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SOME REMARKS ON KLEINIAN GROUPS

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1.

One of the purposes of this note is to answer a question which has been proposed by professor J. Lehner in his book *Discontinuous Groups and Automorphic Functions* [3, p. 136]. The problem may be stated as follows:

Let G be a function group, leaving invariant a simply connected component D of its ordinary set. Assume ∞ is an ordinary point and not a fixed point of an elliptic element of G. Then, is the Ford polygon for G relative to D always connected?

If D were required to be connected only, it would be easy to show that the answer is "no", for instance by means of Schottky groups of genus greater than one. But it is not difficult either, under the additional condition of simple connectivity, to demonstrate that the answer generally is in the negative. To this end quasi-Fuchsian groups will do.

Instead of solving the problem by a concrete example, which is possible but rather tedious, the opportunity is taken to bring up some observations related to the modern theory of Kleinian groups.

Thanks to the book mentioned above, it is not necessary here to discuss the notions and definitions belonging to this subject. The theory of quasi-Fuchsian groups and the boundaries of their deformation spaces is treated in the fundamental papers of L. Bers and B. Maskit, [1] and [5].

2.

All the groups considered in sections 2, 3 and 4 are generated by two Möbius transformations A and B with parabolic commutator $P = ABA^{-1}B^{-1}$. In terms of two-by-two matrices, the following representation is used:

$$A = \begin{pmatrix} z-i & -iz^2 \\ i & z+i \end{pmatrix}$$

$$B = \begin{pmatrix} x & 2x-yz \\ yz^{-1} & x \end{pmatrix}$$

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where x and y are two complex parameters and $z = -2xy(1-x^2-y^2)^{-1}$.

In this representation, the isometric circles of A^{-1} and P are externally tangent; the common point is the fixed point of P.

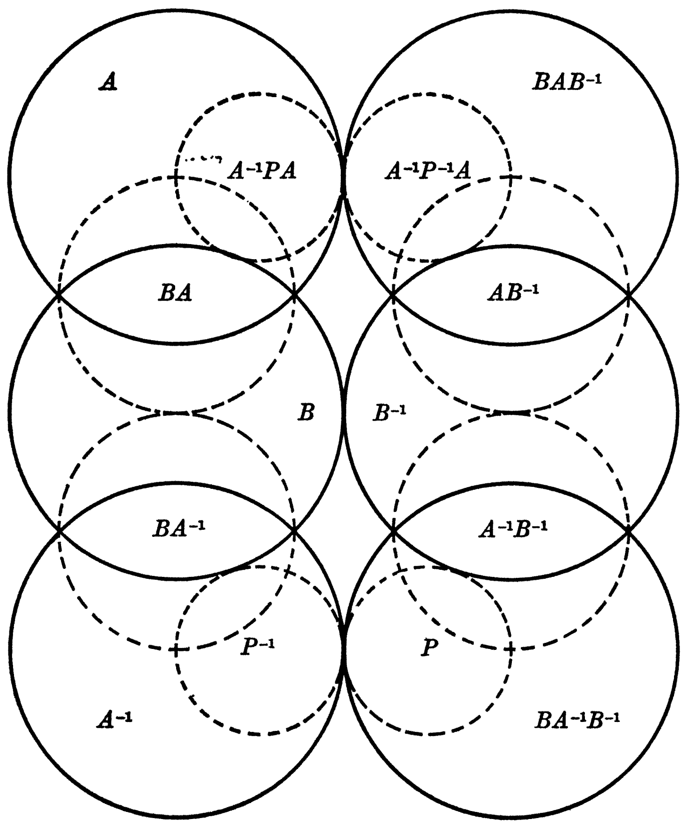
Every group generated by two transformations with parabolic commutator is conjugate to one of the groups above, unless all the elements of the group have a common fixed point.

The Fuchsian groups correspond to the relation

$$x = \sqrt{1+|y|^2}$$

between the parameters, where y can be any complex number with non vanishing real part.

Denote by T the space of quasi-Fuchsian groups in which both A and B have traces with positive real parts. Later it will be shown that T contains groups having Ford polygons with more than two connected components. Since their ordinary sets consist of two quasi-discs, each invariant, this will answer the question of Lehner.



