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INVARIANT DIFFERENTIAL OPERATORS AND POLYNOMIALS OF LIE TRANSFORMATION GROUPS

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Introduction.

Among all differential operators on R^n, those that have constant coefficients play an important role for analysis. They are characterized by their invariance under the transitive group of translations.

More generally, if G is a group acting on a differentiable manifold M, then ti is of great interest for analysis to determine the algebra $\mathscr{D}(M)$ of differential operators on M which are invariant under the action of G (see [8]). In case G is a transitive Lie transformation group of M then M can be identified with the quotient manifold G/H where H is a closed subgroup of G. Nomizu studied the differential geometry of those spaces $M=G/H$ which are *reductive*, that is admit a G-invariant affine connection (cf. [10], [11]). For these spaces the problem of determining $\mathscr{D}(G/H)$ was investigated by S. Helgason [5].

If G is a semisimple Lie group and $G=KAN$ an Iwasawa decomposition let M denote the centralizer of A in K. The space of horocycles in the symmetric space G/K can be identified with G/MN and can, in analogy with the space of hyperplanes in R^n, be viewed as a dual space to G/K. Although the space G/MN is not reductive the algebra $\mathscr{D}(G/MN)$ can be explicitly determined, (see [7]).

In this paper we study $\mathscr{D}(G/H)$ when G is a nilpotent Lie group and G/H not necessarily reductive. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively, and let $\mathfrak{m}$ be a linear subspace of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ (direct sum). Using the technique of [5] and [6] transferring the problem to one about polynomials on $\mathfrak{m}$, we give a general condition for an element of $\mathscr{D}(G)\simeq\mathscr{U}(\mathfrak{g})$ to define an element of $\mathscr{D}(G/H)$. In [2] and [3] central elements of $\mathscr{U}(\mathfrak{g})$ are studied, and it turns out to be simpler to work in the quotient field of $\mathscr{U}(\mathfrak{g})$. In this paper we proceed in the same way considering a subalgebra $\mathscr{R}(\mathfrak{g})$ of $\mathscr{U}(\mathfrak{g})$ instead of $\mathscr{D}(G/H)$, where $\mathscr{D}(G/H)$ is the canonical image of $\mathscr{R}(\mathfrak{g})$. In section 5 we determine the quotient field of the invariant polynomials on $\mathfrak{m}$.

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The reader is referred to [1], [6] and [9] for background material on Lie groups and Lie algebras.

1. Preliminaries.

Let M be a differentiable manifold of dimension m. $C^\infty(M)$ denotes the space of complex C^∞ functions on M. If (φ, U) is a local chart on M and $f \in C^\infty(M)$ we shall sometimes write f^* for the composite function $f \circ \varphi^{-1}$ defined on $\varphi(U)$. Let $x_1, x_2, \ldots, x_m$ be the coordinate functions of φ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a m-tuple of non-negative integers. We put $\partial_i = \partial/\partial x_i$ $(1 \leqq i \leqq m)$ and $D^\alpha = \partial_1^{\alpha_1} \ldots \partial_m^{\alpha_m}$.

$C_c^\infty(M)$ is the subspace of $C^\infty(M)$ consisting of the functions with compact support.

A linear map $D \colon C_c^\infty(M) \to C_c^\infty(M)$ is called a *differential operator* on M if the following condition is satisfied: For each $p \in M$ and each local chart (φ, U) around p there exists a finite set of functions $a_\alpha \in C^\infty(U)$ such that for each $f \in C_c^\infty(M)$ with support contained in U,

$$(1) \qquad \begin{aligned} (Df)(x) &= \textstyle\sum_\alpha a_\alpha(x)[D^\alpha f^*](\varphi(x)) && \text{if } x \in U\,, \\ (Df)(x) &= 0 && \text{if } x \notin U\,. \end{aligned}$$

If D is a differential operator on M then it can be extended in the obvious way to a linear map

$$D \colon C^\infty(M) \to C^\infty(M)\,.$$

A Lie group G is said to be a *Lie transformation group* of M if to each $g \in G$ is associated a diffeomorphism $\tau(g)$ of M such that

(i) $\tau(g_1 g_2) = \tau(g_1)\tau(g_2)$ for all $g_1, g_2 \in G$ and
(ii) the mapping $(g,p) \to \tau(g)p$ is a differentiable mapping of $G \times M$ onto M.

If the action is transitive, M is called a *homogeneous space*. In this case it follows that M is diffeomorphic to the quotient manifold G/H of left cosets gH, where H is the isotropy group of some element in M. The action on G/H is given by left multiplication,

$$\tau(g)(xH) = gxH \quad \text{for all } g, x \in G\,.$$

Now, suppose that G is a transitive Lie transformation group of M.

A differential operator D on M is *G-invariant* if

(2) $$D[\tau(g)f] = \tau(g)(Df) \quad \text{for all } f \in C^\infty(M)$$

and all $g \in G$, where $[\tau(g)f](p) = f(\tau(g)p)$ for all $p \in M$.

We let $\mathscr{D}(M)$ denote the algebra of G-invariant differential operators on M.

For every $D \in \mathscr{D}(M)$, fix $p \in M$ and choose a local chart (φ, U) around p. D has a local expression near p given by (1). Define a polynomial in m variables $X_1, X_2, \ldots, X_m$ by

$$P(X_1, \ldots, X_m) = \textstyle\sum_\alpha a_\alpha(p) X_1^{\alpha_1} \ldots X_m^{\alpha_m}.$$

Then

(3) $$(Df)(p) = [P(\partial_1, \ldots, \partial_m) f^*](\varphi(p))$$

for every $f \in C^\infty(M)$.

Using (2) we find

(4) $$(Df)(\tau(g)p) = [P(\partial_1, \ldots, \partial_m)(\tau(g)f)^*](\varphi(p))$$

for every $f \in C^\infty(M)$ and every $g \in G$. Since the action of G on M is transitive it follows that D is uniquely determined by the polynomial P.

Suppose that V is a linear space of finite dimension over a field K of characteristic 0, let $T(V)$ denote the tensor algebra over V and let J be the ideal in $T(V)$ generated by the set of elements of the form $X \otimes Y - Y \otimes X$, $X, Y \in V$. The factor algebra $\mathscr{S}(V) = T(V)/J$ is called the *symmetric algebra* over V. If $X_1, X_2, \ldots, X_n$ is a basis of V, $\mathscr{S}(V)$ can be identified with the abelian algebra of polynomials in the base elements over K.

