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ALGEBRAS OF CONTINUOUS FUNCTIONS INVARIANT UNDER THE BACKWARD SHIFT

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Introduction.

Let D be a domain in the complex plane which contains the origin and let f be a function analytic on D with power series expansion

$$f(z) = \sum_0^\infty a_n z^n, \qquad |z| < \delta$$

near 0. The backward shift is the operator T which sends f to the analytic function

$$Tf(z) = \sum_1^\infty a_n z^{n-1}, \qquad |z| < \delta$$

near 0 and

$$Tf(z) = (f(z)-a_0)z^{-1}, \qquad z \in D$$

in general. If f is bounded or is continuous on the closure of D, then it is clear that Tf will again have the same property.

In Section 1 we prove that if K is a compact set whose interior is connected and dense in K and if A is a closed subalgebra of $A(K)$ which contains the constants and which is invariant under the backward shift then A lies between $R(X)$ and $A(X)$ where X is a compact set containing K and $\partial X \subseteq \partial K$ or else every function in A extends to be continuous on the sphere and analytic off ∂K. The definitions of $R(K)$ and $A(K)$ and the precise statement of the theorem are in Section 1.

In Section 2 we discuss an extension of the backward shift operator to $C(\Gamma)$, where Γ is a simple closed curve; we consider a closed subalgebra A of $C(\Gamma)$ which contains 1 and which contains $f(x)x^{-1}$ whenever $f \in A$ and $\int f d\mu = 0$ where μ is some finite regular Borel measure on Γ. Theorem 2 shows that such an algebra, if it is not all of $C(\Gamma)$, must consist entirely of analytic functions.

This paper is a continuation of investigations in [2].

1. Invariant subalgebras of $A(K)$.

If K is a compact set in the plane, $R(K)$ denotes the uniform closure on K of the rational functions with poles off K and $A(K)$ the algebra

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of functions which are continuous on K and analytic on the interior of K; clearly, $R(K) \subseteq A(K)$. We shall always assume that the origin is an interior point of K.

Let D be the interior of K which we assume to be connected and dense in K and suppose A is a closed subalgebra of $A(K)$ which contains 1 and is invariant under the backward shift. If $f \in A$, we shall show that $(f(z)-f(a))(z-a)^{-1}$ is in A for all $a \in D$. To see this let λ be a measure on ∂K, the boundary of K, with $\lambda \perp A$. Then

$$F(a) = \int_{\partial K} (f(z)-f(a))(z-a)^{-1} d\lambda(z)$$

is an analytic function of a for $a \in D$ and

$$F^{(n)}(0) = \int \left(f(z) - \sum_0^n a_j z^j\right) z^{-n-1} d\lambda(z)$$

where $f(z) = \sum_0^\infty a_j z^j$ is the power series expansion of f near 0. Since the integrand is just the backward shift applied to f and iterated n times, $F^{(n)}(0) = 0$ for all n and hence $F \equiv 0$. Thus, $(f(z)-f(a))(z-a)^{-1}$ is in A.

Now let

$$I = \{f(z)/z : f \in A \text{ and } f(0) = 0\} .$$

We define a projection π from the maximal ideal space M of A into the sphere S^2 by $\pi(m) = \infty$ if $I \subseteq m$ and

$$\pi(m) = \left(\int f d\mu\right)\left(\int f(z) z^{-1} d\mu(z)\right)^{-1} \tag{1.1}$$

if $f(0) = 0$ but $f(z)z^{-1} \notin m$, and μ is a representing measure on ∂K for m. If $g \in A$, $g(0) = 0$, but $g \notin m$, then

$$0 \neq \int g d\mu \int f(z) z^{-1} d\mu = \int g(z) f(z) z^{-1} d\mu = \int f d\mu \int g(z) z^{-1} d\mu .$$

Hence, if $g(0) = 0$ and $g \notin m$, then $g(z)z^{-1} \notin m$ and so in (1.1) we may use any f with $f(0) = 0$ and $f \notin m$. It is a simple matter to check that π is continuous from M with the weak-* topology into S^2. Further, if $c = \pi(m)$, then replacing f by

$$(f(z)-f(w))(z-w)^{-1} z, \quad w \in D ,$$

in (1.1) and doing a bit of manipulation we find that

$$(c-w) \int (f(z)-f(w))(z-w)^{-1} d\mu(z) = \hat{f}(m) - f(w), \quad f \in A, w \in D . \tag{1.2}$$

This immediately implies that if $\pi(m) = c \in D$, then $m(f) = f(c)$ and hence π is $1-1$ over D.