Notice that the automorphism defined by

$$\begin{aligned} A &\to BA^{-1}B^{-1} \\ B &\to B^{-1} \end{aligned}$$

is induced by the involutory Möbius transformation $z \to -z$. Visually, this appears as a pointwise symmetry of the isometric circles with respect to the origin.

3.

The figure shows the Ford polygon of the Kleinian group, denoted by G_0, which arises by taking $x = 1$ and $y = \sqrt{2}$.

It is not hard to verify this assertion. Firstly, it is checked that the isometric circles of A, B, BA, BA^{-1}, BAB^{-1}, P and $A^{-1}PA$ together with the isometric circles of the inverse transformations make up a configuration as sketched. It is convenient to have the matrices explicitly:

$$A = \begin{pmatrix} \sqrt{2}-i & -2i \\ i & \sqrt{2}+i \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} \sqrt{2}-i & -2i \\ \sqrt{2} & \sqrt{2}-i \end{pmatrix}$$

$$BA^{-1} = \begin{pmatrix} \sqrt{2}+i & 2i \\ \sqrt{2} & \sqrt{2}+i \end{pmatrix}$$

$$BAB^{-1} = \begin{pmatrix} \sqrt{2}+i & -2i \\ i & \sqrt{2}-i \end{pmatrix}$$

$$P = \begin{pmatrix} -1-2i\sqrt{2} & 4 \\ 2 & -1+2i\sqrt{2} \end{pmatrix}$$

$$A^{-1}PA = \begin{pmatrix} -1-2i\sqrt{2} & -4 \\ -2 & -1+2i\sqrt{2} \end{pmatrix}.$$

Secondly, if, in the upper half space, one imagines the corresponding isometric hemi-spheres, it is easily seen that their common exterior points form a polyhedron with identifications, for which the angle relations — which can be read off from the polygon — are satisfied. Hence a theorem of Poincaré — proved by Maskit [6] — applies, showing that it is a fundamental polyhedron for the group generated by the side-pairing transformations. But this group is precisely G_0. Therefore, all other transformations of the group must have isometric hemi-spheres lying exterior to the polyhedron; in particular, the Ford polygon looks as claimed.

One minor difficulty has been passed over so far. Maskit makes in [6] an assumption, which is not satisfied in the present case, namely, that any two tangential faces of the polyhedron must be paired by a parabolic transformation. Here, as in other cases, this restriction can be evaded by a suitable modification of the polyhedron. Then the given polyhedron can be approximated by its deformations, for instance, by Dirichlet polyhedrons with respect to a sequence of points approaching ∞.

4.

From the polygons one can see that the quotient surfaces associated with G_0 are a torus with one puncture and a sphere with three punctures. The torus is represented by the infinite component, the sphere by the two finite components together. The puncture of the torus is paired to one of the punctures on the sphere; they both arise from the parabolic element P. The other pair of punctures arises from the parabolic transformation B.

Clearly, G_0 does not belong to T, the space of quasi-Fuchsian groups. But G_0 belongs to the boundary of T; the cusp determined by B can be opened.

To see this, recall that G_0 is given by $x=1$ and $y=\sqrt{2}$. Starting from these values and then changing x and y a little such that the absolute value of x becomes greater than 1, the isometric circles of B and B^{-1} will move apart, the configuration of circles through the fixed points of P and $A^{-1}PA$ will remain unchanged (the two points may move a little) while the isometric circles either of BA and $A^{-1}B^{-1}$ or of BA^{-1} and AB^{-1} may come out to carry sides of a new Ford polygon. That no other isometric circles can "break out" is an easy consequence of the general incidence relations between isometric circles; in this case, it is enough to know that if the isometric circle of a transformation I is tangent from the inside to the isometric circle of another transformation J, then the isometric circles of J^{-1} and IJ^{-1} are externally tangent.

Denote the new Kleinian group by G. As an example, both components of its Ford polygon may be bounded by sides lying on the isometric circles of

$$A^{-1},\ B,\ BA,\ A,\ BAB^{-1},\ B^{-1},\ A^{-1}B^{-1},\ BA^{-1}B^{-1}\,.$$

Each of the two quotient surfaces of G is a once-punctured torus. Clearly, G must have two invariant components; hence, by a theorem of Marden and of Maskit [5, p. 610], it follows that G is quasi-Fuchsian; therefore, G belongs to T. It is then obvious that G_0 belongs to the boundary of T, since G can be arbitrarily close to G_0.