Given a Lie group G with Lie algebra $\mathfrak{g}$, let $\mathscr{S}(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$. G acts on itself by left multiplication. As noted above, every $D \in \mathscr{D}(G)$ determines a unique polynomial $P \in \mathscr{S}(g)$ such that

(5) $$(Df)(g) = [P(\partial_1, \ldots, \partial_n) f(g \exp(x_1 X_1 + \ldots + x_n X_n))](0)$$

for all $g \in G$ and all $f \in C^\infty(G)$. ($X_1, \ldots, X_n$ is a basis of $\mathfrak{g}$.)

For $P \in \mathscr{S}(\mathfrak{g})$ arbitrary, (5) defines a left invariant differential operator $\lambda(P)$ on G. The map

$$\lambda \colon \mathscr{S}(\mathfrak{g}) \to \mathscr{D}(G)$$

is a *linear* isomorphism. If $Y_1, \ldots, Y_p \in \mathfrak{g}$ then

(6) $$\lambda(Y_1 Y_2 \ldots Y_p) = (p!)^{-1} \textstyle\sum_\sigma Y_{\sigma(1)} Y_{\sigma(2)} \ldots Y_{\sigma(p)}$$

where the right hand side is calculated in $\mathscr{D}(G)$. The sum is taken over all permutations σ of $\{1, \ldots, p\}$.

2. A criterium for a differential operator to be invariant.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let H be a closed subgroup of G. If $\mathfrak{h}$ denotes the Lie algebra of H let $\mathfrak{m}$ be any linear subspace of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \quad \text{(direct sum)} .$$

Choose a basis $X_1,\ldots,X_m,\ldots,X_n$ of $\mathfrak{g}$ such that $X_1,\ldots,X_m$ is a basis of $\mathfrak{m}$ and $X_{m+1},\ldots,X_n$ is a basis of $\mathfrak{h}$. If $\pi\colon G \to G/H$ is the canonical projection map, then

$$\pi(g \exp(x_1X_1 + \ldots + x_mX_m)) \to (x_1,\ldots,x_m) \tag{7}$$

defines a local chart of gH.

For $D \in \mathscr{D}(G/H)$ let $P \in \mathscr{S}(\mathfrak{m})$ be the unique polynomial determined by (3) with respect to this chart when $g=e=$identity element of G. Then by (4)

$$(Df)(gH) = [P(\partial_1,\ldots,\partial_m)f(g \exp(x_1X_1 + \ldots + x_mX_m)H)](0) \tag{8}$$

for all $g \in G$ and all $f \in C^\infty(G/H)$.

Lemma 2.1. *Let $\mathscr{D}(G)\mathfrak{h}$ denote the set of all real linear combinations of elements of the form DT where $D \in \mathscr{D}(G)$ and $T \in \mathfrak{h}$. Then*

$$\mathscr{D}(G) = \mathscr{D}(G)\mathfrak{h} + \lambda(\mathscr{S}(\mathfrak{m})) \quad \textit{(direct sum)} .$$

For a proof see [5, p. 394].

Let $C_0^\infty(G)$ denote the set of C^∞ functions on G which are constant on each coset gH. Then the mapping $f \to \tilde{f}$ where $\tilde{f} = f\circ\pi$, is an isomorphism of the algebra $C^\infty(G/H)$ onto $C_0^\infty(G)$.

Now, suppose that $D \in \mathscr{D}(G/H)$ and let $P \in \mathscr{S}(\mathfrak{m})$ denote the corresponding polynomial defined by (8). Since $\mathscr{S}(\mathfrak{m}) \subset \mathscr{S}(\mathfrak{g})$, $E = \lambda(P)$ is a left invariant differential operation on G. From (5) it is clear that

$$E\tilde{f} = (Df)^\sim \quad \text{for all } f \in C^\infty(G/H) . \tag{9}$$

Because $\tau(h)H = H$ for all $h \in H$, D must satisfy

$$(Df)(ghH) = (Df)(gH) \tag{10}$$

for all $g \in G$ and all $f \in C^\infty(G/H)$. Using (5) and (9) this means that

$$\begin{aligned} (11)\quad & [P(\partial_1,\ldots,\partial_m)\tilde{f}(g \exp(x_1X_1 + \ldots + x_mX_m))](0) \\ &= [P(\partial_1,\ldots,\partial_m)\tilde{f}(gh \exp(x_1X_1 + \ldots + x_mX_m))](0) \\ &= [P(\partial_1,\ldots,\partial_m)\tilde{f}\big(g \exp(x_1 \operatorname{Ad}(h)X_1 + \ldots + x_m \operatorname{Ad}(h)X_m)\big)](0) \\ &= [Q(\partial_1,\ldots,\partial_n)\tilde{f}(g \exp(x_1X_1 + \ldots + x_nX_n))](0) \end{aligned}$$

for all $f \in C^\infty(G/H)$ and all $h \in H$. The polynomial $Q \in \mathscr{S}(\mathfrak{g})$ is given by

(12) $$Q(X_1,\ldots,X_n) = P(\mathrm{Ad}(h)X_1,\ldots,\mathrm{Ad}(h)X_m)\,.$$

(11) can be expressed in the following way:

(13) $\lambda(Q-P)$ restricted to $C_0^\infty(G)$ defines the zero operator.

Because of Lemma 2.1 we can find $D \in \mathscr{D}(G)\mathfrak{h}$ and $R \in \mathscr{S}(\mathfrak{m})$ such that $\lambda(Q-P)=D+\lambda(R)$. Now $Df=0$ for all $f \in C_0^\infty(G)$, and it follows that $\lambda(R)f=0$ for all $f \in C_0^\infty(G)$.

Given $\alpha_1,\ldots,\alpha_m \in \mathrm{R}$, we see by (7) that there exists a function $f \in C^\infty(G/H)$ such that

(14) $$f\big(\pi(\exp(x_1X_1+\ldots+x_mX_m))\big) = e^{\alpha_1x_1+\ldots+\alpha_mx_m}$$

in a neighborhood of $\{H\}$ in G/H.