THEOREM 1. *Let K be a compact set whose interior D is connected and dense in K. Let A be a closed subalgebra of $A(K)$ which contains* 1 *and* *which is invariant under the backward shift. Then either there is a compact set X on the sphere with $X \supseteq K$ and $\partial X \subseteq \partial K$ such that $R(X) \subseteq A \subseteq A(X)$ or else each function in A is continuous on S^2 and analytic off ∂K.*

Proof. Let $A_0 = \{f \in A : f(0) = 0\}$. Let M be the maximal ideal space of A and let $X = \pi(M)$. If $c \notin X$, then

$$I_c = \{(z-c)f(z)z^{-1} : f \in A_0\}$$

is a closed ideal in A which lies in no maximal ideal, for if $I_c \subseteq m$, then let μ represent m. For every $f \in A_0$ we have

$$0 = \int (z-c)f(z)z^{-1}d\mu(z)$$

or

$$c \int f(z)z^{-1}d\mu = \int f d\mu$$

which implies $\pi(m) = c$, contradicting the fact that $c \notin X$. Hence, 1 lies in I_c; equivalenty, $(z-c)^{-1} \in A$ which shows that $R(X) \subseteq A$.

Since π is one-to-one over D, D is an open subset of M; let $Y = M \setminus D$. If $m \in Y$ and $f \in A_0$ with $\hat{f}(m) \neq 0$, then $1/f$ can be approximated near m by polynomials in $f - m(f)$. Thus, $1/\pi$ can be uniformly approximated near m by elements of A. Let B be the closed algebra on Y generated by A and $1/\pi$. By a theorem of C. E. Rickart (see [3; p. 60]) the Silov boundary of B is just the Silov boundery of A; that is, the Silov boundary of B lies in ∂K. Hence, B can be identified with the algebra on ∂K generated by A and $1/z$. This algebra is $R(Z)$ for some compact subset Z of $S^2 \setminus D$ with $\partial Z \subseteq \partial K$.

Now if $\Psi \in M(B)$, then

$$\pi(\Psi) = \Psi(f)\Psi(f(z)/z)^{-1}$$

for $f \in A_0$ and hence $\pi(\Psi) = \Psi(1/z)^{-1}$, because Ψ is multiplicative on B. If $c \notin Z \cup D$, then as before $(z-c)^{-1} \in B$ so that $z/(z-c)$ is invertible in B. If $\pi(\Psi) = c$, then

$$\Psi(z - c/z) = 1 - c\Psi(1/z) = 0$$

which implies that

$$1 = \Psi(1) = \Psi((z-c/z)(z/z-c)) = 0 \cdot \Psi(z/z-c)$$

a contradiction. Hence, $\pi(Y) \subseteq Z$, so that $A \subseteq A(Z) \cup A(D) \subseteq A(X)$. Clearly, $X = Z \cup D$ and the proof is complete.

2. Invariant subalgebras of $C(\Gamma)$.

In this section we extend the backward shift operator to $C(\Gamma)$; specifically, we consider a simple closed curve Γ and a closed subalgebra

A of $C(\Gamma)$ which contains 1 and which contains $f(x)x^{-1}$ whenever $f \in A$ and $\int_\Gamma f d\mu = 0$ where μ is a (fixed) finite regular Borel measure. Theorem 2 provides a characterization of such algebras but it is illustrative to give some examples before beginning the proof. We denote by $A(U)$ those analytic functions on the domain U which are continuous on the closure of U.

EXAMPLE 1. Let D be the bounded component of the complement of Γ and suppose that the origin is in D. Let $\mu = \omega_0$, harmonic measure on Γ for 0, and let A be $A(D)$; or take A to be a closed subalgebra in $A(\Omega)$ containing 1, where Ω is the unbounded component of the complement of Γ, and let μ be multiplicative on A. Note that in the first case the maximal ideal space of A is $D \cup \Gamma$, while in the second it is $\Omega \cup \Gamma$.

