $\mathbf{C} \setminus K$ and let V be a neighborhood of z in K. Then there exists a polynomial which is one-to-one on a neighborhood of K and which assumes its maximum modulus on K at a unique point $z' \in V$.*

LEMMA 2. *Let K be a compact subset of D. Suppose there exists a smooth function $h \in A$ and a point $q \in K$ such that h assumes its maximum modulus on K only at q, $h'(q) \neq 0$, and $h^{-1}(h(q)) \cap K$ is finite. Then*

(3) *there exists a smooth function $f \in A$ and a point $p \in K$ such that $f'(p) \neq 0$ and p is the unique point of K at which f assumes its maximum modulus.*

The next lemma was proved by Björk for the case when $K = \mathsf{T}$ [2, Lemma 2.3].

LEMMA 3. *If K is a compact subset of D which is not totally disconnected, then* (3) *is satisfied.*

PROOF. Let g be a smooth, non-constant function in A. Set

$$G = \{z \in K : g'(z) = 0\} \quad \text{and} \quad L = g^{-1}(g(G)) \cap K .$$

Notice first that if $p_n \to p_0$ and $g(p_n) = g(p_0)$ where $p_n \in K$ and $p_n \neq p_0$ for $n = 1, 2, \ldots$, then $g'(p_0) = 0$. Hence

(4) $g^{-1}(g(p)) \cap K$ is finite for each $p \in K \setminus L$,

since if $g^{-1}(g(p)) \cap K$ were infinite, $g(p)$ would lie in $g(G)$.

Next, let Ω denote the closure of the unbounded component of $\mathsf{C} \setminus g(K)$ and let

$$K_0 = \{z \in K : g(z) \in \Omega\} .$$

Assume it is known that

(5) $K_0 \setminus L \neq \emptyset$,

and let $w \in K_0 \setminus L$. We apply Lemma 1, taking $g(K)$, $g(w)$ and $g(K) \setminus g(G)$ as our set, point and neighborhood respectively. Let $P(z)$ denote the polynomial thus obtained. Then, with the aid of (4), we see that $h \equiv P \circ g$, K and some point $q \in K_0 \setminus L$ satisfy the hypotheses of Lemma 2, the application of which yields the conclusion to the lemma.

Thus, to complete the proof it suffices to show

(6) L is totally disconnected.

(7) K_0 is not totally disconnected.

Since g is smooth, $g' \in \mathscr{A}$. Hence, G is totally disconnected. An elementary version of Sard's theorem [6, VI.30.14] now implies that $g(G)$ is totally disconnected, from which (6) follows. If (7) were false, $g(K_0)$ would be totally disconnected and thus polynomially convex. This would imply

$g(K)=g(K_0)$ [6, I.7.12], contradicting the assumption that K is not totally disconnected.

The significance of the conclusion that $f'(p)\neq 0$ in the preceding lemma is that it implies the existence of a closed neighborhood W of p in D on which the polynomials in f are dense in $(\mathscr{A}|W)^-$. This is easy to see when $p\in D^\circ$ so suppose $p\in \mathrm{T}$. Let W be a closed neighborhood of p in D on which f is one-to-one which is bounded by a closed Jordan curve J. Then $f(J)$ is a closed Jordan curve whose interior is $f(W)\setminus f(J)$, and $f^{-1}|f(W)$ is uniformly approximable by polynomials. Hence, the uniform closure on W of polynomials in f contains the identity function. The assertion follows.

Lemma 4. *Let K be a closed subset of D which is not totally disconnected. Then there exists a function $f\in A$, a point $p\in K$ and open subsets U and W of D with $p\in W$ and $W\cup K\subseteq U$ such that*

(8) *f achieves its maximum modulus over K only at p.*
(9) *If $z\in U$, $z'\in W$ and $z\neq z'$, then $f(z)\neq f(z')$.*

(In particular, f is one-to-one on W.)

(10) *The polynomials in f are dense in $(\mathscr{A}|W)^-$.*

Proof. Lemma 3 and the preceding remarks imply the existence of a function $f\in A$, a point $p\in K$ and an open neighborhood W_0 of p in D which satisfy (8) and (10). Since f achieves its maximum modulus on K only at p, for each $z\in K\setminus W_0$ there exist open neighborhoods U_z of z and W_z of p such that $f(U_z)\cap f(W_z)=\varnothing$. Finitely many of the U_z cover $K\setminus W_0$, say U_{z_i}, $i=1,\ldots,n$. Now, take

$$U = \bigcup_{i=1}^n U_{z_i}\cup W_0 \quad \text{and} \quad W = \bigcap_{i=1}^n W_{z_i}\cap W_0\,.$$

3.