If $R \in \mathscr{S}(\mathfrak{m})$ and $R \neq 0$ we can find $\alpha_1,\ldots,x_m \in \mathrm{R}$ such that $R(\alpha_1,\ldots,\alpha_m) \neq 0$. With such a choice in (14) it follows that

$$\lambda(R)\tilde{f} = R(\alpha_1,\ldots,\alpha_m)\tilde{f}$$

in a neighbourhood of e, and in particular that $\lambda(R)\tilde{f} \neq 0$. This proves the following

Lemma 2.2. *If $D \in \mathscr{D}(G/H)$ let $P \in \mathscr{S}(\mathfrak{m})$ be the unique polynomial defined by* (8). *Putting*

$$[\mathrm{Ad}(h)\cdot P](X_1,\ldots,X_n) = P(\mathrm{Ad}(h)X_1,\ldots,\mathrm{Ad}(h)X_m)\,,$$

we have

(15) $$\lambda(\mathrm{Ad}(h)\cdot P) = D(h)+\lambda(P)$$

for all $h \in H$, where $D(h) \in \mathscr{D}(G)\mathfrak{h}$.

For each $X \in \mathfrak{g}$ let $d(X)$ be the uniquely determined derivation of $\mathscr{D}(G)$ (respectively $\mathscr{S}(\mathfrak{g})$), which extends the endomorphism $\mathrm{ad}(X)$ of $\mathfrak{g}$. This defines a $\mathfrak{g}$-module structure on $\mathscr{D}(G)$ and $\mathscr{S}(\mathfrak{g})$ and the linear map λ becomes a $\mathfrak{g}$ module isomorphism.

For each $g \in G$, the automorphism $\mathrm{Ad}(g)$ of $\mathfrak{g}$ extends uniquely to an automorphism of $\mathscr{D}(G)$. Let the extension also be denoted by $\mathrm{Ad}(g)$. Then $\lambda(\mathrm{Ad}(g)\cdot P)=\mathrm{Ad}(g)\lambda(P)$. Since the order of $d(X)D$ is less than or equal to the order of D for all $X \in \mathfrak{g}$, all differential operators $d(X)^nD$ lie in a finite dimensional subspace of $\mathscr{D}(G)$. This implies the convergence of the following series

(16) $$e^{d(X)}D = \sum_{n=0}^{\infty} \frac{1}{n!}\, d(X)^n D\,.$$

From the uniqueness mentioned above it follows that

$$\mathrm{Ad}(\exp X)D = e^{d(X)}D \tag{17}$$

for all $X \in \mathfrak{g}$ and all $D \in \mathscr{D}(G)$. By differentiation

$$\frac{d}{dt}[e^{td(X)}D]_{t=0} = d(X)D . \tag{18}$$

THEOREM 2.1. *If H is a closed connected subgroup of G, then $P \in \mathscr{S}(\mathfrak{m})$ defines an element of $\mathscr{D}(G/H)$ by* (8) *if and only if $d(T)\lambda(P) \in \mathscr{D}(G)\mathfrak{h}$ for all $T \in \mathfrak{h}$.*

PROOF. First suppose that $D \in \mathscr{D}(G/H)$. If $P \in \mathscr{S}(\mathfrak{m})$ is the corresponding polynomial then Lemma 2.2 and (18) imply

$$d(T)\lambda(P) \in \mathscr{D}(G)\mathfrak{h} \quad \text{for all } T \in \mathfrak{h} .$$

$\mathscr{D}(G)\mathfrak{h}$ is closed with respect to the limit process in (18).

Next suppose that $P \in \mathscr{S}(\mathfrak{m})$ satisfies $d(T)\lambda(P) \in \mathscr{D}(G)\mathfrak{h}$ for all $T \in \mathfrak{h}$. The subalgebra $\mathscr{D}(G)\mathfrak{h}$ is invariant under all derivations $d(T)$, $T \in \mathfrak{h}$. This means that the series

$$e^{d(T)}\lambda(P) - \lambda(P) = \textstyle\sum_{n=1}^{\infty} \frac{1}{n!}\, d(T)^n \lambda(P) \tag{19}$$

converges in $\mathscr{D}(G)\mathfrak{h}$ for all $T \in \mathfrak{h}$. (17), (19) and the fact that H is connected imply that

$$\lambda(\mathrm{Ad}(h)\cdot P) = D(h) + \lambda(P) \tag{20}$$

for all $h \in H$, where $D(h) \in \mathscr{D}(G)\mathfrak{h}$. Put $E = \lambda(P)$ and define a linear transformation D of $C^\infty(G/H)$ by

$$(Df)^{\tilde{}} = E\tilde{f} . \tag{21}$$

D is well defined because of (20). It is clear from the local expression of E that D is a differential operator. The left invariance of E finally shows that $D \in \mathscr{D}(G/H)$.

COROLLARY. *Suppose that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sum). Let σ be the projection of $\mathfrak{g}$ onto $\mathfrak{m}$. Denote by $\delta(X)$, $X \in \mathfrak{g}$, the derivation of $\mathscr{S}(\mathfrak{m})$ extending the endomorphism $\sigma \circ \mathrm{ad}(X)$ of $\mathfrak{m}$. If $D \in \mathscr{D}(G/H)$ and $P \in \mathscr{S}(\mathfrak{m})$ is the polynomial defined by* (8) *let P_D denote the component of highest degree. Then*

$$\delta(T)P_D = 0 \quad \textit{for all } T \in \mathfrak{h} . \tag{22}$$

PROOF. Write $P = P_D + Q$ where Q is of lower degree than P_D. By Theorem 2.1

$$\lambda(d(T)P_D) + \lambda(d(T)Q) \in \mathscr{D}(G)\mathfrak{h} \tag{23}$$

for all $T \in \mathfrak{h}$. But this is only possible if each term in $d(T)P_D$ contains at least one $T \in \mathfrak{h}$. Applying σ to (23) we find

$$\delta(T)P_D = \sigma\cdot[d(T)P_D] = 0$$

for all $T \in \mathfrak{h}$.

THEOREM 2.2. *Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sum) as previously. Denote by $I(\mathfrak{m})$ the subspace of $\mathscr{S}(\mathfrak{m})$ consisting of those P which satisfy $\delta(T)P = 0$ for all $T \in \mathfrak{h}$. Let*

$$\mathfrak{n} = \{X \in \mathfrak{m} \mid [X,T] \in \mathfrak{h} \text{ for all } T \in \mathfrak{h}\}.$$

If $I(\mathfrak{m}) = \mathscr{S}(\mathfrak{n})$ then the mapping $E \to D_E$ of $\mathscr{D}(N) \to \mathscr{D}(G/H)$ defined by

$$(D_E f)(gH) = E_n(n \to f(gnH))_{n=e} \tag{24}$$

for all $f \in C^\infty(G/H)$, where $n \in N$ and N is the normalizer of H in G, is onto and is an algebraic homomorphism.