EXAMPLE 2. Let α be an arc with positive continuous analytic capacity; that is, the set $A(S^2 - \alpha)$ of all functions which are continuous on S^2 and analytic off α contains non-constant functions. This will be true, for example, if α has positive 2-dimensional Lebesgue measure. Let Γ be a simple closed curve containing α and define A to be all those continuous functions f on Γ with $f|\alpha \in A(S^2 \setminus \alpha)$. Then A is not a subset of $A(D)$ or $A(\Omega)$ and $A \neq C(\Gamma)$ since

$$A(S^2 \setminus \alpha)|\alpha \neq C(\alpha) .$$

Further, note that the maximal ideal space of A is $S^2 \cup \{\Gamma \setminus \alpha\}$ and each point $c \in \Gamma \setminus \alpha$ has two homomorphisms over it: one is evaluation of f at c and the other is evaluation at c of the analytic extension to $S^2 \setminus \alpha$ of the restriction of f to α. If the origin is not in α, then there is a measure μ on K such that $\int_\Gamma f d\mu = f(0)$ for all $f \in A$ and if $\int f d\mu = 0$, then clearly $f(z)/z$ is again in A.

A slight variant of this example is to take A to be the restriction to Γ of $A(S^2 \setminus \alpha)$; in this case the maximal ideal space of A is the sphere.

EXAMPLE 3. Let S be a compact (connected) Riemann surface on which there is a meromorphic function F which assumes every value on the sphere exactly twice. For example, S can be a torus.

Let a and b be distinct points of S with $F(a) = F(b) = 0$. Then F is 1–1 in some neighborhood of a and hence there is an $r > 0$ and a neighborhood Δ of a such that F is a homeomorphism of Δ onto $\{|z| < r\}$ and F is a homeomorphism of $\Gamma = \partial\Delta$ onto $\{|z| = r\}$. Let

$$R = S \setminus (\Delta \cup \Gamma)$$

and let B consist of all functions which are continuous on $R \cup \Gamma$ and holomorphic on R. The restriction of B to Γ is an isometry of B onto a closed subalgebra of $C(\Gamma)$. Let ν be a measure on Γ which represents the homomorphism $f \to f(b)$ of B. If $f \in B$ and $\int_\Gamma f d\nu = 0$, then $f/F \in B$.

Let

$$A = \{g \in C(|z|=r) : g = f \circ F^{-1} \text{ where } f \in B\}$$

and let μ be the measure on $|z|=r$ defined by $\mu(E) = \nu(F^{-1}(E))$. Then A is a closed subalgebra of $C(|z|=r)$ and if $g \in A$ and $\int g d\mu = 0$, then $g(z)/z \in A$. Note that the maximal ideal space of A is $R \cup \Gamma$ and F is a continuous function on M which is 1–1 over $\{|z|<r\}$ and 2-to-1 over $\{|z|>r\}$, counting multiplicities.

This example can be carried further. Let K be a compact set in Γ of zero harmonic measure (for a point in R) and let $P = \{z_i\}$ be a discrete set in F^{-1} $(|z|>r)$ whose set of limit points in $R \cup \Gamma$ lies in K. For each $z \in K \cup P$ there is a $\tilde{z}$ in $R \cup \Gamma$ such that $F(z) = F(\tilde{z})$; let

$$B_1 = \{f \in B : f(z) = f(\tilde{z}) \text{ for } z \in K \cup P\}$$

and define A_1 on $\{|z|=r\}$ analogously to A above. Then A_1 is a uniformly closed subalgebra of $C(|z|=r)$ and if $\int f d\nu = 0$, then $f(z)/z$ is in A_1. The maximal ideal space of A_1 is obtained from $R \cup \Gamma$ by identifying the points z and $\tilde{z}$ for $z \in K \cup P$. Note that F is 1-to-1 over $D \cup K \cup \{F(P)\}$ and 2-to-1 over the rest of the sphere, again counting multiplicities.