But, the quasi-Fuchsian groups G, defined above, do not approximate any of the degenerate groups lying close to G_0 on the boundary of T. Their polyhedrons are too nice, the horocycles at the fixed point of P are too large. So already that shows that the cusp at B can be opened without separating the isometric circles of B and B^{-1}. For instance, one may change the parameters a little so that x becomes different from 1 but still of absolute value 1. Then the isometric circles of B and B^{-1} remain externally tangent; besides that, essentially, the polygon can only be deformed as before. The new group will be quasi-Fuchsian, the deformation being small enough, and its Ford polygon will have three components. This answers the question of Lehner.

5.

The boundary group G_0 is regular in the sense of having a finite sided fundamental polyhedron. Such groups are rather well understood. In his paper *The geometry of finitely generated Kleinian groups* [4], Marden has proved that they are quasiconformally stable in the sense of Bers [1]. More generally, the geometrical content of this matter admits of "reopening of cusps" as well. All that is needed to know is, what can happen to the critical cyclic subgroup — in the case of G_0, what can happen to the cyclic group generated by B. From [2], where the isometric polyhedrons of cyclic groups have been described, it follows that G_0 can be approximated from almost all directions by groups belonging to T. If G_0 is approximated by groups on the boundary of T and with parameter values $x \neq 1$, then x must approach 1 tangentially to the real axis. Finally, one may distinguish between controllable and non-controllable directions of deformations, controllable meaning that only finitely many sides of the isometric polyhedron change. For G_0, the non-controllable directions are precisely those in which the real part of x becomes less than 1. The situation is more complicated for groups whose polyhedrons are bounded by infinitely many sides.

6.

The Ford polygon of a finitely generated Kleinian group is bounded by a finite number of sides. This beautiful formulation of Ahlfors' finiteness theorem was first suggested by J. R. McMillan. A detailed treatment will appear elsewhere. The proof depends heavily on the results of Ahlfors. Using that the Ford polygon is locally finite, A. F. Beardon has deduced that the limit points on its boundary must be parabolic fixed points. But the number of such points is finite, and at each of them there is a horocycle making the polygon finite sided in a neighbourhood.

One consequence is that the number of connected components of the Ford polygon is finite. It is an open problem whether the number of limit points lying exterior to all isometric circles may be infinite. The number of connected components of the complement of the Ford polygon is finite; actually, an upper estimate can be given.

THEOREM. *Let G be a Kleinian group which is generated by n transformations. Assume that G has a Ford polygon. Then the number of connected components of the complement of the Ford polygon of G is at most equal to $2n$. If the number is $2n$, then G is a Schottky group.*

PROOF. Let Ph denote the isometric fundamental polyhedron of G. An element of G is called a pairing transformation if there is an edge or a side of Ph lying on its isometric hemi-sphere. Let S denote the set of pairing transformations. If g belongs to S, so does g^{-1}.

The number of components of the complement $\complement Ph$ of Ph relative to the closed upper half space is finite. Two such components K_+ and K_- are called related if there is a pairing transformation g such that the isometric circle of g belongs to K_+ and the isometric circle of g^{-1} belongs to K_-. Two components K and $\tilde{K}$ are called equivalent if they are the two extremities of a simple chain of neighbour-related components. Thereby an equivalence relation is defined on the set of components of $\complement Ph$. The equivalence classes are called blocks. One may observe that each block consists of the components of the complement of the isometric fundamental polyhedron of a subgroup of G. According to the classical combination theorem of F. Klein, G is the free product of these block-subgroups.

Assume that there are l blocks: B_i, $i=1,\ldots,l$. Label the elements of B_i as $K_{i,j}$, $j=1,\ldots,r_i$. Hence r_i is the number of elements of B_i, and the total number of components of $\complement Ph$ is

$$k = \sum_{i=1}^{l} r_i .$$

Accordingly, S is the disjoint union of k sets $S_{i,j}$; the pairing transformation g belongs to $S_{i,j}$ if and only if the isometric circle of g is contained in $K_{i,j}$.