PROOF. Let $D \in \mathscr{D}(G/H)$ and let $P \in \mathscr{S}(\mathfrak{m})$ be the corresponding polynomial. Then $P_D \in I(\mathfrak{m})$. By assymption P_D defines a left invariant differential operator E on N. It is easily shown that D_E given by (24) is well defined and that it is an element of $\mathscr{D}(G/H)$. The corresponding polynomial of D_E is of course P_D. Hence

$$D - D_E \in \mathscr{D}(G/H),$$

and this operator is determined by $P - P_D \in \mathscr{S}(\mathfrak{m})$ which is of lower degree than P. It follows by induction on the degree of P that every $D \in \mathscr{D}(G/H)$ comes from some $E \in \mathscr{D}(N)$.

The multiplicative property $D_{E_1E_2} = D_{E_1}D_{E_2}$ where $E_1, E_2 \in \mathscr{D}(N)$ is obviously satisfied.

EXAMPLE 1. As was shown by S. Helgason in [7] the hypothesis of Theorem 2.2 are satisfied for the space of horocycles in a symmetric space. Let G be a connected semisimple Lie group and let

$$G = KAN$$

be an Iwasawa decomposition of G. Put $M =$ centralizer of A in K. Denote by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{m}$ the Lie algebras of G, K, A, N and M, respectively. The set of horocycles in G/K is G/MN. The normalizer of MN

(respectively N) in G is MAN. In both cases $\mathscr{D}(G/MN)$ (respectively $\mathscr{D}(G/N)$) is determined by $\mathscr{D}(A)$ (respectively $\mathscr{D}(MA)$).

If H is a normal subgroup then of course $\mathfrak{n}(\mathfrak{h})=\mathfrak{g}$ and $I(\mathfrak{m})=\mathscr{S}(\mathfrak{m})$ where $\mathfrak{n}(\mathfrak{h})$ is the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$.

EXAMPLE 2. Let $\mathfrak{g}$ be the 3-dimensional real Lie algebra generated by x, y and z where

$$[x,y] = z, \quad [x,z] = [y,z] = 0 .$$

$\mathfrak{g}$ is nilpotent. Putting $\mathfrak{h}=\mathsf{R}x$ then $\mathfrak{n}(\mathfrak{h})=\mathsf{R}x+\mathsf{R}z \neq \mathfrak{g}$. The polynomial

$$P(y,z) = Q_n(z)y^n+\ldots+Q_1(z)y+Q_0(z)$$

is invariant if and only if the polynomials $Q_1(z)=\ldots=Q_n(z)=0$, hence $I(\mathfrak{m})=$ all polynomials in z.

EXAMPLE 3. As the following example shows Example (2) does not give the general situation for nilpotent Lie algebras. Let $\mathfrak{g}$ be the 4-dimensional Lie algebra generated by x, y, z, w where

$$[x,y] = z, \quad [x,w] = y ,$$
$$[x,z] = [y,z] = [w,z] = 0 .$$

Putting $\mathfrak{h}=\mathsf{R}x$ then $\mathfrak{n}(\mathfrak{h})=\mathsf{R}x+\mathsf{R}z$. Simple computation shows that a polynomial $P(z,y,w) \in I(\mathfrak{m})$ if and only if it is generated by z and y^2-2zw. The polynomial y^2-2zw is G-invariant and the associated left invariant differential operator D defined by (5) belongs to the center of $\mathscr{D}(G)$ where G is a Lie group with Lie algebra $\mathfrak{g}$.

3. Some algebraic tools.

Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic 0. Let $\mathscr{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. If G is a Lie group with Lie algebra $\mathfrak{g}$, we will identify $\mathscr{U}(\mathfrak{g})$ and $\mathscr{D}(G)$.

In [4] I. M. Gelfand and A. A. Kirillov proved

LEMMA 3.1. *$\mathscr{U}(\mathfrak{g})$ is a Noetherian ring without null divisors.*

From this it easily follows that

LEMMA 3.2. *$\mathscr{U}(\mathfrak{g})$ is an Ore ring without null divisors. (Ore ring means a ring with the following property: For all $a,b \in \mathscr{U}(\mathfrak{g})$, $a \neq 0$, $b \neq 0$ there exist $x,y \in \mathscr{U}(\mathfrak{g})$, $x \neq 0$, $y \neq 0$ such that $xa=yb$.)*

If $\mathscr{R}$ is an Ore ring without null divisors, define the quotient field in the following way.

Consider all expressions of the forms $a^{-1}b$ and ba^{-1} where $a,b \in \mathscr{R}$ and $a \neq 0$. Identify $a^{-1}b$ and cd^{-1} if $ac=bd$. Since $\mathscr{R}$ is an Ore ring it follows that every "right expression" cd^{-1} can be put into a left expression $a^{-1}b$. Also every couple of fractions $a^{-1}b$, $c^{-1}d$ can be reduced to one with common denominator. For expressions with common denominator define the operations addition, subtraction and division:

$$a^{-1}b_1 \pm a^{-1}b_2 = a^{-1}(b_1 \pm b_2) ,$$
$$(a^{-1}b_1)^{-1}(a^{-1}b_2) = b_1{}^{-1}b_2 .$$

Finally, multiplication by $a^{-1}b$ can be considered as division by the inverse element $b^{-1}a$.

Denote by $C(\mathfrak{g})$ the quotient field of $\mathscr{U}(\mathfrak{g})$. For any subset $\mathscr{A} \subset \mathscr{U}(\mathfrak{g})$ let $\mathscr{K}(\mathscr{A})$ be the subalgebra of $C(\mathfrak{g})$ generated by $\mathscr{A}$.