THEOREM 2. *Let Γ be a simple closed curve with complementary components D and Ω, $0 \in D$ and $\infty \in \Omega$, and let μ be a finite regular Borel measure on Γ. Let A be a closed subalgebra of $C(\Gamma)$ which contains* 1 *and which contains $x^{-1}f(x)$ whenever $f \in A$ and $\int_\Gamma f d\mu = 0$. Then one of the following holds*:

(i) $A = C(\Gamma)$,

(ii) $A \subseteq A(D)$,

(iii) $A \subseteq A(\Omega)$,

(iv) *there is a compact set K in Γ such that $A|K \subseteq A(S^2 \setminus K)$*,

(v) *if M is the maximal ideal space of A, then $M \setminus \Gamma$ is a connected one-dimensional analytic variety on which the functions in A are analytic; there is a meromorphic function π on $M \setminus \Gamma$ which is one-to-one over D and two-to-one on $\Omega \setminus P$, where P is a discrete set in Ω. Further, $\pi^{-1}(c)$ has at most two points for each $c \in \Gamma$ and the set*

$$K = \{c \in \Gamma : \pi^{-1}(c) \text{ is a singleton}\}$$

has harmonic measure zero and K contains all the limit points of P.

PROOF. Suppose cases (ii)–(v) do not hold; we show $A=C(\Gamma)$. Let

$$A_0 = \{f \in A : \textstyle\int_\Gamma f d\mu = 0\}.$$

If $1 \in A_0$, then $x^{-1} \in A$ and hence $A=A(\Omega)$ or $A=C(\Gamma)$ by Wermer's maximality theorem. Hence, there is no loss in assuming that $\int_\Gamma d\mu = 1$. Let

$$I = \{x^{-1}f(x) : f \in A_0\}.$$

If 1 lies in the closed ideal generated by I, then $x \in A$ and hence $A=A(D)$ or $A=C(\Gamma)$, again by Wermer's theorem. Hence, I lies in at least one maximal ideal m. Let $d\beta$ be any complex representing measure on Γ for m. Since $x^{-1} \notin A$ we may choose β so that $\int x^{-1}d\beta \neq 0$. Since β represents m we have

$$0 = \int_\Gamma (f(x)-a_0)x^{-1}d\beta(x)$$

for each $f \in A$ where $a_0 = \int_\Gamma f d\mu$. Hence

$$\int_\Gamma f(x)x^{-1}d\beta(x) = a_0 \int_\Gamma x^{-1}d\beta(x).$$

Let $f \in A_0$ and $g \in A$. Then $f(x)g(x)x^{-1} \in m$ and hence

$$0 = \int g(x)f(x)x^{-1}d\beta(x) = \int gf d\mu \int x^{-1}d\beta(x).$$

Thus, $\int gf d\mu = 0$ when $f \in A_0$ and $g \in A$. Hence,

$$\int fg\, d\mu = \int f d\mu \int g\, d\mu \quad (f,g \in A).$$

Hence, A_0 is an ideal in A and μ is multiplicative on A. Let $Sf(x) = (f(x)-a_0)/x$. Now for $\lambda \in D$ let

$$A_\lambda = \{(z-\lambda)z^{-1}f(z) : f \in A_0\};$$

then A_λ is an ideal in A and, if $1 \in A_\lambda$ for some λ, then $(z-\lambda)^{-1}$ lies in A and hence $A=A(\Omega)$ or $A=C(\Gamma)$ once again by Wermer's maximality theorem. Suppose L is a linear functional on A with $L(A_\lambda)=0$ and $L(1)=0$. Then for $f \in A$ we have

$$0 = L((z-\lambda)z^{-1}(f-a_0)),$$

so that

$$\lambda L(Sf) = L(f-a_0) = L(f).$$

Iterating this we find that

$$L(f) = \lambda^n L(S^n f),$$

so that when λ is so small that $|\lambda|\,\|S\| < 1$, we have $L=0$. Hence, if $|\lambda|\,\|S\| < 1$, A_λ must be a maximal ideal of A since it has codimension 1. Let m_λ be the corresponding homomorphism. Let

$$F(\lambda) = \int_\Gamma z(z-\lambda)^{-1} d\mu(z) ;$$

F is analytic in D and $F(0)=1$; thus the zeros of F are discrete. For $f \in A$ and $F(\lambda) \neq 0$ define

$$Pf(\lambda) = F(\lambda)^{-1} \int (z-\lambda)z^{-1}f(z)\,d\mu(z) .$$

Then $P1 \equiv 1$ and $Pg(\lambda)=0$ if $g \in A_\lambda$. For $|\lambda|$ sufficiently small,