$C(X)^*$, the dual space of $C(X)$ where X is a compact Hausdorff space, is identified with the space of all regular, Borel measures on X. If S is a subspace of $C(X)$, $S^\perp$ is the space of all $\mu\in C(X)^*$ such that $\int f d\mu=0$ for each $f\in S$. If E is a Borel subset of X, the restriction of μ to E, denoted μ_E, is the measure defined by $\mu_E(S)=\mu(E\cap S)$. $\operatorname{Supp}\mu$ denotes the support of μ, the smallest closed subset K of X such that $\mu=\mu_K$.

If K_1 and K_2 are closed subsets of D, $K_2\subseteq K_1$, we define

$$\varrho(K_1,K_2) = \sup\{\inf\{|p-q| : q\in K_2\} : p\in K_1\}\,.$$

PROOF OF THEOREM 2. Let g be a locally A-approximable function on D, and let $\mu \in A^{\perp}$. We want to show $\int g d\mu = 0$. Let $\mathscr{S}$ denote the set of all A-convex subsets K of D which support a measure $\nu\langle K\rangle \in A^{\perp}$ such that

$$\int g d\mu = \int g d\nu\langle K\rangle .$$

We define a sequence of sets $\{K_n\}$ from $\mathscr{S}$ by taking $K_1 = D$ and choosing $K_{n+1} \in \mathscr{S}$ such that $K_{n+1} \subseteq K_n$ and

$$\varrho(K_n, K_{n+1}) \geqq \tfrac{1}{2} \sup\{\varrho(K_n, K) : K \subseteq K_n, K \in \mathscr{S}\} .$$

Clearly, $\varrho(K_n, K_{n+1}) \to 0$ as $n \to \infty$. Set $K_0 = \bigcap_{n=1}^{\infty} K_n$.

We will prove below that K_0 is totally disconnected but first we will show that this implies that $\int g d\mu = 0$. We can find finitely many, pairwise disjoint open subsets of D, U_i, $i = 1, \ldots, n$ which cover K_0 such that $g \mid U_i \in (A \mid U_i)^-$ for each i. Choose m so large that $K_m \subseteq \bigcup_{i=1}^{n} U_i$. If we define $J_i = K_m \cap U_i$ and let ν_i denote the restriction of $\nu\langle K_m\rangle$ to J_i, then it is easy to show, using the Shilov idempotent theorem [3], [6], that $\nu_i \in A^{\perp}$. It follows that $\int g d\nu_i = 0$, $i = 1, \ldots, n$, and hence $\int g d\mu = 0$.

Suppose now that K_0 is not totally disconnected. We apply Lemma 4 to $K = K_0$ and obtain $f \in A$, $p \in K_0$ and open subsets U and W of D with $p \in W$ and $W \cup K \subseteq U$ which satisfy (8), (9) and (10). We may assume $f(p) > 0$. Choose $b < f(p)$ so that

$$\{z \in K_0 : \operatorname{Re} f(z) \geqq b\} \subseteq W .$$

Let

$$P_n = \{z \in K_n : \operatorname{Re} f(z) \geqq b\} \quad \text{and} \quad Q_n = \{z \in K_n : \operatorname{Re} f(z) \leqq b\}$$

for $n = 0, 1, 2, \ldots$. Since $\bigcap_{n=1}^{\infty} P_n = P_0$, we may choose a positive integer N for which $K_N \subseteq U$, $P_N \subseteq W$ and

$$2\varrho(K_N, K_{N+1}) < \inf\{|p - z| : z \in Q_N\} .$$

The inequality implies $\varrho(K_N, Q_N) > 2\varrho(K_N, K_{N+1})$. It suffices now to prove that $Q_N \in \mathscr{S}$, for then this last inequality will yield a contradiction with the choice of K_{N+1}.