4. Comparison of $\mathscr{D}(G_0/H)$ and $\mathscr{D}(G/H)$ when $H \subset G_0 \subset G$.

Let H, G, $\mathfrak{h}$ and $\mathfrak{g}$ be as in section 2. If $D \in \mathscr{D}(G/H)$ put $E(D)=\lambda(P)$ where $P \in \mathscr{S}(\mathfrak{m})$ is the polynomial defined by (8) corresponding to D. Moreover let

$$J(\mathfrak{m}) = \{P \in \mathscr{S}(\mathfrak{m}) \mid P \text{ corresponds to some } D \in \mathscr{D}(G/H)\} .$$

Finally put

$$\mathscr{R}(\mathfrak{g}) = \mathscr{U}(\mathfrak{g})\mathfrak{h} + \lambda(J(\mathfrak{m})) . \tag{25}$$

For $E \in \mathscr{R}(\mathfrak{g})$ define

$$[\varrho(E)f](gH) = E_x[x \to f(gxH)]_{x=e}$$

for all $f \in C^\infty(G/H)$, $g \in G$.

Lemma 4.1. *$\mathscr{R}(\mathfrak{g})$ is a subalgebra of $\mathscr{U}(\mathfrak{g})$ and ϱ is an algebraic homomorphism of $\mathscr{R}(\mathfrak{g})$ onto $\mathscr{D}(G/H)$ with kernel $\mathscr{U}(\mathfrak{g})\mathfrak{h}$.*

Proof. If $E_1,E_2 \in \mathscr{R}(\mathfrak{g})$ then E_1E_2 leaves $C_0{}^\infty(G)$ invariant and this means that $E_1E_2=D+\lambda(P)$ for some $D \in \mathscr{U}(\mathfrak{g})\mathfrak{h}$ and some $P \in J(\mathfrak{m})$ (Lemma 2.1). Therefore $\mathscr{R}(\mathfrak{g})$ is closed under multiplication and the algebra property follows. The last part of the statement is immediate by the construction.

Now suppose that G_0 is a closed, connected and normal subgroup of G with Lie algebra $\mathfrak{g}_0$ such that $\mathfrak{h} \subset \mathfrak{g}_0 \subset \mathfrak{g}$. Let $\mathfrak{m}_0$ be a complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}_0$.

THEOREM 4.1. *If* $\mathfrak{g}_0$ *is an ideal of codimension* 1 *in* $\mathfrak{g}$ *and* x *an element of* $\mathfrak{g}$ *not in* $\mathfrak{g}_0$, *then for* $\mathfrak{m} = \mathfrak{m}_0 + \mathsf{R}x$ *either*

(i) $J(\mathfrak{m}) = J(\mathfrak{m}_0)$ *or*

(ii) *there exist elements* $a_1 \in \lambda(J(\mathfrak{m}_0))$, $a_1 \neq 0$ *and* $a_2 \in \mathcal{U}(\mathfrak{g}_0)$ *such that such that* $a = xa_1 + a_2 \in \lambda(J(\mathfrak{m}))$.

Moreover $\mathcal{K}(\mathcal{R}(\mathfrak{g})) \subset \mathcal{K}(a, a_1^{-1}, \mathcal{R}(\mathfrak{g}_0))$ *and* a *is transcendental over* $C(\mathfrak{g}_0)$.

PROOF. Suppose that $J(\mathfrak{m}) \neq J(\mathfrak{m}_0)$. Choose the element

$$x^n b_n + x^{n-1} b_{n-1} + \ldots + b_0 \in \lambda(J(\mathfrak{m}))$$

where $b_0, b_1, \ldots, b_n \in \mathcal{U}(\mathfrak{g}_0)$, $n > 0$ and $b_n \neq 0$. We apply $d(T)$, $T \in \mathfrak{h}$ to this element noting that

$$(26)\qquad \begin{aligned} d(T)(x^m) &= x^{m-1}[d(T)x] + x^{m-2}[d(T)x]x + \ldots + [d(T)x]x^{m-1} \\ &= mx^{m-1}[d(T)x] + x^{m-2}c_{m,m-2} + \ldots + xc_{m,1} + c_{m,0} \end{aligned}$$

where $c_{m,m-2}, c_{m,m-3}, \ldots, c_{m,0} \in \mathcal{U}(\mathfrak{g}_0)$. Using Theorem 2.1 and (26) we find

$$(27)\qquad \begin{aligned} & d(T)(x^n b_n + x^{n-1} b_{n-1} + \ldots + b_0) \\ &= x^n[d(T)b_n] + nx^{n-1}[d(T)x]b_n + x^{n-2}c_{n,n-2}b_n + \ldots + c_{n,0}b_n + \\ &\quad + x^{n-1}[d(T)b_{n-1}] + (n-1)x^{n-2}[d(T)x]b_{n-1} + \\ &\quad + x^{n-3}c_{n-1,n-3}b_{n-1} + \ldots + \\ &\quad + c_{n-1,0}b_{n-1} + \ldots + d(T)b_0 \in \mathcal{U}(\mathfrak{g})\mathfrak{h} \end{aligned}$$

for all $T \in \mathfrak{h}$. Collecting different powers of x we find

$$(28)\qquad d(T)b_n \in \mathcal{U}(\mathfrak{g})\mathfrak{h} \quad \text{and} \quad n[d(T)x]b_n + d(T)b_{n-1} \in \mathcal{U}(\mathfrak{g})\mathfrak{h}$$

for all $T \in \mathfrak{h}$.

Since $\mathcal{U}(\mathfrak{g})\mathfrak{h}$ is an ideal in $\mathcal{R}(\mathfrak{g})$ we conclude by (28) that

$$(29)\qquad d(T)(nxb_n + b_{n-1}) \in \mathcal{U}(\mathfrak{g})\mathfrak{h} \quad \text{for all } T \in \mathfrak{h}\,.$$

Now put $a_1 = nb_n$ and $a_2 = b_{n-1}$. This proves the first part of (ii).