$$Pf(\lambda) = m_\lambda(f)$$

and thus $P(fg)=P(f)P(g)$ holds for $|\lambda|$ small. Since all of $P(f)$, $P(g)$, and $P(fg)$ are analytic on D with the except of a discrete set, this equality holds where all three functions are defined. At a point λ where $F(\lambda) \neq 0$ we have

$$|Pf(\lambda)| \leqq C\|f\|$$

where C does not depend on f. Thus, $|Pf(\lambda)|^n \leqq C\|f\|^n$ for all n and so $|Pf| \leqq \|f\|$ where $F \neq 0$. This implies that Pf is analytic on all of D and that $P(fg)=P(f)P(g)$ on all of D, for $f,g \in A$.

Let σ be a measure on Γ which is orthogonal to A and set

$$G(z) = \int \big(f(x)-Pf(z)\big)(x-z)^{-1} d\sigma(x) .$$

Then G is holomorphic in D; for $|\lambda|$ small we have $f-Pf(\lambda) \in A_\lambda$ and hence

$$\big(f-Pf(\lambda)\big)(x-\lambda)^{-1} \in A$$

which implies that G vanishes for $|\lambda|$ small. Hence, G is identically zero and thus

$$\big(f(x)-Pf(\lambda)\big)(x-\lambda)^{-1} \in A \quad \text{for all } \lambda \in D .$$

Let π be the projection from the maximal ideal space M of A into S^2 given by (1.1). As in Section 1, π is continuous and for $\lambda \in D$ and $f \in A$ we have

$$\big(\pi(m)-\lambda\big) \int_\Gamma \big(f(x)-Pf(\lambda)\big)(x-\lambda)^{-1} d\nu(x) = \hat{f}(m)-Pf(\lambda)$$

where ν is a representing measure for m on Γ. Hence, π is one-to-one over D and if $c=\pi(m) \in D$, then m is evaluation of Pf at c. If $m \in M \setminus D$, then the proof of Theorem 1 shows that $1/\pi$ is locally approximable by A in a neighborhood of m.

Fix a number R so large that $\Gamma \subset \{|z|<R\}$ and let $Y=\pi^{-1}(|z| \leqq R)$. Let B be the uniform algebra on Y generated by A and π. Then the maximal ideal space of B is Y and the Silov boundary of B is $\Gamma \cup \pi^{-1}(|z|=R)$ since π is locally approximable by A.

It is now apparent that D is a π-regular component of multiplicity one for the uniform algebra B and hence by a theorem of J.-E. Björk [1; Theorem 1.7], $\pi^{-1}(\lambda)\setminus\Gamma$ has at most one point for each $\lambda\in\Gamma$ and, if $\pi^{-1}(\lambda)\setminus\Gamma$ is not empty then M has analytic structure at that point.

Let c_0 be any point of Γ. Let L be in the cluster set of Pf at c_0; then there is a sequence $w_n\in D$ with $w_n\to c_0$ and $Pf(w_n)\to L$. Let

$$\alpha = \{g\in A : Pg(w_n)\to 0\};$$

α is a proper ideal in A and hence lies in a maximal ideal m. Let μ represent m; then if $h\in A$,

$$(h(x)-Ph(0))x^{-1}(x-c_0)$$

lies in α and hence in m so that

$$0 = \int(h(x)-Ph(0))x^{-1}(x-c_0)\,d\mu(x)$$

or

$$\hat{h}(m)-Ph(0) = c_0\int(h(x)-Ph(0))x^{-1}\,d\mu(x).$$

This implies that $\pi(m)=c_0$. Therefore, if $\lim Pf(w_n)=L$, then $0=m(f-L)$ and hence the cluster set of Pf at c_0 lies in the range of $\hat{f}$ on the fiber over c_0. Hence, since the range of $\hat{f}$ on $\pi^{-1}(\lambda_0)$ can have at most 2 points and since the cluster set of Pf at λ_0 is connected, Pf is continuous from within D at each point of Γ; that is, $Pf\in A(D)$.