The idea is to construct a certain space of deformations of G. It consists of groups with isometric fundamental polyhedrons which look like Ph locally. These groups will be called similarity deformations of G and the space will be denoted by $\Sigma(G)$. First a class of mappings of S onto other sets of Möbius transformations has to be defined.

A Möbius transformation M which does not keep ∞ fixed may be written as

$$M = \begin{pmatrix} Qc & b \\ c & -Pc \end{pmatrix}, \quad c \neq 0,$$

where P is the centre of the isometric circle of M and Q is the centre of the isometric circle of M^{-1}. Both circles have radius equal to $|c|^{-1}$. The argument of c may be called the rotation parameter of M, and b is determined by

$$1+bc+PQc^2 = 0 .$$

A parameter set $\{\lambda_i, q_{i,j}, \varphi_{i,j}\}$ consists of l positive real numbers λ_i together with k complex numbers $q_{i,j}$ and k purely imaginary numbers $\varphi_{i,j}$. Associate to each such set a mapping of S onto another set S' of Möbius transformations: If M belongs to $S_{i,j}$ and M^{-1} belongs to $S_{i,h}$ then the image M' is defined as

$$M' = \begin{pmatrix} Q'c' & b' \\ c' & -P'c' \end{pmatrix}$$

where

$$\begin{aligned} Q' &= q_{i,h}+\lambda_i Q \exp\varphi_{i,h} \\ P' &= q_{i,j}+\lambda_i P \exp\varphi_{i,j} \\ c' &= c\lambda_i^{-1} \exp[-\tfrac{1}{2}(\varphi_{i,h}+\varphi_{i,j})] . \end{aligned}$$

It is immediately verified that the configuration of isometric circles of the elements of $S'_{i,j}$ is the image of the configuration of isometric circles of the elements of $S_{i,j}$ under an euclidian similarity transformation determined by λ_i, $q_{i,j}$ and $\varphi_{i,j}$. Remark that $(M')^{-1}=(M^{-1})'$.

If A and B are two elements of $S_{i,j}$ and A^{-1} lies in $S_{i,a}$ and B^{-1} lies in $S_{i,b}$, then AB^{-1} must belong to $S_{i,b}$ and BA^{-1} must belong to $S_{i,a}$, provided that AB^{-1} and BA^{-1} are elements of S at all. In that case a computation shows that $(AB^{-1})'=A'(B^{-1})'$ and that $(BA^{-1})'=B'(A^{-1})'$.

Each parameter set induces k similarity mappings, $K_{i,j} \to K'_{i,j}$. If the images $K'_{i,j}$ are disjoint, then the group G' generated by S' is Kleinian because the isometric circle of any element of G' must belong to some $K'_{i,j}$. The complement of the isometric fundamental polyhedron Ph' of G' has $\{K'_{i,j}\}$ as its components. Since G is generated by S and the relations in G can be read off from the identifications of the pairing transformations, it follows that G' and G are isomorphic. In particular, G' is generated by n Möbius transformations.

Let O be the open subset of R^{3k+l} consisting of those points $\{\lambda_i, q_{i,j}, \varphi_{i,j}\}$ which define disjoint components $K'_{i,j}$. To each connected component of O corresponds a deformation space of groups $\{G'\}$; one of these contains G and is denoted $\Sigma(G)$. Since G is generated by n Möbius transformations, it follows that the real dimension of $\Sigma(G)$ is at most equal to $6n$. On the other hand, the real dimension of $\Sigma(G)$ is at least $3k$: there are l block scale parameters λ_i, another $2k$ real parameters come from the translations $q_{i,j}$, and in each block B_i, the rotations defined by the $\varphi_{i,j}$ contribute with at least $r_i - 1$ real dimensions. Hence $3k \leqq \mathrm{real}\dim \Sigma(G) \leqq 6n$; thus $k \leqq 2n$. The case $k = 2n$ is equivalent to G being a Schottky group for which each $S_{i,j}$ consists of a single transformation.

The final remark is that the above theorem can be proved without use of McMillan's version of the finiteness theorem. It is not necessary a priori to know that the number of components $K_{i,j}$ is finite. One may always replace G by another group G' for which the components $K'_{i,j}$ lie sufficiently apart from one another to make the dimension argument possible.

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