Suppose that

$$d = x^p d_p + x^{p-1} d_{p-1} + \ldots + d_0 \in \lambda(J(\mathfrak{m}))$$

where $d_p, d_{p-1}, \ldots, d_0 \in \mathscr{U}(\mathfrak{g}_0)$ and $d_p \neq 0$. We want to prove that d is contained in the subalgebra $\mathscr{K}(a, a_1{}^{-1}, \mathscr{R}(\mathfrak{g}_0))$ of $C(\mathfrak{g})$. This is obviously satisfied if $p=0$. Suppose that it is proved for all integers $<p$. Using the first part of the proof we know that $d_p \in \lambda(J(\mathfrak{m}_0))$. Therefore

$$da_1{}^p - d_p a^p \in \mathscr{R}(\mathfrak{g}) .$$

But

$$\begin{aligned} da_1{}^p - d_p a^p &= (x^p d_p + \ldots + d_0)a_1{}^p - d_p(xa_1 + a_2)^p \\ &= \textstyle\sum_{k<p} x^k d_k' \end{aligned}$$

where $d_k' \in \mathscr{U}(\mathfrak{g}_0)$ for all $1 \leqq k \leqq p-1$. By the induction hypothesis

$$da_1{}^p - d_p a^p \in \mathscr{K}(a, a_1{}^{-1}, \mathscr{R}(\mathfrak{g}_0)) .$$

We know that $d_p \in \mathscr{R}(\mathfrak{g}_0)$ and this proves that $d \in \mathscr{K}(a, a_1{}^{-1}, \mathscr{R}(\mathfrak{g}_0))$. It remains to consider the $\mathscr{U}(\mathfrak{g})$ part of $\mathscr{R}(\mathfrak{g})$. That is, we must prove

$$\mathscr{U}(\mathfrak{g})\mathfrak{h} \subset \mathscr{K}(a, a_1{}^{-1}, \mathscr{R}(\mathfrak{g}_0)) .$$

First note that $a = xa_1 + a_2 = a_1 x + a_3$ for some $a_3 \in \mathscr{U}(\mathfrak{g}_0)$. Consider the subset $x\mathscr{U}(\mathfrak{g}_0)\mathfrak{h}$ of $C(\mathfrak{g})$.

$$x\mathscr{U}(\mathfrak{g}_0)\mathfrak{h} = (a_1{}^{-1}a - a_1{}^{-1}a_3)\mathscr{U}(\mathfrak{g}_0)\mathfrak{h} \subset \mathscr{K}(a, a_1{}^{-1}, \mathscr{R}(\mathfrak{g}_0))$$

because $a_3\mathscr{U}(\mathfrak{g}_0)\mathfrak{h} \subset \mathscr{R}(\mathfrak{g}_0)$.

Suppose that $x^k\mathscr{U}(\mathfrak{g}_0)\mathfrak{h} \subset \mathscr{K}(a, a_1{}^{-1}, \mathscr{R}(\mathfrak{g}_0))$ for all $1 \leqq k < n$. For $k=n$

$$\begin{aligned} x^n\mathscr{U}(\mathfrak{g}_0)\mathfrak{h} &= (a_1{}^{-1}a - a_1{}^{-1}a_3)x^{n-1}\mathscr{U}(\mathfrak{g}_0)\mathfrak{h} \\ &= a_1{}^{-1}ax^{n-1}\mathscr{U}(\mathfrak{g}_0)\mathfrak{h} - a_1{}^{-1}a_3x^{n-1}\mathscr{U}(\mathfrak{g}_0)\mathfrak{h} . \end{aligned}$$

By the induction hypothesis

$$a_1{}^{-1}ax^{n-1}\mathscr{U}(\mathfrak{g}_0)\mathfrak{h} \subset \mathscr{K}(a, a_1{}^{-1}, \mathscr{R}(\mathfrak{g}_0)) .$$

Consider the last term. Since

$$a_3x^{n-1} = x^{n-1}a_3 + x^{n-2}a'_{n-2} + \ldots + a'_0$$

where $a'_{n-2}, a'_{n-3}, \ldots, a'_0 \in \mathscr{U}(\mathfrak{g}_0)$,

$$\begin{aligned} a_3x^{n-1}\mathscr{U}(\mathfrak{g}_0)\mathfrak{h} &\subset x^{n-1}\mathscr{U}(\mathfrak{g}_0)\mathfrak{h} + x^{n-2}\mathscr{U}(\mathfrak{g}_0)\mathfrak{h} + \ldots + \mathscr{U}(\mathfrak{g}_0)\mathfrak{h} \\ &\subset \mathscr{K}(a, a_1{}^{-1}, \mathscr{R}(\mathfrak{g}_0)) . \end{aligned}$$

This means that $x^n\mathscr{U}(\mathfrak{g}_0)\mathfrak{h} \subset \mathscr{K}(a, a_1{}^{-1}, \mathscr{R}(\mathfrak{g}_0))$ for all n and hence that

$$\mathscr{U}(\mathfrak{g})\mathfrak{h} \subset \mathscr{K}(a, a_1{}^{-1}, \mathscr{R}(\mathfrak{g}_0)) .$$

Finally suppose that there exists a relation

$$a^q e_q + a^{q-1}e_{q-1} + \ldots + e_0 = 0$$

where $e_q, e_{q-1}, \ldots, e_0 \in \mathscr{U}(\mathfrak{g}_0)$ and $e_q \neq 0$. Exchanging a and xa_1+a_2 we find

$$x^q a_1{}^q e_q + x^{q-1} e'_{q-1} + \ldots + e'_0 = 0$$

where $e'_{q-1}, \ldots, e'_0 \in \mathscr{U}(\mathfrak{g}_0)$. But this is only possible if $a_1{}^q e_q = 0$ which is a contradiction, and the proof is complete.

Now suppose that $\mathfrak{h}$ is a subalgebra of a nilpotent Lie algebra $\mathfrak{g}$. $\mathfrak{n}(\mathfrak{h})$ denotes the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$. Put

$$\mathfrak{n}_r(\mathfrak{h}) = \mathfrak{n}(\mathfrak{n}_{r-1}(\mathfrak{h})) \quad \text{for } r \geqq 1$$

where $\mathfrak{n}_0(\mathfrak{h}) = \mathfrak{h}$. Let r_0 be the greatest possible integer such that $\mathfrak{n}_{r_0}(\mathfrak{h}) \neq \mathfrak{g}$. This is of course only possible if $\mathfrak{h} \neq \mathfrak{g}$. We exclude the case $\mathfrak{g} = \mathfrak{h}$. $\mathfrak{n}_r(\mathfrak{h})$ is an ideal in $\mathfrak{n}_{r+1}(\mathfrak{h})$. In particular $\mathfrak{n}_{r_0}(\mathfrak{h})$ is an ideal in $\mathfrak{g}$.