Let

$$K = \{c\in\Gamma : f(c)=Pf(c) \text{ for all } f\in A\};$$

K is a closed subset of Γ. If $K=\Gamma$, then $A\subseteq A(D)$, in contradiction to our initial assumption. Note that π is 2 to 1 over the points of $\Gamma\setminus K$. Let

$$V = \Omega\cap\{|z|<R\}.$$

Then by [3; Theorem 4] V is a π-regular component of B of multiplicity no more than 2 and if α is an open arc in Γ, then $\pi^{-1}(D\cup\alpha\cup V)\setminus\Gamma$ is a one-dimensional analytic variety on which the functions in B (and hence the functions in A) are analytic. If V has multiplicity 1, then Pf extends analytically across $\Gamma\setminus K$ to V. The cluster set argument given earlier shows that Pf is actually continuous on $V\cup\Gamma$. Letting $R\to\infty$ we find that $f|K\in A(S^2\setminus K)$ if K is non-empty or Pf is bounded and holomorphic on S^2 if K is empty. The latter is impossible since $\hat{f}=Pf$ must separate the points of D and the former has been ruled out by assumption. This allows only the possibility that V is π-regular of multiplicity 2.

Let S consist of all points $w\in\Omega$ such that $\pi^{-1}(w)$ is a single point in

M. Then S is closed and the only limit points of S which lie in Γ are in K. According to the basic theorem on analytic structure in the maximal ideal space [3], π is a 2-sheeted analytic cover of $\pi^{-1}(V)$ and each function in B, and hence each function in A, is analytic on $\pi^{-1}(V)$. Further, $S \cap V$ is discrete and for each point $p \in V \setminus S, \pi^{-1}(p)$ consists of two points p_1 and p_2 and there are disjoint neighborhoods W_1 and W_2 of p_1 and p_2, respectively, such that π is a homeomorphism of W_i onto a disc in the plane. Let $c \in \Gamma \setminus K$ and let m be the (unique) element of $\pi^{-1}(c)$ such that $m \notin \Gamma$. Then m has a neighborhood W which is mapped homeomorphically by π onto a disc centered at c and each f in A is analytic on W. This implies that Pf can be continued analytically across each point of $\Gamma \setminus K$. Note that K must have harmonic measure zero with respect to ∞ since the two sheets over V come together at K and the analytic functions $\hat{f}_i$ on these two sheets agree with f at all points of K. Let $R \to \infty$. This completes the proof.

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A minor variation of the proof of Theorem 2 yields the following.

Theorem 3. *Let α be an arc, $0 \notin \alpha$, and let μ be a finite regular Borel measure on α. Suppose A is a closed subalgebra of $C(\alpha)$ such that $1 \in A$ and whenever $f \in A$ and $\int_\alpha f d\mu = 0$, then $f(x)x^{-1} \in A$. Then either $A = C(\alpha)$ or $A \subseteq A(S^2 \setminus \alpha)$.*

One application of Theorem 3 is the following.

Proposition 4. *Let X_1 be an arc and X_2 a simple closed curve such that X_1 lies in the bounded component of $S^2 \setminus X_2$ Let $X = X_1 \cup X_2$ and let*

$$A = \{f \in C(X) : f|X_2 \in P(X_2)\}.$$

Then A is a maximal subalgebra of $C(X)$.

Proof. Let $f \in A$, $f \notin P(X)$; by subtracting an element of $P(X_2)$ from f we obtain a function in A which vanishes on X_2 and is not identically zero. Let $p \in X_1$ and let

$$A_1 = \{f \in A : f \text{ is constant on } X_2 \text{ and } f(p) = f(X_2)\}.$$

Let δ be the point mass at p. A_1 is a closed subalgebra of $C(X_1)$ and if $\int f d\delta = 0$, then $zf(z)$ and $z^{-1}f(z)$ both are in A_1. Hence, $A_1 = C(X_1)$ by Theorem 3.

Let B be a closed subalgrbea of $C(X)$ which properly contains A. Let $g \in B$, $g \notin A$. By the above we may subtract from g a function in A_1 which agrees with g on X_1 and hence B contains a function h which is zero on X_1 and whose restriction to X_2 is not in $P(X_2)$. By Wermer's theorem, this function h together with $P(X_2)$ generates $C(X_2)$. Hence, B contains every continuous function on X which vanishes on X_1 so that $B = C(X)$.

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