Let $\nu = \nu\langle K_N\rangle$. Since $\operatorname{supp} \nu \subseteq K_N$, we have $\nu \in [(A \mid K_N)^-]^{\perp}$. Set $F = f(K_N)$ and define a measure τ on F by setting $\tau(S) = \nu(f^{-1}(S))$. Then $\tau \in R(F)^{\perp}$, where $R(F)$ denotes the closure on F of the rational functions with poles in the complement of F. For if $r \in R(F)$, then $r \circ f \in (A \mid K_N)^-$, and hence

$$\int r d\tau = \int r \circ f d\nu = 0 .$$

By (9), $f(K_N \cap W) = F \setminus f(K_N \setminus W)$ and $f(K_N \setminus P_N) = F \setminus f(P_N)$. Thus, if

$$G_1 = f(K_N \cap W) \quad \text{and} \quad G_2 = f(K_N \setminus P_N) ,$$

then $\{G_1, G_2\}$ is a covering of F by open sets. By the Bishop splitting lemma [3, II.10.2] there exist $\tau_i \in R(F)^{\perp}$, $i=1,2$, such that $\operatorname{supp}\tau_i \subseteq G_i$, $i=1,2$, and $\tau=\tau_1+\tau_2$.

By means of (9) we can define a measure ν' on $K_N \cap W$ by setting $\nu'(S)=\tau_1(f(S))$. If $S \subseteq P_N$, then $S=f^{-1}(f(S))$ and $f(S) \subseteq W_1 \setminus W_2$; hence

$$\nu(S) = \tau(f(S)) = \tau_1(f(S)) = \nu'(S) .$$

Thus

$$(11) \quad \nu(S) = \nu'(S) \quad \text{if } S \subseteq P_N .$$

Moreover, $\nu' \in A^{\perp}$; this follows from (10) and the observation that if h is a polynomial, then

$$\int h(f) d\nu' = \int h d\tau_1 = 0 .$$

We now define $\nu''=\nu-\nu'$. Then $\nu'' \in A^{\perp}$,

$$\int g d\nu'' = \int g d\nu$$

and, by (11), $\operatorname{supp}\nu'' \subseteq Q_N$. Since Q_N is clearly A-convex, the proof that $Q_N \in \mathscr{S}$ is complete.

Björk has observed that Theorem 2 can be stated in a slightly sharper way as a consequence of the following fact: If A satisfies (1), the proof in [2] of Björk's theorem also shows that every measure on T in $A^{\perp}$ is absolutely continuous. Say that f is weakly locally A-approximable if

(i) to each $p \in \mathrm{T}$ there is an open neighborhood U of p in D and a sequence $\{a_n\}$ from A such that for each $z \in U \cap D^{\circ}$,

$$\lim a_n(z) = f(z) ,$$

while $\|a_n\|_U \leqq C$ for some constant C depending on f and U,

(ii) f is locally A-approximable in D°.

Then, if A satisfies (1) and f is weakly locally A-approximable, $f \in A$.

Proof of Theorem 1. Theorem 1 is an immediate consequence of Theorem 2, the remarks following Lemma 3, and the following lemma which is implicit in Bishop's argument (see [1, the proof of Lemma 2]).

Lemma 5. *Let B be a closed, point-separating subalgebra of $\mathscr{A}$ containing the constants. Let $p \in D^{\circ}$, and suppose $f'(p)=0$ for each $f \in B$. Then there exists a closed neighborhood V of p in D° and a positive integer N such that $(B \mid V)^{-} \supseteq (z-p)^N \mathscr{A} \mid V$.*

BIBLIOGRAPHY

1. E. Bishop, *Subalgebras of functions on a Riemann surface*, Pacific J. Math. 8(1959), 29–50.
2. J.-E. Björk, *Holomorphic convexity and analytic structures in Banach algebras*, Ark. Mat. 9 (1971), 39–54.
3. T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewoodcliff, 1969.
4. T. W. Gamelin, *Polynomial approximation on thin sets, in Symposium on several Complex Variables*, Park City, Utah, 1970 (Lecture Notes in Mathematics 184), Springer-Verlag Berlin, Heidelberg, New York, 1971.
5. B. S. Lund, *Algebras of analytic functions on the unit disk*, (preprint) University of New Brunswick, Fredericton, N. B. Canada.
6. E. L. Stout, *The Theory of Uniform Algebras*, Bogden and Quigley, Tarrytown-on-Hudson, New York, 1971.
7. J. Wermer, *Rings of analytic functions*, Ann. of Math. 67 (1958), 497–516.
8. J. Wermer, *The hull of a curve in* $\mathbf{C}^n$, Ann. of Math. 68 (1958), 550–561.

WELLESLEY COLLEGE, WELLESLEY, MASSACHUSETTS, U.S.A.