If $\mathfrak{g}$ is nilpotent and $\mathfrak{k}$ is an ideal in $\mathfrak{g}$, form the quotient algebra $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{k}$. Then $\tilde{\mathfrak{g}}$ is nilpotent and we can find a series of ideals in $\tilde{\mathfrak{g}}$

$$(0) = \tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}}_1 \subset \ldots \subset \tilde{\mathfrak{g}}_n = \tilde{\mathfrak{g}} \tag{30}$$

such that $\dim(\tilde{\mathfrak{g}}_{i+1}/\tilde{\mathfrak{g}}_i) = 1$ for $i = 0, 1, \ldots, n-1$. If $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ denotes the canonical projection map put $\mathfrak{g}_i = \pi^{-1}(\tilde{\mathfrak{g}}_i)$. Then

$$\mathfrak{k} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \ldots \subset \mathfrak{g}_n = \mathfrak{g} \tag{31}$$

is a series of ideals increasing in dimension by 1 by passing from $\mathfrak{g}_k$ to $\mathfrak{g}_{k+1}$ $(0 \leqq k \leqq n-1)$.

Returning to the situation $\mathfrak{h} \subset \mathfrak{g}$, $\mathfrak{h} \neq \mathfrak{g}$ where $\mathfrak{g}$ is nilpotent we can, by repeated use of the technique described in (30) and (31), find an increasing series of subalgebras of $\mathfrak{g}$

$$\mathfrak{h} \subset \mathfrak{n}(\mathfrak{h}) = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \ldots \subset \mathfrak{g}_m = \mathfrak{g} \tag{32}$$

where $\dim \mathfrak{g}_{j+1} = \dim \mathfrak{g}_j + 1$ for $j = 0, 1, \ldots, m-1$. Let x_j be an element of $\mathfrak{g}_j$ not in $\mathfrak{g}_{j-1}$ for $j = 1, \ldots, m$. $\mathfrak{m}_j$ denotes a complementary linear subspace of $\mathfrak{h}$ in $\mathfrak{g}_j$. These can be chosen in such a way that $\mathfrak{m}_{j+1} = \mathfrak{m}_j + \mathsf{R}x_{j+1}$ for $0 \leqq j \leqq m-1$.

Since every subalgebra of codimension 1 in a nilpotent Lie algebra is an ideal, Theorem 4.1 applies to every pair $(\mathfrak{g}_j, \mathfrak{g}_{j+1})$ for $j = 0, 1, \ldots, m-1$.

THEOREM 4.2. *Suppose that $\mathfrak{h}$ is a subalgebra of a nilpotent Lie algebra $\mathfrak{g}$. Assume $\mathfrak{h} \neq \mathfrak{g}$. Then either*

(i) $\mathfrak{n}(\mathfrak{h}) = \mathfrak{g}$ *or*
(ii) $m \geqq 1$ *in* (32).

In case (ii) *let* $j_1 < j_2 < \ldots < j_q$ *be the indices such that* $J(\mathfrak{m}_j) \neq J(\mathfrak{m}_{j-1})$, *so that by Theorem* 4.1, *for each* $j \in \{j_1, j_2, \ldots, j_q\}$ *we can find* $a_{j1} \in \lambda(J(\mathfrak{m}_{j-1}))$, $a_{j1} \neq 0$ *and* $a_{j2} \in \mathscr{U}(\mathfrak{g}_{j-1})$ *such that* $a_j = x_j a_{j1} + a_{j2} \in \lambda(J(\mathfrak{m}_j))$. *Then*

$$\mathscr{K}(\mathscr{R}(\mathfrak{g})) \subset \mathscr{K}(a_{j_1}, \ldots, a_{j_q}, a_{j_1 1}^{-1}, \ldots, a_{j_q 1}^{-1}, y_1, \ldots, y_r)[\bmod \mathscr{U}(\mathfrak{g})\mathfrak{h}]$$

where $y_1, \ldots, y_r$ *is a basis of* $\mathfrak{m}_0$.

The theorem is easily proved by induction on the dimension of $\mathfrak{g}$ using Theorem 4.1.

5. Invariant polynomials.

In view of the corollary of Theorem 2.1 it is important to determine the invariant polynomials $I(\mathfrak{m})$ where $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sum). But as we will see it turns out to be easier if we instead of $I(\mathfrak{m})$ consider its quotient field.

THEOREM 5.1. *Suppose that* $\mathfrak{h}$ *is a subalgebra of a Lie algebra* $\mathfrak{g}$ *such that* $\mathfrak{h} \subset \mathfrak{g}_0 \subset \mathfrak{g}$ *where* $\mathfrak{g}_0$ *is an ideal of codimension* 1 *in* $\mathfrak{g}$. *Choose an element* x *in* $\mathfrak{g}$ *not in* $\mathfrak{g}_0$ *and put* $\mathfrak{m} = \mathfrak{m}_0 + \mathsf{R}x$ *where* $\mathfrak{m}_0$ *is any complementary linear subspace of* $\mathfrak{h}$ *in* $\mathfrak{g}_0$.
Then either

(i) $I(\mathfrak{m}) = I(\mathfrak{m}_0)$ *or*
(ii) *there exist elements* $a_1 \in I(\mathfrak{m}_0)$, $a_1 \neq 0$ *and* $a_2 \in \mathscr{S}(\mathfrak{m}_0)$ *such that* $a = xa_1 + a_2 \in I(\mathfrak{m})$. $I(\mathfrak{m})$ *as a subalgebra of the quotient field* $C(\mathfrak{m})$ *is contained in the subalgebra* $\mathscr{K}(a, a_1^{-1}, I(\mathfrak{m}_0))$ *generated by* a, a_1^{-1} *and* $I(\mathfrak{m}_0)$.

In case (ii) *we also have*:

(iii) $C(\mathfrak{m})$ *is generated by* a *and* $I(\mathfrak{m}_0)$. $C(\mathfrak{m})$ *is a transcendental extension of* $C(\mathfrak{m}_0)$.

PROOF. If $I(\mathfrak{m}) \neq I(\mathfrak{m}_0)$ choose $P = x^n b_n + \ldots + b_0 \in I(\mathfrak{m})$ where $b_0, \ldots, b_n \in \mathscr{S}(\mathfrak{m}_0)$, $n > 0$ and $b_n \neq 0$. Applying $\delta(T)$, $T \in \mathfrak{h}$ to P we find

$$\begin{aligned} 0 &= \sigma \cdot [d(T)(x^n b_n + \ldots + b_0)] \\ &= \sigma \cdot (x^n[d(T)b_n] + nx^{n-1}[d(T)x]b_n + x^{n-1}[d(T)b_{n-1}] + \ldots + d(T)b_0) . \end{aligned}$$

But this is only possible if

$$\delta(T)b_n = 0 \quad \text{and} \quad \delta(T)(nxb_n + b_{n-1}) = 0$$

for all $T \in \mathfrak{h}$.

Define $a_1 = nb_n$ and $a_2 = b_{n-1}$. Then $a = xa_1 + a_2 \in I(\mathfrak{m})$. Following the lines of the proof of Theorem 4.1, (ii) follows. To prove (iii) we first note that $\mathscr{S}(\mathfrak{m})$ is abelian without null divisors and the quotient fields can be formed. See [1, p. 24]. $a_1 \in I(\mathfrak{m}_0)$ and therefore $a_1 \in C(\mathfrak{m}_0)$. From (ii) it is clear that $C(\mathfrak{m})$ is generated by a and $I(\mathfrak{m}_0)$. The last part of (iii) is proved in the same manner as in Theorem 4.1.

THEOREM 5.2. *Suppose that* $\mathfrak{h} \subset \mathfrak{n}(\mathfrak{h}) = \mathfrak{g}_0 \subset \ldots \subset \mathfrak{g}_m \subset \mathfrak{g}$ *as in Theorem 4.2, where* $\mathfrak{g}$ *is a nilpotent algebra. If* $m \geqq 1$ *let* $j_1 < j_2 < \ldots < j_q$ *be the indices such that* $I(\mathfrak{m}_j) \neq I(\mathfrak{m}_{j-1})$. *For each* $j \in \{j_1, \ldots, j_q\}$ *we can find* $a_{j1} \in I(\mathfrak{m}_{j-1})$, $a_{j1} \neq 0$ *and* $a_{j2} \in \mathscr{S}(\mathfrak{m}_{j-1})$ *such that* $a_j = x_j a_{j1} + a_{j2} \in I(\mathfrak{m}_j)$. *Then*

(i) $\mathscr{K}(I(\mathfrak{m})) \subset \mathscr{K}(a_{j_1}, \ldots, a_{j_q}, a_{j_1 1}^{-1}, \ldots, a_{j_q 1}^{-1}, y_1, \ldots, y_r)$

where $y_1, \ldots, y_r$ *is a basis of* $\mathfrak{m}_0$ *and*

(ii) *the quotient field of* $I(\mathfrak{m})$ *is the field generated by the algebraic independent elements* $a_{j_1}, \ldots, a_{j_q}, y_1, \ldots, y_r$.

(iii) *By* (ii) *we have* $a_{j_1 1} = b_1 (b_1')^{-1}, \ldots, a_{j_q 1} = b_q (b_q')^{-1}$ *where* $b_q, b_q' \in \mathscr{K}(a_{j_1}, \ldots, a_{j_{q-1}}, y_1, \ldots, y_r)$.

Putting $b = b_1 b_2, \ldots, b_q$ *we have*

$$I(\mathfrak{m}) \subset \mathscr{K}(a_{j_1}, \ldots, a_{j_q}, y_1, \ldots, y_r, b^{-1}) .$$

PROOF. (i) and (ii) are consequences of Theorem 5.1. To prove (iii) we use induction on the dimension of $\mathfrak{g}$. Note that

$$a_{j_1 1}^{-1} = b_1' b^{-1} b_2 b_3 \ldots b_q \in \mathscr{K}(a_{j_1}, \ldots, a_{j_{q-1}}, y_1, \ldots, y_r, b^{-1}) .$$

Similar results for $a_{j_2 1}^{-1}, \ldots, a_{j_q 1}^{-1}$ hold. (iii) now follows from (ii).

THEOREM 5.3. *Let* $\mathfrak{h}$ *be a subalgebra of a nilpotent Lie algebra* $\mathfrak{g}$. *Denote by* n *the dimension of* $\mathfrak{g}$, h *the dimension of* $\mathfrak{h}$ *and by* n_1 *the transcendence degree of* $C(\mathfrak{m})$ *over the scalar field* $\mathfrak{k}$ *of* $\mathfrak{g}$. *If* $n = n_1 + h$ *then* $\mathfrak{h}$ *is an ideal of* $\mathfrak{g}$.

PROOF. $n_1 = r + q$ where $r = \dim \mathfrak{m}_0$ and q is as in Theorem 5.2. We prove the theorem by induction on the degree of $\mathfrak{g}$. If $n = 1$ then either $\mathfrak{h} = \mathfrak{g}$ or $\mathfrak{h} = (0)$. In both cases the statement is satisfied. Suppose that it has been proved for all nilpotent Lie algebras of dimension $< n$, and let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n. $\mathfrak{h}$ is a subalgebra. If $n = n_1 + h$ then either $q = 0$ or $q > 0$. We only have to consider the possibility $q > 0$. With the notation of Theorem 5.2 this means that $j_q = m$.

Choose $a_{m1} \in I(\mathfrak{m}_{m-1})$, $a_{m1} \neq 0$ and $a_{m2} \in \mathscr{S}(\mathfrak{m}_{m-1})$ such that $x_m a_{m1} + a_{m2} \in I(\mathfrak{m})$. Then for $T \in \mathfrak{h}$

$$\begin{aligned} (33) \qquad 0 &= \delta(T)[x_m a_{m1} + a_{m2}] \\ &= \sigma \cdot ([d(T)x_m]a_{m1} + x_m[d(T)a_{m1}] + d(T)a_{m2}] \end{aligned}$$

where $\sigma \cdot P(y_1, \ldots, y_r, x_1, \ldots, x_m) = P(\sigma(y_1), \ldots, \sigma(x_m))$ for all $P \in \mathscr{S}(\mathfrak{m})$. By induction hypothesis $\delta(T)a_{m2} = 0$. We also know that $\delta(T)a_{m1} = 0$ for all $T \in \mathfrak{h}$. (33) therefore reduces to $[\delta(T)x_m]a_{m1} = 0$, but this is only possible if $[T, x_m] \in \mathfrak{h}$ for all $T \in \mathfrak{h}$. The theorem follows.

COROLLARY. *Let G be a nilpotent Lie group with Lie algebra $\mathfrak{g}$ and let H be the connected Lie subgroup of G corresponding to a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Suppose that $\mathfrak{n}(\mathfrak{h})$ is of codimension* 1 *in $\mathfrak{g}$. Then $I(\mathfrak{m}) = \mathscr{S}(\mathfrak{m}_0)$ and $\mathscr{D}(G/H)$ is determined by* (24).

PROOF. Using the previous notation we must have $n_1 = r$ and the first part follows. To complete the proof apply Theorem 2.2.

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